Notes on solutions and MATHEMATICA files

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These notes provide some explanatory remarks on the data files and MATHEMATICA notebooks that accompany the paper *Kerr black holes with synchronised hair: an analytic model and dynamical formation*.

1 The Ansatz

1.1 The metric

Following the conventions in [1, 2], we consider an axially symmetric line element

$$ds^2 = -e^{2F_0}N(R)dt^2 + e^{2F_1}\left[\frac{dR^2}{N(R)} + R^2 d\theta^2\right] + e^{2F_2}R^2\sin^2\theta(d\varphi - \mathcal{W}dt)^2 \ , \tag{1.1}$$

where

$$N(R) = 1 - \frac{r_H}{R} \ , \tag{1.2}$$

and F_0, F_1, F_2, W are functions of (R, θ). The coordinates, R, θ, φ are spherical-like, with $r_H \leqslant R < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$, while t is the time coordinate. The constant $r_H \geqslant 0$ is an input parameter corresponding to the event horizon radial coordinate.

The numerical work is done, however, in terms of a new radial coordinate

$$r \equiv \sqrt{R^2 - r_H^2} \tag{1.3}$$

such that $0 \leqslant r < \infty$, the horizon being located at $r = 0$ in the new coordinate. We also found it convenient to introduce a new off-diagonal function associated with rotation,

$$W(r, \theta) \equiv (r^2 + r_h^2)\mathcal{W}(r, \theta). \tag{1.4}$$

Therefore the line element (1.1) becomes

$$ds^2 = -e^{2F_0}\frac{r^2}{g(r)}H(r)dt^2 + e^{2F_1}\left[\frac{dr^2}{H(r)} + g(r)d\theta^2\right] + e^{2F_2}g(r)\sin^2\theta\left[d\varphi - \frac{W}{g(r)}dt\right]^2 , \tag{1.5}$$

with the auxiliary functions

$$H(r) \equiv \frac{\sqrt{r^2 + r_H^2}}{\sqrt{r^2 + r_H^2} + r_H} \ , \qquad g(r) \equiv r^2 + r_H^2 \ , \tag{1.6}$$

a form which is also used in the MATHEMATICA notebook to compute M_ψ, M_H, J_ψ, J_H.

1.2 The matter fields

The scalar field ansatz contains a single function Z which depends only on r, θ [1]

$$\Phi = Z(r,\theta)e^{i(m\varphi - wt)} \ . \tag{1.7}$$

The Proca field ansatz is more involved [2], and is defined in terms of four functions (H_i, V) which depend also on r, θ

$$A = [iV(r,\theta)dt + H_1(r,\theta)dr + H_2(r,\theta)d\theta + iH_3(r,\theta)\sin\theta d\varphi]\, e^{i(m\varphi - wt)} \ . \tag{1.8}$$

In both the above relations, $m \in \mathbb{Z}^+$ is the azimuthal harmonic index, whereas $w > 0$ is the frequency of the field.

The Einstein-matter field equations are solved with appropriate boundary conditions, which are compatible with an approximate construction of the solutions on the boundary of the domain of integration. For details on this part, see [1, 2, 3].

2 Details on the data

2.1 Input files

We provide four solutions of black holes with scalar hair (BHsSH) and six solutions of black holes with Proca hair (BHsPH). For each solution, we provide three files:

- ”res.txt” containing the input parameters (in particular r_H and w).
- ”funct.dat” containing the solution (numerical output on a given grid).
- ”fx-inf.txt” containing the 1st order radial derivatives the functions F_0, W evaluated at infinity (see below for details).

The input data for the BHsSH here is

$$\begin{aligned} &S1: w = 0.99,\ r_H = 0.300691 \\ &S2: w = 0.993,\ r_H = 0.298641 \\ &S3: w = 0.995,\ r_H = 0.282417 \\ &S4: w = 0.997,\ r_H = 0.245501 \end{aligned}$$

while for the BHsPH one takes

$$\begin{aligned} &P1: w = 0.93,\ r_H = 0.296913 \\ &P2: w = 0.94,\ r_H = 0.30475 \\ &P3: w = 0.96,\ r_H = 0.310005 \\ &P4: w = 0.973,\ r_H = 0.298034 \\ &P5: w = 0.984,\ r_H = 0.265868 \\ &P6: w = 0.992,\ r_H = 0.212392 \end{aligned}$$

All relevant parameters (*e.g.* mass and angular momentum) are computed in the MATHEMATICA notebooks. Therein, we also provide a comparison with the analytic predictions based on the quasi-Kerr horizon model provided in the paper.

2.2 The structure of files "funct.dat" containing the solutions

Both type of BHs possess a reflection symmetry with respect to the $z = 0$ plane (where $z = r\cos\theta$) [1, 2]. Therefore we shall provide data for $0 \leqslant \theta \leqslant \pi/2$ only.

We supply data in a generic format which is used for both types of BHs. The solutions are given on a grid with

$$n_x \times n_y \text{ points} \tag{2.9}$$

covering the region $(0 \leqslant x \leqslant 1) \times (0 \leqslant \theta \leqslant \pi/2)$, where x is a compactified coordinate, related to r in (1.3) via

$$x \equiv \frac{r}{1+r} \ . \tag{2.10}$$

Also,

$$\begin{aligned} n_x &= 261, \quad n_y = 35 \ \text{ for BHsPH,} \\ n_x &= 401, \quad n_y = 40 \ \text{ for BHsSH.} \end{aligned}$$

For BHsPH, the structure of the file 'funct.dat' is the following:

$$\begin{aligned}
&\{x_1, \theta_1; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
&\{x_2, \theta_1; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
&... \\
&\{x_{n_x}, \theta_1; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
\\
&\{x_1, \theta_2; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
&\{x_2, \theta_2; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
&... \\
&\{x_{n_x}, \theta_2; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}, \\
&... \\
&\{x_{n_x}, \theta_{n_y}; F_1, F_2, F_0, W, H_1, H_2, H_3, V\}.
\end{aligned}$$

For BHsSH, the structure of the file 'funct.dat' is the following:

$$\begin{aligned}
&\{x_1, \theta_1; F_1, F_2, F_0, Z, W\}, \\
&\{x_2, \theta_1; F_1, F_2, F_0, Z, W\}, \\
&...
\end{aligned}$$

$$\{x_{n_x}, \theta_1; F_1, F_2, F_0, Z, W\},$$

$$\{x_1, \theta_2; F_1, F_2, F_0, Z, W\},$$
$$\{x_2, \theta_2; F_1, F_2, F_0, Z, W\},$$
...
$$\{x_{n_x}, \theta_2; F_1, F_2, F_0, Z, W\},$$
...
$$\{x_{n_x}, \theta_{n_y}; F_1, F_2, F_0, Z, W\}.$$

The MATHEMATICA notebooks reads these data and interpolates it, reconstructing the profiles of all functions in the problem.

2.3 The mass and angular momentum computation

For each solution, the file "fx-inf.txt" contains the first derivatives all functions in the problem with respect to the compactified coordinate $x \equiv r/(1+r)$, evaluated at $x = 1$ (*i.e.* at spatial infinity). This data is used in the MATHEMATICA notebooks to compute the ADM mass M and the angular momentum J.

For the general metric ansatz (1.1), M and J are read from the asymptotic sub-leading behaviour of the metric functions:

$$g_{tt} = -e^{2F_0} N(R) + e^{2F_2} \mathcal{W}^2 R^2 \sin^2\theta = -1 + \frac{2M}{R} + \dots\ ,$$
$$g_{\varphi t} = -e^{2F_2} \mathcal{W} R^2 \sin^2\theta = -\frac{2J}{R} \sin^2\theta + \dots .$$

In practice, the asymptotic behaviour can be re-expressed in terms of the function W rather than $\mathcal{W}$, and small r, rather than R. Then, one possibility is to do a (large-r) fit of the functions $F_0(r)$, $W(r)$ and extract the coefficients of the $1/r$ terms. We have found, however, that a more accurate procedure is to compute M and J is from the first derivatives with respect to the compactified coordinate x of the functions F_0 and W evaluated at infinity (*i.e.* at $x = 1$).

This can be justified as follows. We consider a generic function $U(r)$ which decays as $1/r$ at infinity,

$$U(r) = \frac{c}{r} + \dots\ . \tag{2.11}$$

The same asymptotic behaviour expressed in term of the compactified radial coordinate $x = r/(1+r)$ reads

$$U(x) = \frac{c(1-x)}{x} + \dots\ . \tag{2.12}$$

Therefore the value of the constant c is given by the first derivative of $U(x)$ evaluated at infinity ($x = 1$)

$$\left.\frac{dU}{dx}\right|_{x=1} = -c \ . \tag{2.13}$$

Turning now to the mass and angular momentum stored in the field ($M_\psi,\ J_\psi$), they are computed in the MATHEMATICA notebooks by evaluated the volume integrals of the corresponding components of the energy-momentum tensor. The horizon mass and angular momentum are computed by Komar integrals at the horizon (see *e.g.* [4]).

References

[1] C. A. R. Herdeiro and E. Radu, “Kerr black holes with scalar hair,” Phys. Rev. Lett. **112** (2014) 221101 [arXiv:1403.2757 [gr-qc]].

[2] C. Herdeiro, E. Radu and H. Runarsson, “Kerr black holes with Proca hair,” Class. Quant. Grav. **33** (2016) no.15, 154001 [arXiv:1603.02687 [gr-qc]].

[3] C. Herdeiro and E. Radu, “Construction and physical properties of Kerr black holes with scalar hair,” Class. Quant. Grav. **32** (2015) no.14, 144001 [arXiv:1501.04319 [gr-qc]].

[4] P. K. Townsend, “Black holes: Lecture notes,” gr-qc/9707012.