Absorption of scalars by d–dimensional string corrected black holes

Filipe Moura

Centro de Matemática,

Universidade do Minho,

Braga, Portugal

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and work to appear.

Leading α' corrections

- Effective action in the Einstein frame

$$\frac{1}{16\pi G}\int\sqrt{-g}\left[\mathcal{R}-\frac{4}{d-2}\left(\partial^\mu\phi\right)\partial_\mu\phi+\lambda\,\mathrm{e}^{\frac{4}{d-2}(1+w)\phi}Y(\mathcal{R})\right]\mathrm{d}^dx,$$

$Y(\mathcal{R})$: scalar polynomial in the Riemann tensor with conformal weight w.
λ : suitable power of α', up to a numerical factor.

- Field equations

$$\begin{aligned}\mathcal{R}_{\mu\nu} &+ \lambda\,\mathrm{e}^{\frac{4}{d-2}(1+w)\phi}\left(\frac{\delta Y(\mathcal{R})}{\delta g^{\mu\nu}}+\frac{1}{d-2}Y(\mathcal{R})g_{\mu\nu}-\frac{1}{d-2}g_{\mu\nu}g^{\rho\sigma}\frac{\delta Y(\mathcal{R})}{\delta g^{\rho\sigma}}\right)=0;\\ \nabla^2\phi &- \frac{\lambda}{2}\,\mathrm{e}^{\frac{4}{d-2}(1+w)\phi}\,Y(\mathcal{R})=0.\end{aligned}$$

Background black hole solution

- Asymptotically flat, spherically symmetric metric in the Einstein frame of the type

$$d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2};$$

- General assumption for the α' corrected solution:

$$f(r) = f_0(r)\left(1 + \frac{\lambda}{R_H^{2n}} f_c(r)\right),\; g(r) = f_0(r)\left(1 + \frac{\lambda}{R_H^{2n}} g_c(r)\right).$$

- Tangherlini solution: $f_0(r) =: f_0^T(r) = 1 - \left(\frac{R_H}{r}\right)^{d-3};$
- The most general form we will be considering, in order to encode string effects:

$$f_0(r) = c(r)\left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right).$$

Scattering Theory

- Classical result in EH gravity - for *any* spherically symmetric black hole in arbitrary d, the absorption cross–section of minimally–coupled massless scalar fields equals the area of the black hole horizon in the low–frequency limit (Das, Gibbons, Mathur, 1997):

$$\sigma = A_H = 4GS.$$

- The result is analogous for higher spin fields.
- Is such result generalized with the inclusion of higher–derivative corrections?
- (Related to the α' corrections of $\frac{\eta}{s}\Big|_{\alpha'=0} = \frac{1}{4\pi}$.)

A test scalar field

- Let's consider a **massless** minimally coupled test scalar field $\mathcal{H}$.
- Without α' corrections, it obeys the KG equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \left[\sqrt{-g}g^{\mu\nu}\partial_\nu\mathcal{H}\right] = 0.$$

- This field can be redefined and expanded as

$$\Phi(t,r,\theta) = k(r)\mathcal{H}(t,r,\theta) = \sum_\ell \Phi_\ell(t,r)Y_{\ell 0\ldots 0}(\theta).$$

$Y_{\ell,\varphi_1,..,\varphi_{d-3}}(\theta)$: spherical harmonics defined over S^{d-2}.
$Y_{\ell 0\ldots 0}(\theta) \sim C_\ell^{\frac{d-3}{2}}(\cos\theta)$ (Gegenbauer polynomials).
$k_0(r) = r^{\frac{d-2}{2}}$.

The scalar field equation

- $\mathcal{H}$ obeys therefore a field equation of the type

$$\partial_t^2 \mathcal{H} - F^2(r)\, \partial_r^2 \mathcal{H} + P(r)\, \partial_r \mathcal{H} + Q(r)\, \mathcal{H} = 0.$$

$F(r), P(r), Q(r)$ are functionals of the metric and its derivatives, namely of the functions $f(r), g(r)$.

- For pure gravity (in the absence of α' corrections) in d dimensions, it is not difficult to obtain such functionals:

$$\begin{aligned} F &= \sqrt{fg}, \\ P &= -f\left[(d-2)\frac{g}{r} + \frac{1}{2}\left(f' + g'\right)\right], \\ Q &= \frac{\ell\,(\ell + d - 3)}{r^2} f + \frac{(g-f)f'}{r}. \end{aligned}$$

A field equation with a potential

Taking

$$k(r)=\frac{1}{\sqrt{F}}\exp\left(-\int\frac{P}{2F^2}\,dr\right)$$

and replace $\partial/\partial r$ by $\partial/\partial r_*$, with $dr_*=\frac{dr}{F(r)}$, the equation for Φ may be written as a wave equation with a potential $V\left[f(r),g(r)\right]$:

$$\frac{\partial^2\Phi}{\partial r_*^2}-\frac{\partial^2\Phi}{\partial t^2}=\left(Q+\frac{F'^2}{4}-\frac{FF''}{2}-\frac{P'}{2}+\frac{P^2}{4F^2}+\frac{PF'}{F}\right)\Phi\equiv V\left[f(r),g(r)\right]\Phi.$$

For solutions with $f(r)=g(r)$, a potential analogous to $V\left[f(r),g(r)\right]$ has been obtained in d dimensions (Cardoso, Lemos (2002)).

The field equation close to the horizon

- Close to the horizon, an **incoming** regular scalar field can only depend on t and r through their nonsingular combination, the incoming Eddington–Finkelstein coordinate $v = t + r_* : \left.\frac{\partial \mathcal{H}_{in}}{\partial u}\right|_{\text{horizon}} = 0.$
- Also, an **outgoing** regular scalar field can only depend on t and r through their nonsingular combination, the outgoing Eddington–Finkelstein coordinate $u = t - r_* :$ $\left.\frac{\partial \mathcal{H}_{out}}{\partial v}\right|_{\text{horizon}} = 0.$
- Combining the two possible solutions, $\left.\frac{\partial}{\partial u}\frac{\partial}{\partial v}\mathcal{H}\right|_{\text{horizon}} = 0.$
- This statement is **independent of the action:** it always gives us close to the horizon a second order field equation for the massless scalar (Paulos (2010)).

The λ–corrected scalar field equation

- At infinity, curvature corrections vanish and we also have a second order field equation for the massless scalar.
- This way $\mathcal{H}$ still obeys

$$\partial_t^2 \mathcal{H} - F^2(r)\, \partial_r^2 \mathcal{H} + P(r)\, \partial_r \mathcal{H} + Q(r)\, \mathcal{H} = 0.$$

$F(r), P(r), Q(r)$ are functionals of the metric and its derivatives with explicit λ corrections:

$$F = F_0 + \lambda F_1, P = P_0 + \lambda P_1, Q = Q_0 + \lambda Q_1,$$

which give a λ–corrected potential. Also $H = H_0 + \lambda H_1, k(r) = k_0(r) + \lambda k_1(r)$.

Example of a λ–corrected potential

- At order $\lambda = 0$, the potential in d–dimensons found by Cardoso and Lemos is the same that governs tensor type metric perturbations (Ishibashi, Kodama (2003)). That does not need to be the case in the presence of λ corrections.
- We get for the tensor perturbations

$$
\begin{aligned}
F &= \sqrt{fg}\left(1+\alpha'\,\frac{f'-g'}{4r}\right),\\
P &= -f\left[(d-2)\frac{g}{r}+\frac{1}{2}\left(f'+g'\right)\right.\\
&+ \left.\frac{\alpha'}{4r^2}\left(4(d-4)\frac{g(1-g)}{r}+rg'\left(f'-g'\right)-4gg'+2(d-2)gf'\right)\right]\\
Q &= \frac{\ell\left(\ell+d-3\right)}{r^2}f+\frac{(g-f)f'}{r}\\
&+ \frac{\alpha'}{2r^2}\left[\frac{\ell\left(\ell+d-3\right)}{r}f\left(2\frac{1-g}{r}+f'\right)+(g-f)f'^2\right].
\end{aligned}
$$

The string-corrected tensor potential

$$
\begin{aligned}
V_{\mathsf{T}}[f(r),g(r)] &= \frac{1}{16r^2fg}\Big[(16\ell(\ell+d-3)f^2g+r^2f^2f'^2+3r^2g^2f'^2-2r^2f(f+g)f'g' \\
&- 4r^2fg(g-f)f''+16rfg^2f'+4r(d-6)f^2gf' \\
&+ 4(d-2)rf^2gg'+4(d-4)(d-2)f^2g^2\Big] \\
&+ \frac{\alpha'}{32r^4fg}\Big[32\ell(\ell+d-3)f^2(1-g)g+16\ell(d+\ell-3)f^2gf'r \\
&+ 3r^3g^2f'^2\left(f'-g'\right)-r^3f^2f'^2\left(f'-g'\right)-2r^3fgf'\left(f'-g'\right)g' \\
&+ 2r^3fg^2\left(-3f'f''+2g'f''+f'g''\right)-4r^3f^2gf'\left(f''-g''\right) \\
&- 2r^3f^2gg'\left(f''-g''\right)-4r^3f^2g^2\left(f^{(3)}-g^{(3)}\right) \\
&+ 18r^2fg^2f'^2-12r^2f^2gf'^2-10r^2f^2gg'^2-2r^2fg^2f'g' \\
&+ 2r^2(4d-13)f^2gf'g'+8r^2f^2g^2f''+8(d-5)r^2f^2g^2g'' \\
&+ 4r(d-4)^2f^2g^2(f'+g')+8rf^2g^2(g'-f') \\
&+ 8(d-4)rf^2g(f'+g'-4gg')+16(d-5)(d-4)f^2g^2(1-g)\Big]\,.
\end{aligned}
$$

Scattering of scalars by black holes

- We work in the low-frequency regime, $R_H \omega \ll 1$.
- This allows us to use the technique of **matching solutions** near the event horizon to solutions at asymptotic infinity (Unruh (1976); Moura, Schiappa (2006); Halmark, Natário, Schiappa (2007)).
- Only contribution to the cross section at low frequency: s–wave, with $\ell = 0$. This way we only consider $\mathcal{H}_0 =: H$.
- Solutions of the form $H(r,t) = e^{i\omega t} H(r)$.

Near-horizon solution (I)

- The potential $V[f(r), g(r)]$ vanishes.
- The master equation reduces to a simple free–field equation

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right)\Big(k(r)H(r)\Big) = 0$$

whose solutions are purely incoming plane–waves in the tortoise coordinate:

$$H(r_*) = A_{\text{near}} e^{i\omega r_*}.$$

Near-horizon solution (II)

Very close to the event horizon, $r \simeq R_H$, one has

$$r_*(r) = \int \frac{1}{f_0(r)} \left(1 - \frac{\lambda}{R_H^{2n}} \frac{f_c(r) + g_c(r)}{2}\right) dr$$

$$\simeq: \frac{R_H}{(d-3)c(R_H)} \left(1 - \frac{\lambda}{R_H^{2n}} \frac{f_c(R_H) + g_c(R_H)}{2}\right) \log\left(\frac{r - R_H}{R_H}\right) + \mathcal{O}\left(r - R_H\right)$$

and then

$$H(r) \simeq A_{\text{near}} \left(1 + i \frac{R_H \omega}{(d-3)c(R_H)} \left(1 - \frac{\lambda}{R_H^{2n}} \frac{f_c(R_H) + g_c(R_H)}{2}\right) \log\left(\frac{r - R_H}{R_H}\right)\right).$$

Asymptotic infinity solution (I)

- We consider asymptotically flat black holes.
- At asymptotic infinity, again $V[f(r), g(r)]$ vanishes.
- The master equation reduces to a free–field equation whose solutions are incoming or outgoing plane–waves in the tortoise coordinate.
- In the original radial coordinate,

$$H(r) = (r\omega)^{(3-d)/2} \left[A\, J_{(d-3)/2}(r\omega) + B\, N_{(d-3)/2}(r\omega) \right].$$

Asymptotic infinity solution (II)

- At low–frequencies, with $r\omega \ll 1$, such solution becomes

$$H(r) \simeq A_{\text{asymp}} \frac{1}{2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)} + B_{\text{asymp}} \frac{2^{\frac{d-3}{2}}\Gamma\left(\frac{d-3}{2}\right)}{\pi\left(r\omega\right)^{d-3}} + \mathcal{O}\left(r\omega\right).$$

- In order to compute the absorption cross–section, one now needs to relate A_{near}, A_{asymp} and B_{asymp}.

Intermediate region solution (I)

- $V(r) \gg \omega^2$, $r\omega \ll 1$ (low–frequency constraint), $\frac{r-R_H}{R_H} \gg (R_H\omega)^2$.
- To order zero in λ,

$$\left[-f_0 \frac{d}{dr}\left(f_0 \frac{d}{dr}\right) + f_0 \left(\frac{(d-2)(d-4)f_0}{4r^2} + \frac{(d-2)f_0'}{2r}\right)\right]\left(k_0 H_0\right) = 0,$$

where, for any $f(r)$,
$k_0(r) = \frac{1}{\sqrt{f}} \exp\left(\int \left(\frac{d-2}{2r} + \frac{f'}{2f}\right)\, dr\right) = r^{\frac{d-2}{2}}.$

- This equation may be written as

$$\frac{d}{dr}\left(r^{d-2} f_0(r) \frac{d}{dr} H_0(r)\right) = 0.$$

Intermediate region solution (II)

- The equation for $H_0(r)$ can be written as:

$$H_0'' - \frac{P_0}{F_0^2} H_0' - \frac{Q_0}{F_0^2} H_0 = 0.$$

- Equation for $H_1(r)$:

$$H_1'' - \frac{P_0}{F_0^2} H_1' - \frac{Q_0}{F_0^2} H_1 = R(r),$$

$$R(r) = -\left(\frac{F_1}{F_0}\right)^2 H_0'' + \frac{P_1}{F_0^2} H_0' + \frac{Q_1}{F_0^2} H_0.$$

- This is a **non-homogeneous** version of the **(linear)** equation for $H_0(r)$.

Intermediate region solution (III)

- General solution for $H_0(r)$:

$$H_0(r) = A^0_{\text{inter}} + B^0_{\text{inter}} \int \frac{d\,r}{r^{d-2} f_0(r)}.$$

- General solution for $H_1(r)$ (and for $H(r)$):

$$H_1(r) = A^1_{\text{inter}}(r) + B^1_{\text{inter}}(r) \int \frac{d\,r}{r^{d-2} f_0(r)}.$$

(**variation of constants**).

- Two independent solutions for $H_0(r)$:

$$h_1(r) = 1,\;\; h_2(r) = \int \frac{d\,r}{r^{d-2} f_0(r)}.$$

Intermediate region solution (IV)

- The **wronskian matrix** is

$$W(r) = \begin{bmatrix} h_1(r) & h_2(r) \\ h_1'(r) & h_2'(r) \end{bmatrix} = \begin{bmatrix} 1 & \int \frac{d\,r}{r^{d-2} f_0(r)} \\ 0 & \frac{1}{r^{d-2} f_0(r)} \end{bmatrix}.$$

- A particular solution given by

$$H_1^{\mathsf{part}}(r) = v_1(r) h_1(r) + v_2(r) h_2(r),$$

with

$$\begin{bmatrix} v_1(r) \\ v_2(r) \end{bmatrix} = \int R(r)\, W^{-1}(r) \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\,r.$$

Intermediate region solution (V)

- The most general solution: add to $H_1^{\text{part}}(r)$ the most general solution to the homogeneous equation, including the contributions H_0, H_1 as $H = H_0 + \lambda H_1$:

$$H(r) = A_{\text{inter}} + B_{\text{inter}} \int \frac{d\,r}{r^{d-2} f_0(r)} + \lambda H_1^{\text{part}}(r).$$

- It can be shown that the function $H_1^{\text{part}}(r)$ is well defined, namely the indefinite integrals

$$\begin{aligned} v_1(r) &= -\int R(r) r^{d-2} f_0(r) h_2(r)\, d\,r, \\ v_2(r) &= \int R(r) r^{d-2} f_0(r)\, d\,r. \end{aligned}$$

Intermediate region solution (VI)

- The function $H_1^{\text{part}}(r)$ has been explicitly checked to be **finite** and **subleading** when compared to $H_0(r)$.
- Near the horizon,

$$H(r) \simeq A_{\text{inter}} + \frac{B_{\text{inter}}}{(d-3)R_H^{d-3}c\,(R_H)} \log\left(\frac{r-R_H}{R_H}\right) + \cdots$$

- At asymptotic infinity, one finds

$$H(r) \simeq A_{\text{inter}} - \frac{B_{\text{inter}}}{d-3}\frac{1}{r^{d-3}} + \cdots$$

Calculation of the fluxes

- Matching coefficients:

$$
\begin{aligned}
A_{\text{as}} &= 2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)A_{\text{inter}} = 2^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)A_{\text{near}}, \\
B_{\text{as}} &= -\frac{\pi\omega^{d-3}}{2^{\frac{d-3}{2}}(d-3)\Gamma\left(\frac{d-3}{2}\right)}B_{\text{inter}} \\
&= -\frac{i\pi\left(R_H\omega\right)^{d-2}}{2^{\frac{d-1}{2}}\Gamma\left(\frac{d-1}{2}\right)}\left(1-\frac{\lambda}{R_H^{2n}}\frac{f_c(R_H)+g_c(R_H)}{2}\right)A_{\text{near}}.
\end{aligned}
$$

- Near the black hole horizon the flux per unit area is

$$
J_{\text{near}} = \frac{1}{2i}\left(H^\dagger(r_*)\frac{dH}{dr_*} - H(r_*)\frac{dH^\dagger}{dr_*}\right) = \omega\left|A_{\text{near}}\right|^2.
$$

- At infinity, the flux per unit area is

$$
J_{\text{as}} = \frac{1}{2i}\left(H^\dagger(r)\frac{dH}{dr} - H(r)\frac{dH^\dagger}{dr}\right) = \frac{2}{\pi}r^{2-d}\omega^{3-d}\left|A_{\text{as}}B_{\text{as}}\right|.
$$

The absorption cross–section

- General formula:
$$\sigma = \frac{\int r^{d-2} J_{\text{asymp}} d\Omega_{d-2}}{J_{\text{near}}} = \frac{2}{\pi}\omega^{2-d}\frac{\left|A_{\text{asymp}}B_{\text{asymp}}\right|}{\left|A_{\text{near}}\right|^2}\Omega_{d-2}.$$
- In our case,
$$\sigma = A_H\left(1 - \frac{\lambda}{R_H^{2n}}\,\frac{f_c(R_H) + g_c(R_H)}{2}\right).$$
- σ is still given in terms of information at the horizon;
- is it related to the α'–corrected black hole entropy?

Thermodynamics: temperature

- Wick–rotate to Euclidean time $t = i\tau$; the resulting manifold has no conical singularities as long as τ is a periodic variable, with a period $\beta = \frac{1}{T}$.
- Smoothness condition: $2\pi = \lim_{r\to R_H} \frac{\beta}{g^{-\frac{1}{2}}(r)} \frac{df^{\frac{1}{2}}(r)}{dr}$, or

$$T = \lim_{r\to R_H} \frac{\sqrt{g}}{2\pi} \frac{d\sqrt{f}}{d\,r}.$$

- In our case,

$$T = \frac{f_0'(R_H)}{4\pi} \left(1 + \frac{\lambda}{R_H^{2n}} \frac{f_c(R_H) + g_c(R_H)}{2}\right).$$

The α' correction to T is the same we obtained to σ, but with opposite sign.

Thermodynamics: entropy

- Wald entropy: $S = -2\pi G \int_\Sigma \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} \sqrt{h}\, d\Omega_{d-2}$;
- $\varepsilon_{tr} = \sqrt{\frac{f}{g}}$;
- For $Y(\mathcal{R}) = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$,
  $8\pi G \frac{\partial \mathcal{L}}{\partial \mathcal{R}^{\mu\nu\rho\sigma}} \varepsilon^{\mu\nu} \varepsilon^{\rho\sigma} = \left(-\frac{f}{g} + \text{e}^{\frac{4}{d-2}\phi} \frac{\alpha'}{4} f'' \right) \frac{g}{f}$;
- At order $\lambda = 0$, $\mathcal{R}^{trtr} = \frac{1}{2} f'' = -\frac{1}{2R_H^2}(d-3)(d-2)$, $\phi = 0$, $f = g = f_0$;
- One gets $S = \frac{A_H}{4} \left(1 + (d-3)(d-2) \frac{\alpha'}{4R_H^2} \right)$;
- α'–corrections increase S for every value of d.
- This is a general result for solutions which are α'–corrections to the Tangherlini black hole.

The Callan-Myers-Perry black hole

- For $Y(\mathcal{R}) = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}$;
- The only free parameter is the horizon radius R_H (secondary hair), which is not changed;
- $f_0(r) =: f_0^T(r) = 1 - \left(\frac{R_H}{r}\right)^{d-3}$;
- $f_c(r) = g_c(r) = f_c^{CMP}(r) := -\frac{(d-3)(d-4)}{2} \left(\frac{R_H}{r}\right)^{d-3} \frac{1-\left(\frac{R_H}{r}\right)^{d-1}}{1-\left(\frac{R_H}{r}\right)^{d-3}}.$
- α'–corrected cross section:

$$\sigma = A_H \left(1 + \frac{(d-1)(d-4)}{2} \frac{\alpha'}{4R_H^2}\right).$$

- One finds $\sigma \neq 4GS$.

Some questions

- Cornalba et. al. (2006) found out that $\sigma = 4GS$, to all orders in α', for fundamental strings in the (small) black hole phase (BPS states of heterotic strings compactified on $S^1 \times T^5$).
- Recently, Kuperstein/Murthy (2010) also found such agreement, to first order in α', for 1/4 BPS $\mathcal{N} = 4$ supersymmetric black holes in $d = 4, 5$.
- Open questions: does that result only hold for supersymmetric black holes? What could be the minimal amount of supersymmetry for it to eventually hold? Does it hold for generic dimensions?

Things to do

- Take the near–extremal limit and apply the formula for supersymmetric black holes;
- Verify the agreement with the shear viscosity obtained by the "pole method" (Paulos (2010)).
- Maybe derive a general solution for d–dimensional spherically symmetric α'–corrected black holes to all orders?
- **See you next meeting!**