QFT on quantum Bianchi I space-time

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Motivation

- QFT in curved space-time is a theory wherein matter is treated quantum-mechanically, but gravity is treated classically in agreement with GR.
- This provides a good approximate description in circumstances where the quantum effects of gravity do not play a dominant role.
- This suggests that the background classical space-time in the Planck regime (near the singularity) has to be replaced by a quantum background geometry.
- Discrete approaches to quantum gravity lead to a breakdown of the usual structure of space-time at around the Planck scale, with possible violations of Lorentz symmetry.
- This can have phenomenological implications, such as a deformation of the dispersion relations for propagating particles (modes of a matter field) on this background.

Lorentz symmetry breaking

Phenomenological Approach to Lorentz invariance violation (LIV) issues has been studied by Amelino-Camelia et al. [2]. Unbroken mass-shell constraint (for massless particles) is given by:

$$E^2 - p^2 = 0 \tag{1}$$

Breaking energy scale: $E_{\mathrm{Pl}} \sim 1.2 \times 10^{19}\, GeV$, at leading order,

$$p^2 = E^2 \left[1 + \sigma \frac{E}{E_{\mathrm{Pl}}} + \mathcal{O}\left(\frac{E^2}{E_{\mathrm{Pl}}^2} \right) \right]. \tag{2}$$

where σ depends on the QG theory. Thinking of this as a dispersion relation, we obtain the measured velocity is

$$v = \frac{dE}{dp} = 1 - \sigma \frac{E}{E_{\mathrm{Pl}}}. \tag{3}$$

This deviation from the 'conventional' speed of light ($c = 1$) can be observed in GRB's or other highly energetic particles.

QFT on LQC space-time

Attempt to define QFT on LQC spacetime has been done by Ashtekar, Kaminski, Lewandowski [3]:

- The construction is carried out in FRW space-time.
- The analysis involves a single mode $\vec{k}$ of a massless scalar field ϕ.

Therein, by comparing QFT on classical and semiclassical limit of the QFT on quantum space-time an effective metric $\bar{g}_{ab}$ is emerged.
In principle, the resulting geometry could depend on each field's mode $\vec{k}$; the quanta of different energy and momentum would 'feel' different geometries, and hence an (apparent) Lorentz violation could be obtained:

- For FRW case, $\bar{g}_{ab}$ does not depend on $\vec{k}$; therefore, no LIV.
- Justification can be: FRW is conformally flat, thus, the massless particles do not feel the difference with Minkowski geometry ($\bar{g}_{ab}p^a p^b = 0 \quad \Leftrightarrow \quad \Omega^2 \bar{g}_{ab}p^a p^b = 0$).

The idea: QFT on anisotropic geometries

The Bianchi type I space-time metric:

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \sum_{i=1}^{3} a_i^2(t)(dx^i)^2, \qquad (4)$$

Why Bianchi I cosmology?

- Possible idea is to consider more gravitational dof's, such as in anisotropic cosmological models. Therefore, it is the simplest anisotropic models for cosmology.
- Anisotropy may lead to the Lorentz symmetry breaking.
- It is interesting from the point of view of BKL conjecture.
- LQC of Bianchi I model is available: Ashtekar and Wilson-Ewing [4].

Classical background geometry

We consider the background space-time manifold to be topologically $M = \mathbb{R} \times \mathbb{T}^3$. In terms of Ashtekar $SU(2)$ connection variables, $A_a^i = c^i \delta_a^i$, and $E_i^a = p_i \delta_i^a$: It holds then the phase space variables, p_i, of gravity Γ_{gr}:

$$p_1 = a_2 a_3, \quad p_2 = a_3 a_1, \quad p_3 = a_1 a_2. \tag{5}$$

It is convenient to work with a harmonic time function τ: $N_\tau d\tau = N_t dt$ and $N_\tau = \sqrt{|p_1 p_2 p_3|} = V$. In terms of (τ, x^i), the BI metric becomes then

$$g_{\mu\nu} dx^\mu dx^\nu = |p_1 p_2 p_3| \left[-d\tau^2 + \sum_{i=1}^{3} \frac{(dx^i)^2}{p_i^2} \right]. \tag{6}$$

Indeed, Gauss and Vector constraints are already reduced, and hence, we are left with the (homogeneous part of) scalar constraint only:

$$C_{\mathrm{gr}} = \int_{\mathcal{V}} d^3x \ N_\tau \mathcal{C}_{\mathrm{gr}} = -\frac{1}{8\pi G \gamma^2} (p_1 p_2 c_1 c_2 + p_2 p_3 c_2 c_3 + p_3 p_1 c_3 c_1). \tag{7}$$

In LQC, the massless scalar field T and its conjugate momentum P_T coordinatise the phase space of matter, denoted by Γ_T. The energy density is $\rho_T = P_T^2/2V^2$: the contribution of T to the scalar constraint reads

$$C_T = \int_{\mathcal{V}} d^3x N_\tau \mathcal{C}_T = \frac{P_T^2}{2} \, , \tag{8}$$

Total scalar constraint is obtained as

$$C_{\text{geo}} = C_{\text{gr}} + C_T . \tag{9}$$

The τ-evolution of any phase space function

$$dT/d\tau = \{T, C_{\text{geo}}\} = P_T , \quad dP_T/d\tau = \{P_T, C_{\text{geo}}\} = 0. \tag{10}$$

So that, $T = P_T \tau$: thus, T is a good *relational time*. Using $N_T dT = N_\tau d\tau$: $N_T = \sqrt{|p_1 p_2 p_3|}/P_T$. Therefore, in terms of (T, x^i):

$$g_{\mu\nu} dx^\mu dx^\nu = |p_1 p_2 p_3| \left[-\frac{1}{P_T^2} dT^2 + \sum_{i=1}^{3} \frac{(dx^i)^2}{p_i^2} \right] . \tag{11}$$

The kinematical Hilbert space of Bianchi I model, $\mathcal{H}_{\rm kin}$, is given as: $\mathcal{H}_{\rm gr} \otimes \mathcal{H}_{\rm T}$, where

- $\mathcal{H}_{\rm gr}$: the Hilbert space of the gravitational sector is spanned by $\hat{p}_i$-eigenstates $|\vec{\lambda}\rangle := |\lambda_1, \lambda_3, \lambda_3\rangle$.
- $\mathcal{H}_{\rm T} = L_2(\mathbb{R}, dT)$: the Hilbert space of scalar field is quantized according to Schroedinger picture.

The scalar constraint operator $\widehat{C}_{\rm geo}$ is well-defined on $\mathcal{H}_{\rm kin}$:

$$\widehat{C}_{\rm geo} = -\frac{1}{2}(\hbar^2 \partial_T^2 \otimes \mathbb{I}) - \frac{1}{2}(\mathbb{I} \otimes \Theta) \ . \tag{12}$$

Physical states $\Psi_o(T, \vec{\lambda}) \in \mathcal{H}_{\rm kin}$ are those lying in the kernel of $\widehat{C}_{\rm geo}$, which turn out to be the (positive frequency) solutions to

$$\begin{aligned} -i\hbar\partial_T \Psi_o(T, \vec{\lambda}) &= \sqrt{|\Theta|}\Psi_o(T, \vec{\lambda}) \\ &=: \widehat{H}_o \Psi_o(T, \vec{\lambda}) \ . \end{aligned} \tag{13}$$

Classical field on classical background

The classical background $M = \mathbb{R} \times \mathbb{T}^3$, equipped with (x_0, x^j):

$$g_{\mu\nu} dx^\mu dx^\nu = -N_{x_0}^2(x_0) dx_0^2 + \sum_{i=1}^{3} a_i^2(x_0)(dx^i)^2. \tag{14}$$

where $x^j \in \mathbb{T}^3$, with $x_0 \in \mathbb{R}$ being a generic time coordinate.
Matter: A real (inhomogeneous) scalar field $\phi(x_0, \vec{x})$ on this background space-time, whose Lagrangian is

$$\mathcal{L}_\phi = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2). \tag{15}$$

For the pair (ϕ, π_ϕ), the classical solutions of the equation of motion can be expanded in:

$$\phi(x_0, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\vec{k}\in\mathcal{L}} \phi_{\vec{k}}(x_0) e^{i\vec{k}\cdot\vec{x}},$$

$$\pi_\phi(x_0, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{\vec{k}\in\mathcal{L}} \pi_{\vec{k}}(x_0) e^{i\vec{k}\cdot\vec{x}}, \tag{16}$$

For the pair (ϕ, π_ϕ), Poisson bracket reads,

$$\{\phi_{\vec{k}}, \pi_{\vec{k}'}\} = \delta_{\vec{k},-\vec{k}'}. \tag{17}$$

Notice that, $(k_1, k_2, k_3) \in (2\pi\mathbb{Z})^3$ span a 3-dimensional lattice $\mathcal{L}$.
Considering a mode decomposition:

$$\begin{aligned} \phi_{\vec{k}} &= \tfrac{1}{\sqrt{2}}\left(\phi^{(1)}_{\vec{k}} + i\phi^{(2)}_{\vec{k}}\right), \\ \pi_{\vec{k}} &= \tfrac{1}{\sqrt{2}}\left(\pi^{(1)}_{\vec{k}} + i\pi^{(2)}_{\vec{k}}\right). \end{aligned} \tag{18}$$

Since the reality conditions are satisfied: $\phi_{\vec{k}} = \overline{\phi_{-\vec{k}}}$ and $\pi_{\vec{k}} = \overline{\pi_{-\vec{k}}}$, thus, not all variables in Eq. (18) are independent.
Since there exist relations between the "positive" and "negative" modes $\vec{k}$ and $-\vec{k}$, the lattice $\mathcal{L}$ can be splitted into:

$$\begin{aligned} \mathcal{L}_+ &= \{\vec{k} : k_3 > 0\} \cup \{\vec{k} : k_3 = 0, k_2 > 0\} \cup \{\vec{k} : k_3 = k_2 = 0, k_1 > 0\}, \\ \mathcal{L}_- &= \{\vec{k} : k_3 < 0\} \cup \{\vec{k} : k_3 = 0, k_2 < 0\} \cup \{\vec{k} : k_3 = k_2 = 0, k_1 < 0\}. \end{aligned}$$

Let us define the following real variables for all values of $\vec{k} \in \mathcal{L}$:

$$q_{\vec{k}} = \begin{cases} \phi^{(1)}_{\vec{k}} & \text{if } \vec{k} \in \mathcal{L}_+ \\ \phi^{(2)}_{-\vec{k}} & \text{if } \vec{k} \in \mathcal{L}_- \end{cases} , \quad p_{\vec{k}} = \begin{cases} \pi^{(1)}_{\vec{k}} & \text{if } \vec{k} \in \mathcal{L}_+ \\ \pi^{(2)}_{-\vec{k}} & \text{if } \vec{k} \in \mathcal{L}_- \end{cases} \tag{19}$$

Therefore, we can obtain the Hamiltonian of the test fields as a collection of decoupled harmonic oscillators:

$$\begin{aligned} H_\phi(x_0) &:= \sum_{\vec{k}\in\mathcal{L}} H_{\vec{k}}(x_0) = \frac{N_{x_0}}{2\sqrt{|p_1 p_2 p_3|}} \\ &\times \sum_{\vec{k}\in\mathcal{L}} \left[p_{\vec{k}}^2 + \left(\sum_{i=1}^{3} (p_i k_i)^2 + |p_1 p_2 p_3| m^2 \right) q_{\vec{k}}^2 \right], \end{aligned} \tag{20}$$

one of them for each $\vec{k}$. In order to pass to quantum theory, we will henceforth fucus on a single mode $q_{\vec{k}}$.

Quantum field on classical background

In order to pass to quantum theory, for each mode $\vec{k}$:

- the Hilbert space of the matter is $\mathcal{H}_{\vec{k}} = L_2(\mathbb{R}, dq_{\vec{k}})$;
- the dynamical variables become operators:

$$\widehat{q}_{\vec{k}}\psi(q_{\vec{k}}) = q_{\vec{k}}\psi(q_{\vec{k}}), \quad \widehat{p}_{\vec{k}}\psi(q_{\vec{k}}) = -i\hbar\partial/\partial q_{\vec{k}}\psi(q_{\vec{k}}). \tag{21}$$

- time x_0-evolution is generated by the time-dependent Hamiltonian operator $\widehat{H}_{\vec{k}}(x_0)$ via Schroedinger equation:

$$\begin{aligned} i\hbar\partial_{x_0}\psi(x_0, q_{\vec{k}}) &= \frac{N_{x_0}(x_0)}{2\sqrt{|p_1(x_0)p_2(x_0)p_3(x_0)|}} \\ &\quad \times \left[\widehat{p}_{\vec{k}}^2 + \left(\sum_{i=1}^{3}(p_i k_i)^2 + |p_1 p_2 p_3| m^2\right)\widehat{q}_{\vec{k}}^2\right]\psi(x_0, q_{\vec{k}}) \\ &=: \widehat{H}_{\vec{k}}(x_0)\psi(x_0, q_{\vec{k}}) \end{aligned} \tag{22}$$

Dispersion relation of the test field

A prediction of many approaches to quantum gravity comes from the study of *in vacuo* "dispersion relation" (i.e., the relation between the frequency ω and the wave-vector $\vec{k}$ of a mode of a field).

For each mode $\vec{k} \in \mathcal{L}$, the x_0-evolution of each pair of variables $(q_{\vec{k}}, p_{\vec{k}})$:

$$\frac{dq_{\vec{k}}}{dx_0} = \{q_{\vec{k}}, H_{\vec{k}}\} = \frac{N_{x_0}}{\sqrt{|p_1 p_2 p_3|}} p_{\vec{k}}$$

$$\frac{dp_{\vec{k}}}{dx_0} = \{p_{\vec{k}}, H_{\vec{k}}\} = -\frac{N_{x_0}}{\sqrt{|p_1 p_2 p_3|}} \times \left(\sum_{i=1}^{3} (p_i k_i)^2 + |p_1 p_2 p_3| m^2 \right) q_{\vec{k}}. \quad (23)$$

Let us define:

$$\beta = -\frac{d}{dx_0} \ln \left(\frac{N_{x_0}}{\sqrt{|p_1 p_2 p_3|}} \right), \quad (24)$$

$$\omega_{\vec{k}}^2 = \frac{N_{x_0}^2}{|p_1 p_2 p_3|} \left(\sum_{i=1}^{3} (p_i k_i)^2 + |p_1 p_2 p_3| m^2 \right). \quad (25)$$

Then, Hamilton equations give:

$$\frac{d^2 q_{\vec{k}}}{dx_0^2} + \beta \frac{dq_{\vec{k}}}{dx_0} + \omega_{\vec{k}}^2 q_{\vec{k}} = 0. \tag{26}$$

We can write this equation of motion in a *normal form* as

$$\frac{d^2 \mathcal{Q}_{\vec{k}}}{dx_0^2} + \Omega_{\vec{k}}^2 \mathcal{Q}_{\vec{k}} = 0, \tag{27}$$

where $\mathcal{Q}_{\vec{k}}$ and $\Omega_{\vec{k}}^2$ are

$$\mathcal{Q}_{\vec{k}} := q_{\vec{k}} \exp\left(\frac{1}{2}\int^{x_0} \beta(x_0') dx_0'\right), \tag{28}$$

$$\Omega_{\vec{k}}^2 = \left(\omega_{\vec{k}}^2 - \frac{\beta^2}{4} - \frac{1}{2}\frac{d\beta}{dx_0}\right). \tag{29}$$

For choice of harmonic time $x_0 = \tau$, $\beta = 0$ (since $N_{x_0} = \sqrt{|p_1 p_2 p_3|}$), and for a massless scalar field ϕ, Eq. (27) reduces to

$$\frac{d^2 q_{\vec{k}}}{d\tau^2} + \omega^2_{\tau,\vec{k}} q_{\vec{k}} = 0, \tag{30}$$

with $\omega^2_{\tau,\vec{k}} = \sum_{i=1}^{3} (p_i k_i)^2$.

For the wave 4-vector $k_\mu = (\omega_{\tau,\vec{k}}, \vec{k})$ of the quantum field, a cosmological observer (with 4-velocity $u^\mu = (\sqrt{-g_{00}^{-1}}, 0, 0, 0)$) measures a frequency

$$\Omega_{\tau,\vec{k}} := u^\mu k_\mu = \frac{\omega_{\tau,\vec{k}}}{\sqrt{|p_1 p_2 p_3|}} \ . \tag{31}$$

The observed 3-velocity of the mode is then

$$V^i = \frac{d\Omega_{\tau,\vec{k}}}{dk_i} = \frac{1}{\sqrt{|p_1 p_2 p_3|}} \frac{k_i p_i^2}{\Omega_{\tau,\vec{k}}} \ . \tag{32}$$

Then, the norm of this vector reads

$$\|V\|^2 = \sum_i \frac{1}{p_i^2} \left(\frac{k_i p_i^2}{\omega_{\tau,\vec{k}}} \right)^2 = 1. \tag{33}$$

The velocity of quanta of massless scalar field, measured by cosmological observers, is precisely the velocity of light, $c = 1$; this confirm the local Lorentz symmetry on the classical BI space-time.

Is the issue of local Lorentz symmetry held on the quantum BI geometry?

QFT on quantum BI background

For a single mode $\vec{k}$, the kinematical Hilbert space becomes

$$\mathcal{H}_{\rm kin}^{(\vec{k})} = \mathcal{H}_{\rm geo} \otimes L^2(\mathbb{R}, dq_{\vec{k}}).$$

The scalar constraint:

$$\begin{aligned} \widehat{C}_{\tau,\vec{k}} &:= \widehat{C}_{\rm geo} + \widehat{H}_{\tau,\vec{k}} = -\frac{\hbar^2}{2}(\partial_T^2 \otimes \mathbb{I}_{\rm gr} \otimes \mathbb{I}_{\vec{k}}) \\ &\quad - \frac{1}{2}(\mathbb{I}_T \otimes \Theta \otimes \mathbb{I}_{\vec{k}}) + (\mathbb{I}_T \otimes \widehat{H}_{\tau,\vec{k}}). \end{aligned} \tag{34}$$

where

$$\widehat{H}_{\tau,\vec{k}} = \frac{1}{2}\left[\widehat{p}_{\vec{k}}^2 + \left(\sum_{i=1}^{3} \widehat{p}_i^2 k_i^2 + |\widehat{p}_1 \widehat{p}_2 \widehat{p}_3| m^2\right) \widehat{q}_{\vec{k}}^2\right], \tag{35}$$

Physical states, $\Psi(T, \vec{\lambda}, q_{\vec{k}})$ on $\mathcal{H}_{\rm phys}^{(\vec{k})} = Ker\ (\widehat{C}_{\tau,\vec{k}})$, being the space of "positive frequency" solutions to

$$-i\hbar\partial_T \Psi(T, \vec{\lambda}, q_{\vec{k}}) = \left[\widehat{H}_o^2 - 2\widehat{H}_{\tau,\vec{k}}\right]^{1/2} \Psi(T, \vec{\lambda}, q_{\vec{k}}). \tag{36}$$

Considering $\widehat{H}_{\tau,\vec{k}}$ as a (mode-dependent) perturbation to $\widehat{H}_o^2$, we can use the operator identity:

$$(A+B)^{1/2} = A^{1/4}\left(1+\frac{1}{2}A^{-1/2}BA^{-1/2}+...\right)A^{1/4}, \tag{37}$$

for $A=\widehat{H}_o^2$ and $B=-2\widehat{H}_{\tau,\vec{k}}$, to obtain:

$$\begin{aligned} -i\hbar\partial_T\Psi(T,\vec{\lambda},q_{\vec{k}}) &= \left[\widehat{H}_o - \widehat{H}_o^{-\frac{1}{2}}\widehat{H}_{\tau,\vec{k}}\widehat{H}_o^{-\frac{1}{2}}\right]\Psi(T,\vec{\lambda},q_{\vec{k}}) \\ &=: \left[\widehat{H}_o - \widehat{H}_{T,\vec{k}}\right]\Psi(T,\vec{\lambda},q_{\vec{k}}). \end{aligned} \tag{38}$$

Here we have used the test field approximation where the backreaction of the scalar field on geometry was disregarded.

Effective BI geometry

- QFT on classical BI geometry:

$$
\begin{aligned}
i\hbar\partial_{x_0}\psi(x_0, q_{\vec{k}}) &= \frac{N_{x_0}(x_0)}{2\sqrt{|p_1(x_0)p_2(x_0)p_3(x_0)|}} \\
&\times \left[\widehat{p}_{\vec{k}}^2 + \left(\sum_{i=1}^{3}(p_i k_i)^2 + |p_1 p_2 p_3| m^2\right)\widehat{q}_{\vec{k}}^2\right]\psi(x_0, q_{\vec{k}}).
\end{aligned}
$$

- QFT on quantum BI geometry:

$$
-i\hbar\partial_T\Psi(T, \vec{\lambda}, q_{\vec{k}}) = \left[\widehat{H}_o - \widehat{H}_o^{-\frac{1}{2}}\widehat{H}_{\tau,\vec{k}}\widehat{H}_o^{-\frac{1}{2}}\right]\Psi(T, \vec{\lambda}, q_{\vec{k}}).
$$

To compare, we need to take the classical limit for the geometrical dof's:

- Pass to the interaction picture (geometrical dof's described in Heisenberg picture);
- Using the Born-Oppenheimer approximation (assuming the geometrical dof's as 'heavy').

Interaction picture

The physical state of the system:

$$\Psi(T,\vec{\lambda},q_{\vec{k}}) = \Psi_o(T,\vec{\lambda}) \otimes \psi(T,q_{\vec{k}}), \tag{39}$$

where the geometry evolves through $\widehat{H}_o$, i.e., $-i\hbar\partial_T\Psi_o = \widehat{H}_o\Psi_o$:

$$\Psi_o(T,\vec{\lambda}) = e^{iT\widehat{H}_o/\hbar}\Psi_o(0,\vec{\lambda}). \tag{40}$$

Then, the Sheroedinger for QFT on quantum geometry becomes:

$$i\hbar\partial_T\psi(T,q_{\vec{k}}) = \frac{1}{2}\Bigg[\langle\widehat{H}_o^{-1}\rangle\widehat{p}_{\vec{k}}^2 + \langle\widehat{H}_o^{-\frac{1}{2}}\left(\sum_{i=1}^{3}\widehat{p}_i^2(T)k_i^2\right.$$
$$\left. +|\widehat{p}_1(T)\widehat{p}_2(T)\widehat{p}_3(T)|m^2\right)\widehat{H}_o^{-\frac{1}{2}}\rangle\widehat{q}_{\vec{k}}^2\Bigg]\psi(T,q_{\vec{k}}), \tag{41}$$

where $\langle\widehat{A}(T)\rangle$ denotes the expectation value on the quantum state of geometry $\Psi_o(0,\vec{\lambda})$ of gravitational operator

$$\widehat{A}(T) = e^{-iT\widehat{H}_o/\hbar}\widehat{A}e^{iT\widehat{H}_o/\hbar}\ . \tag{42}$$

By setting $x_0 = T$, the Shroedinger equation becomes:

$$i\hbar\partial_T\psi(T, q_{\vec{k}}) = \frac{\bar{N}_T(T)}{2\sqrt{|\bar{p}_1\bar{p}_2\bar{p}_3|}}\left[\widehat{p}_{\vec{k}}^2 + \left(\sum_{i=1}^{3}(\bar{p}_i k_i)^2 + |\bar{p}_1\bar{p}_2\bar{p}_3|m^2\right)\widehat{q}_{\vec{k}}^2\right]\psi(T, q_{\vec{k}}),$$

for an effective BI metric $\bar{g}_{\mu\nu}$ of the form:

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -\bar{N}^2(T)dT^2 + |\bar{p}_1\bar{p}_2\bar{p}_3|\sum_{i=1}^{3}\frac{(dx^i)^2}{\bar{p}_i^2} \ . \tag{43}$$

where $\bar{N}$ and $\bar{p}_i$ satisfy,

$$\bar{N}(T) = \langle\widehat{H}_o^{-1}\rangle\sqrt{|\bar{p}_1\bar{p}_2\bar{p}_3|}, \tag{44}$$

$$\frac{\bar{N}(T)}{\sqrt{|\bar{p}_1\bar{p}_2\bar{p}_3|}}\bar{p}_i^2 = \langle\widehat{H}_o^{-\frac{1}{2}}\widehat{p}_i^2(T)\widehat{H}_o^{-\frac{1}{2}}\rangle, \tag{45}$$

$$\bar{N}(T)m^2 = m^2\frac{\langle\widehat{H}_o^{-\frac{1}{2}}|\widehat{p}_1(T)\widehat{p}_2(T)\widehat{p}_3(T)|\widehat{H}_o^{-\frac{1}{2}}\rangle}{\sqrt{|\bar{p}_1\bar{p}_2\bar{p}_3|}} \ . \tag{46}$$

There is a unique solution for $m = 0$:

$$\bar{N}(T) = \langle \widehat{H}_o^{-1} \rangle^{1/4} \left(\prod_{i=1}^{3} \langle \widehat{H}_o^{-\frac{1}{2}} \widehat{p}_i^2(T) \widehat{H}_o^{-\frac{1}{2}} \rangle \right)^{\frac{1}{4}}, \tag{47}$$

$$\bar{p}_i = \left[\frac{\langle \widehat{H}_o^{-1/2} \widehat{p}_i^2(T) \widehat{H}_o^{-1/2} \rangle}{\langle \widehat{H}_o^{-1} \rangle} \right]^{\frac{1}{2}}. \tag{48}$$

Therefore, the effective BI space-time is emerged in terms of expectation values of the gravitational operators on the quantum geometry state Ψ_o, whose components do not depend on modes $\vec{k}$.

Lorentz symmetry on the effective geometry

The wave equation on the effective geometry (for $m=0$):

$$\frac{d^2\mathcal{Q}_{\vec{k}}}{dT^2}+\Omega^2_{T,\vec{k}}\mathcal{Q}_{\vec{k}}=0, \tag{49}$$

where $\mathcal{Q}_{\vec{k}}$ denotes the modified modes $q_{\vec{k}}$:

$$\mathcal{Q}_{\vec{k}}:=\frac{q_{\vec{k}}}{\sqrt{\langle\widehat{H}_o^{-1}\rangle}}\ . \tag{50}$$

and $\Omega^2_{T,\vec{k}}(T)$ is the (modified) dispersion relation of the test field on the effective geometry:

$$\Omega^2_{T,\vec{k}}(T)=\left[\langle\widehat{H}_o^{-1}\rangle\sum_i k_i^2\langle\widehat{H}_o^{-\frac{1}{2}}\widehat{p}_i^2(T)\widehat{H}_o^{-\frac{1}{2}}\rangle\right.$$
$$\left.-\frac{1}{4}\left(\frac{d\ln\langle\widehat{H}_o^{-1}\rangle}{dT}\right)^2+\frac{1}{2}\frac{d^2\ln\langle\widehat{H}_o^{-1}\rangle}{dT^2}\right]. \tag{51}$$

Since $\langle \widehat{H}_0^{-1} \rangle$ is independent of the time T, thus

$$\Omega^2_{T,\vec{k}}(T) = \langle \widehat{H}_o^{-1} \rangle \sum_i k_i^2 \langle \widehat{H}_o^{-\frac{1}{2}} \widehat{p}_i^2(T) \widehat{H}_o^{-\frac{1}{2}} \rangle. \tag{52}$$

Finally, the 3-velocity of modes propagating on the effective geometry can be obtained as

$$\begin{aligned} \|V\|^2 &= -\sum_i \frac{\bar{g}_{ii}}{\bar{g}_{00}} \left(\frac{d\Omega_{T,\vec{k}}}{dk_i} \right)^2 \\ &= \frac{1}{\Omega^2_{T,\vec{k}}} \sum_i k_i^2 \langle \widehat{H}_o^{-1} \rangle \langle \widehat{H}_o^{-\frac{1}{2}} \widehat{p}_i^2(T) \widehat{H}_o^{-\frac{1}{2}} \rangle \\ &= 1. \end{aligned} \tag{53}$$

This equation confirms our expectation that, **no Lorentz-violation** is presented in our model herein.

Born-Oppenheimer approximation: Is Lorentz Invariance held in the presence of the next order correction?

In standard **quantum mechanics**: $-i\partial_t \Psi = \hat{H}\Psi = (\hat{H}_n + \hat{H}_e)\Psi$,

- Heavy degrees of freedom: nucleus n,
- Light degrees of freedom: electron e.

On the (Coulomb) background, solve the eigenequation for $\hat{H}_e$:

$$\hat{H}_e \chi_i(e) = \epsilon_i(n)\chi_i(e) \tag{54}$$

Substitute back, and solve the eigenequation for $\hat{H}$:

$$\Phi_\alpha = \sum_i \varphi_{i,\alpha}(n)\chi_i(e), \qquad \left[\hat{H}_n + \epsilon_i(n)\right]\varphi_{i,\alpha}(n) = E_\alpha \varphi_{i,\alpha}(n). \tag{55}$$

Then, the "corrected" state of the system reads

$$\Psi_0 = \sum_\alpha c_\alpha \Psi_\alpha^0 \quad \Rightarrow \quad \Psi = \sum_\alpha c_\alpha \Psi_\alpha \varphi_{i,\alpha}\chi_i. \tag{56}$$

In our model, consider the Hamiltonian of the system:

$$-i\hbar\partial_{\widetilde{T}}\Psi = \left[\frac{1}{2}\Theta - \widehat{H}_{\tau,\vec{k}}\right]\Psi \ , \tag{57}$$

- Heavy degrees of freedom: geometry ($\vec{\lambda}$),
- Light degrees of freedom: matter ($q_{\vec{k}}$).

On the background Ψ_0, solve the eigenequation for $\widehat{H}_{\tau,\vec{k}}$:

$$\widehat{H}_{\tau,\vec{k}}\chi_i(q_{\vec{k}}) = \epsilon_i(p)\chi_i(q_{\vec{k}}) \tag{58}$$

Substitute back, and solve the eigenequation for $\widehat{H}$:

$$\Phi_\alpha = \sum_i \varphi_{i,\alpha}(\vec{\lambda})\chi_i(q_{\vec{k}}), \qquad \left[\frac{1}{2}\Theta - \widehat{H}_{\tau,\vec{k}}\right]\varphi_{i,\alpha}(\vec{\lambda}) = E_\alpha\varphi_{i,\alpha}(\vec{\lambda}). \tag{59}$$

Then, the "corrected" state of the system becomes

$$\Psi_0 = \sum_\alpha c_\alpha \Psi^0_\alpha \quad \Rightarrow \quad \Psi = \sum_\alpha c_\alpha \Psi_\alpha \varphi_{i,\alpha}\chi_i. \tag{60}$$

The state of the system:

$$\Psi(\widetilde{T}, \lambda, q_{\vec{k}}) = \Psi_o(\widetilde{T}, \lambda) \otimes \psi(\widetilde{T}, q_{\vec{k}}) + \delta\Psi(\widetilde{T}, \lambda, q_{\vec{k}}), \tag{61}$$

$$\delta\Psi \;=\; \sum_{\alpha, i} f_{\alpha i} \varphi_\alpha^o \otimes \chi_i \propto k. \tag{62}$$

Therefore, the effective geometry is extended as

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -(1 + \xi k \ell_{\mathrm{Pl}})^2 \langle \widehat{p}^2 \rangle^{3/2} d\widetilde{T}^2 + \sqrt{\langle \widehat{p}^2 \rangle} d\vec{x}^2. \tag{63}$$

Then, the dispersion relation for mode $\vec{k}$ on the background $\bar{g}_{\mu\nu}$ gives

$$\|V\| = 1 + \frac{\xi}{2} k \ell_{\mathrm{Pl}} \; . \tag{64}$$

Lorentz violation occurs at around $E \sim E_{\mathrm{Pl}}$ (GRB bound $\sim 10^{-2} E_{\mathrm{Pl}}$).

Conclusion and discussion

What we have seen:

- We have developed, the first steps of the QFT on Bianchi LQC space-time.
- We discussed the concept of the “effective geometry” (different than the effective scenario of LQC) felt by quanta of matter.
- We showed that, no Lorentz-violation in Bianchi I space-time exists at 0th order (test field approximation).
- There exists possible Lorentz-violation when the backreaction is taken into account.

Further investigation:

- Try to include the massive fields,
- Refine the QFT part: consider an infinite number of modes.

Obrigado

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