Singularities and Quantum Gravity

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Space-time Singularities

- Motivation
- The geometry of Singularities
- Singularities and Quantum Gravity
- Gravitational collapse and QG
- Discussion

Space-time Singularities

- **Motivation**
- The geometry of Singularities
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Singularities in physics

The space-time singularity is a completely different thing from the singularities which appear in classical electromagnetism or field theory, in general. These are normally expressible by distributions. In general relativity this is not the case, because of the nonlinearity of the gravitational force. The space-time singularity is characterized by the incompleteness of the (non-)geodesics curves. We shall explain this type of singularity in more details in this talk. Then, we will discuss on existence of a modified theory which may be free of such singular solutions.

Geometry on a manifold

Geometry on a manifold

Singular points of manifold

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Singular points of manifold

General Relativity: 4-dim.
Space-time geometry

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Singular points of manifold

General Relativity: 4-dim. Space-time geometry

Singular spherically symmetric solutions

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General Relativity: 4-dim. Space-time geometry

Singular spherically symmetric solutions

Singularity theorems
(Penrose-Hawking-Gerach)

Why singularities? Do they exist in real universe?

Still mystries about our unknown universe !

Is Einstein's theory correct at small limit near classical singularities?

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What is a singularity on the space-time geometry?

- **The definition based on 'g-boundary'**

  (Penrose-Hawking-Geroch, 1968)

- **The definition based on 'b-boundary'**

  (Schmidt, 1971)

What is a singularity on the space-time geometry?

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  (Schmidt, 1971)

Singularities on g-boundary

The singularity theorems were proved using a definition of singularities based on 'g-boundary':

Defining a topological space, the g-boundary whose points are equivalence classes of incomplete non-space-like geodesics. The points of the g-boundary are then the singular points of space-time. Time-like and null geodesic incompleteness are considered as conditions for space-time to be singularity-free, since time-like geodesic incompleteness implies there could be freely moving observers and particles whose histories did not exist after, or before, a finite interval of proper time, and null geodesics are histories of zero rest-mass particles.

What is a singularity on the space-time geometry?

- The definition based on 'g-boundary'

  (Penrose-Hawking-Geroch, 1968)

- The definition based on 'b-boundary'

  (Schmidt, 1971)

Singularities on b-boundary

How one can overcome the problem of the definition based on g-completeness?

*What one needs is some generalization of concept of an affine parameter to "all curves" (geodesic or non-geodesic): One could then define a notion of completeness by requiring that, "**every curve**" of finite legth as measured by such parameter had an endpoint. This idea was formulated by Schmidt (1971) in an elegant manner in which a new boundary, namely, b-boundary were defined for spacetime manifold.*

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Singularities and Quantum Gravity

Geometry:

- Singularity theorems are mainly statements about differential geometry as they refer to properties of geodesics on curved manifold; Einstein's equation is used only at one place, relating positive energy to positive curvature, which then implies focusing effects in the Raychaudhuri equation. For a general solution of the singularity problem one should thus focus on geometry, not just on dynamics.
- An alternative geometry is automatically provided by background independent quantizations, such as canonical quantizations. Such theories are not based on objects in a background space-time but they quantize full metric components as the non-perturbative dynamical objects. For instance, most versions employ wave functions supported, e.g., on the space of spatial metrics q_{ab} which is the configuration space of canonical general relativity. Geometrical objects then become operators acting on these wave functions with properties generally very different from classical smooth geometry.
- A ***canonical formulation*** is defined by introducing a foliation of space-time into a family of spatial slices in terms of a time function t such that the slices are

$$\Sigma_t : \; t = const.$$

• Choosing a time evolution vector field t^a such that $t^a \nabla_a t = 1,$ which determines how points on different spatial slices are identified (along integral curves of t^a) to result in spatial fields "evolving" in coordinate time t .

• One can write t^a into a normal and tangential part as: $t^a = Nn^a + N^a$
One can split off the time components from the inverse space-time metric by definiing the inverse spatial metric q^{ab} given by

$$g^{ab} = q^{ab} - n^a n^b = q^{ab} - \frac{1}{N^2}(t^a - N^a)(t^b - N^b)$$

And inverting it results in the space-time metric

$$ds^2 = -N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$$

• Then, evolution equations do result for the spatial metric $q_{ab}(t)$, interpreted as a time dependent field through its values on different Σ_t .

• The phase space coordinates given by spatial metric components q_{ab} and their momenta are subject to constraint equations implementing their dynamics. When quantized, wave functions on the space of metrics (if a metric representation is chosen) arise which are subject to differential equations depending on the quantization scheme.

- There are then no test objects in a background space-time, but wave functions or gravitational observables are the fundamental dynamical entities. Differential geometry becomes applicable only in a classical approximation to describe space-time, and only in classical regions would geodesics be defined at all. Geodesic incompleteness is thus inapplicable as a criterion for singularities in quantum gravity. This suggests a natural answer to why incompleteness occurs so generally in general relativity: geodesics themselves are only valid as long as a differentiable classical geometry can be assumed.

- When quantum geometry becomes relevant, geodesics stop and have to be replaced by something more appropriate. Only classical geometry would end, but not quantum gravity with its own version of quantum geometry. Such a scenario looks promising but it has to be developed and verified in detail, requiring explicit candidates for background independent quantum gravity such as a canonical quantization. The question then remains:

How do we address or even define singularities in such a context?!

Elements from Loop Quantum Gravity

- The Area of a surface S with co-normal n_a as the usual definition of fluxes is , $A(S) = \int_S dy^2 \sqrt{E^a_i n_a E^b_i n_b}$: Then the area spectrum is determined as

$$\hat{A}(S) f_{g,j} = \frac{1}{2} \sum_{p \in S \cap g} \sqrt{j_p(j_p+1)} f_{g,j}$$

- The volume operator is also obtained by quantizing the classical expression

$$V(R) = \int_R d^3x \sqrt{|\det E|}$$

- Then, the inverse densitized triad can be written in terms of new variables and the volume operator as

$$\frac{E^b_i E^c_j}{\sqrt{|\det(E)|}} \epsilon^{ijk} \epsilon_{abc} = \frac{4}{\kappa\gamma} \{A^i_a, V\}$$

- The Hamiltonian in terms of Ahtekar variables:

$$\mathcal{H} = \int dt N \left(-4e^{-1} \mathrm{Tr}(F_{ab} P^a P^b) - 2e^{-1}(1+\gamma^2) P^a_i P^b_j K^i_{[a} K^j_{b]} \right) + \mathcal{H}_{\mathrm{matt}}$$

Elements from Loop Quantum Gravity

The Hamiltonian constraint then is given by:

$$C^{\mathrm{gr}} = -4e^{-1}\mathrm{Tr}(F_{ab}P^aP^b) - 2e^{-1}(1+\gamma^2)P_i^aP_j^bK_{[a}^iK_{b]}^j,$$

In terms of holonomy

$$h_e[A] = P\exp - \int_e A \in SU(2)$$

one can rewrite the Hamiltonian constraint as:

$$C^{\mathrm{gr}} = -\frac{8\pi}{\mu_0^3}\sum_{IJK}\epsilon^{IJK}\,\mathrm{Tr}\left[h_Ih_Jh_I^{-1}h_J^{-1}h_{[IJ]}\,h_K^{-1}\{h_K,V\}\right],$$

After construction the Kinematical Hilbert space, one has to quantize the model, using the method of LQG and solve the quantum Hamiltonian constrants

$$\widehat{C}|\psi\rangle = (\widehat{C}^{\mathrm{gr}} + \widehat{C}^{\mathrm{matt}})|\psi\rangle = 0$$

for the physical state by using the Ashtekar-Lewandowski measure in the related Hilbert space of LQG.

Quantum Hyperbolicity: There exist a viewpoint is the natural extension of generalized hyperbolicity from matter fields on space-time to the fundamental object of quantum space-time itself, which is therefore called the principle of quantum hyperbolicity:

If a state can uniquely be extended across or around all submanifolds of classically singular configurations they do not pose obstructions to quantum evolution in any sense. If, however, a state such as a wave function on the space of metrics cannot be extended uniquely across a classically singular submanifold, there would still be a boundary to quantum evolution and quantum space-time would remain incomplete.

Quantum Geometry: On the other hand, beside the dynamics, the geometry of space-time near the classical singularity must be regular. In this case, one should find the quantum counterpart of the classical quantity in which the space-time is singular; such as inverse triads (in canonical formalism), semiclassical energy density, etc. In next part of our talk we will discuss about this case with examples within a the quantum fate for gravitational collapse.

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Gravitational Collapse

- Quantum evaporation of a naked singularity;

  *Goswami, Joshi, and Singh, PRL **96**, 031302 (2006).*

- LTB collapse in loop quantum gravity

  *Bojowald et al. PRD **80**, 084002 (2009); PRD **78**, 064057 (2008).*

- Black hole singularities in loop quantum gravity

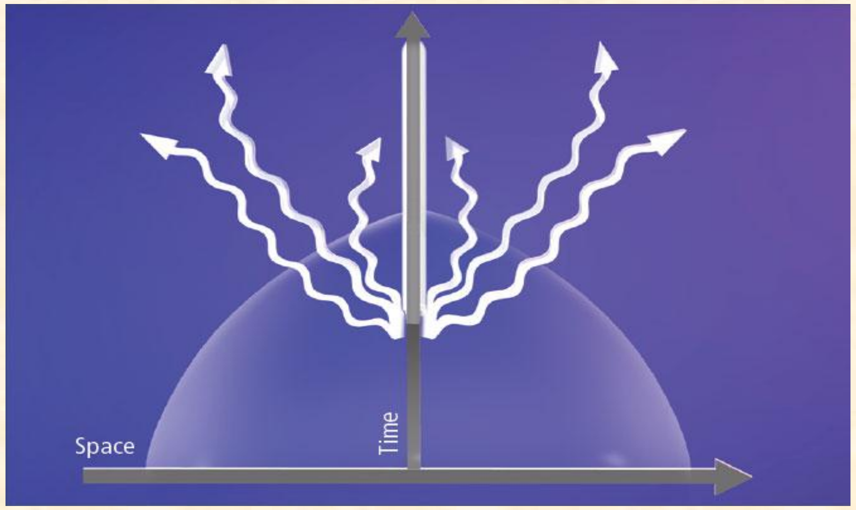
  Cf. see papers by L. Modesto, also Ashtekar-Bojowald

- Quantum fate of a tachyon field collapse

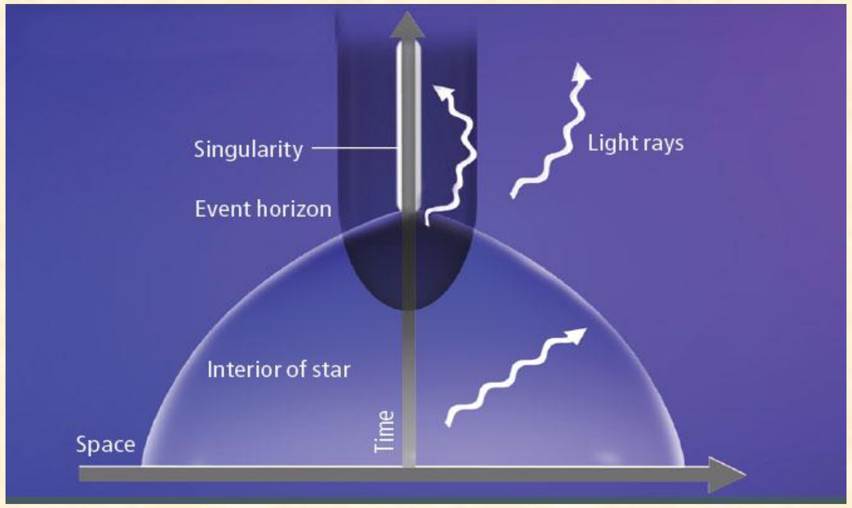
  YT, P. Vargas Moniz, J. Marto, A. Ziaie [arXiv: gr-qc/1105.0445].

Gravitational Collapse of a tachyon matter field

(YT, P. Vargas Moniz, J. Marto, A. Ziaie)



Naked Singularity



Black Hole

The space-time metric for a gravitational

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2 d\Omega^2,$$

Einstein's equations can be written as

$$\rho = \frac{F_{,r}}{R^2 R_{,r}}, \qquad p = \frac{-\dot{F}}{R^2 \dot{R}}, \qquad \dot{R}^2 = \frac{F}{R}.$$

The deviation of a congruence of null geodesics, in the chosen space-time is given by Θ . A trapped surface is then defined as a compact two-dimensional (smooth) space-like surface S having the property that the parameter Θ of outgoing future-directed null geodesics, which are orthogonal to S, is everywhere negative on S .

$$\Theta = \frac{2}{r}\left(1 - \sqrt{\frac{F}{R}}\right)\xi^r$$

Considering the interior space-time (inside the matter) as a marginal FRW space-time, given by

$$ds_{-}^{2} = -dt^{2} + a^{2}(t)[dr^{2} + r^{2}d\Omega^{2}],$$

And the exterior space-time (outsdie the matter) is given by a generalized Vaydia metric

$$ds_{+}^{2} = -\left(1 - \frac{2M(u)}{r}\right)du^{2} - 2dudr + r^{2}d\Omega^{2},$$

In which matching condition is satisfied at the boundary of two regions. Furthermore, by integration the Einstein's equation, the mass function for interior region is obtained as:

$$F = \frac{1}{3}\rho(t)R^{3}.$$

From this equation, one can define the final state of collapse as:

$$\Theta \to -\infty \quad \text{approaching the singularity,} \quad \text{if} \quad F > R\ ; \quad \text{(Black hole).}$$

$$\Theta < +\infty \quad \text{approaching the singularity,} \quad \text{if} \quad F < R\ ; \quad \text{(Naked singularity).}$$

Now let us consider a ***homogeneous*** tachyon field $\phi(t)$ with the Lagrangian given by

$$L = -a^3 V(\phi)\sqrt{1-\dot{\phi}^2},$$

as the matter content for collapsing system: Then, the dynamical evolution of matter in collapse is given by:

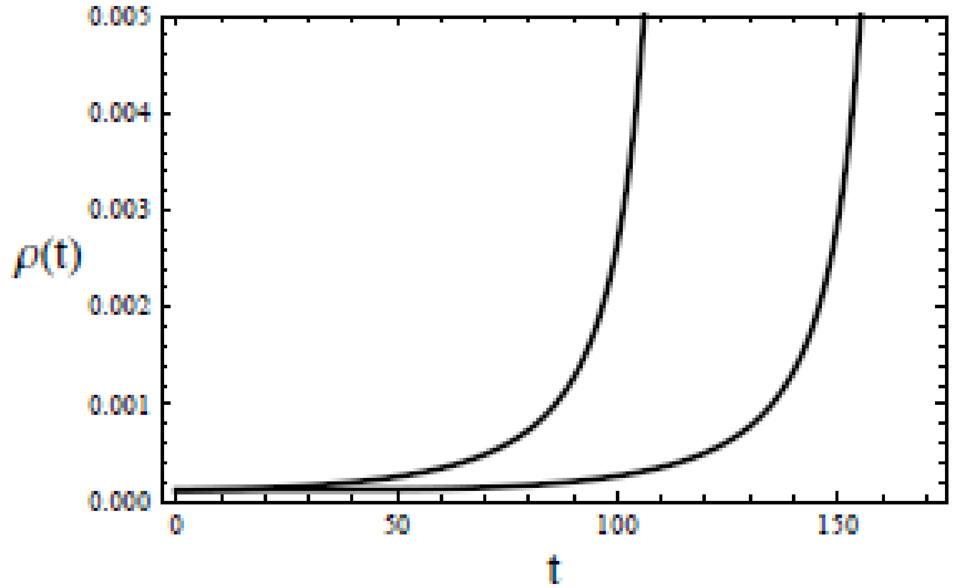
$$\rho(\phi) = 3H^2 = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}, \quad -2\dot{H} = \frac{V(\phi)\dot{\phi}^2}{\sqrt{1-\dot{\phi}^2}}$$

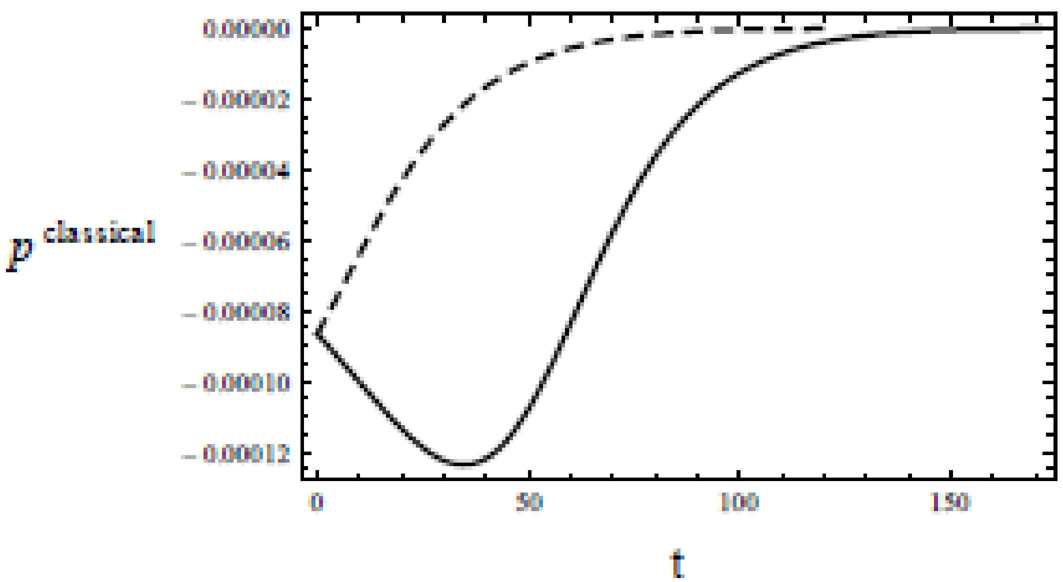
$$\ddot{\phi} = -(1-\dot{\phi}^2)\left[3H\dot{\phi} + \frac{V_{,\phi}}{V}\right]$$

Where we will consider the tachyon potential to be of inverse square form i.e. $V(\phi) \sim \phi^{-2}$
Using a ***dynamical system analysis***, we can get analytically a ***dust-like*** solutions as

$$p \approx 0, \quad \text{and} \quad \rho \propto 1/a^3$$

is described in the following numerical plots; this case corresponds to a ***black hole*** *formation* as collapse endstate.





Quantum fate of tachyon field collapse: Inverse triad correction

Hamiltonian for a tachyon field can be written as:

$$H_\phi(\phi,\pi_\phi) = \mathrm{p}^{3/2}\sqrt{V^2 + \mathrm{p}^{-3}\pi_\phi^2},$$

Where π_ϕ is the conjugate momentum for the tachyon field ϕ given by

$$\pi_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{a^3 V(\phi)\dot{\phi}}{\sqrt{1-\dot{\phi}^2}},$$

Where $\mathrm{p}^{3/2} \equiv a^3$ is a volume operator. Furthermore, one can obtain the energy density and pressure for a matter field in this formulation using the following relation:

$$\rho_\phi = \mathrm{p}^{-3/2} H_\phi, \quad p_\phi = -\mathrm{p}^{-3/2}\left(\frac{2}{3}\mathrm{p}\frac{\partial H_\phi}{\partial \mathrm{p}}\right).$$

From **LQC** the term p^{-1}, in equation of classical Hamiltonian, associated to the operator , π_ϕ replaced by the eigenvalues of the inverse densitized triad given by

$$d_{j,l}(\mathrm{p}) \equiv \mathrm{p}_j^{-1} D_l(\mathrm{p}/\mathrm{p}_j),$$

Where (j,l) are two half-integer free quantization parameters. Also

$$\mathrm{p}_j \equiv (1/3)\gamma j \ell_{\mathrm{Pl}}^2 = a_i^2 j/3 \equiv a_*^2,$$

Where a_* is a critical scale at which the eigenvalue of the inverse scale factor has power-law dependence on the scale factor; $a_i \equiv \sqrt{\gamma}\ell_{\mathrm{Pl}}$ is the scale above which a classical continues space-time can be defined and below which the space-time is discrete. The constant $\gamma = 0.13$ is the Barbero-Immirzi parameter.

For the approximation $\ell = 3/4$, the relation $D(q)$ for with $q = \mathrm{p}/\mathrm{p}_j = a^2/a_*^2$ reads

$$D(q) = (8/77)^6 q^{3/2}\left\{7\left[(q+1)^{11/4} - |q-1|^{11/4}\right] - 11q\left[(q+1)^{7/4} - \mathrm{sgn}(q-1)\,|q-1|^{7/4}\right]\right\}^4,$$

This relation for the semiclassical limit $a_i < a \ll a_*$ becomes

$$D(q) = \left(\frac{12}{7}q\right)^4.$$

Using the constraint equation $H^{loop} = -3\dot{a}^2 a + H_\phi^{loop}$, for the total Hamiltonian of the system, the semiclassical Friedmann constraint can be obtained as

$$\rho_\phi^{loop} = \frac{Vq^{3/2}\,|D(q)|^{3/2}}{\sqrt{q^3\,|D(q)|^3 - \dot{\phi}^2}} = 3H^2,$$

Whose gives the modified energy density of tachyon field in semiclassical regime. Furthermore, the modified pressure in this regime reads,

$$p_\phi^{loop} = -\frac{Vq^{3/2}\,|D(q)|^{3/2}}{\sqrt{q^3\,|D(q)|^3 - \dot{\phi}^2}}\left[1 + \frac{\dot{\phi}^2 p_j\,|D(q)|_{,\mathrm{p}}}{q^2\,|D(q)|^4}\right],$$

From this and the relation for effective energy to the equation of conservation

$$\dot{\rho}_{\phi}^{loop} + 3H(\rho_{\phi}^{loop} + p_{\phi}^{loop}) = 0$$

one gets the modified equation of motion as

$$\ddot{\phi} - 3H\dot{\phi}\left[1 + \frac{1}{2}\left(2 - \frac{\dot{\phi}^2}{q^3 \,|\, D(q)\,|^3}\right)\frac{d\ln D}{d\ln a}\right] + \left(q^3 \,|\, D(q)\,|^3 - \dot{\phi}^2\right)\frac{V_{,\phi}}{V} = 0$$

Within the regime $a_i < a \ll a_*$, the modified energy, pressure, and the field reduces to:

$$\rho_{\phi}^{loop} = 3H^2 = \frac{V(\phi)}{\sqrt{1 - A^{-1}q^{-15}\dot{\phi}^2}},$$

$$-p_{\phi}^{loop} = 2\dot{H} + 3H^2 = \frac{V(\phi)}{\sqrt{1 - A^{-1}q^{-15}\dot{\phi}^2}}\left[1 + \frac{4\dot{\phi}^2}{Aq^{15}}\right],$$

$$\ddot{\phi} - 12H\dot{\phi}\left[\frac{7}{2} - A^{-1}q^{-15}\dot{\phi}^2\right] + \left[Aq^{15} - \dot{\phi}^2\right]\frac{V_{,\phi}}{V} = 0.$$

Where $A \equiv (12/7)^{12}$ is a constant. On the other hand, substituting the relation of energy density $\rho_{\phi} = \mathrm{p}^{-3/2}H_{\phi}$, in the classical equation for the mass function (in Hamiltonian formulation) one get

$$F = \frac{1}{3}r^3\mathrm{p}^{3/2}\sqrt{V^2 + \mathrm{p}^{-3}\pi_{\phi}^2}.$$

Applying the quantum correction (inverse triad type), the modified equation for the mass function reads:

$$F^{loop} = \frac{1}{3} r^3 \text{p}^{3/2} \sqrt{V^2 + |\, d_{j,l}(\text{p}) |^3 \, \pi_\phi^2} \, .$$

For the semiclassical limit which $a_i < a \ll a_*$, the equation above reduces to:

$$F^{loop} = \frac{1}{3} r^3 \frac{a^3 V(\phi)}{\sqrt{1 - A^{-1} q^{-15} \dot{\phi}^2}} \, .$$

Where, similar to the classical one, it can be easily written as $F^{loop} = 1/3 \rho_\phi^{loop} R^3$.

We study the physical solutions for the collapsing system in semiclassical limit, by using dynamical system alaysis (for details of alaysis see arXiv: gr-qc/1105.0445).

There is just *one fixed point solution* which all of its eigenvalues are real but one is zero, the rest being negative; This implies that this is a nonlinear autonomous system with a non-hyperbolic point.

A centre manifold analysis shows that, this fixed point is a ***stable*** *point.*

The solutions:

• The effective energy density in the neighborhood of the fixed point (i.e. classical singularity) can be approximated as

$$\rho_{\phi}^{loop} \simeq V(\phi)\left(1+\frac{1}{2}\frac{\dot{\phi}^2}{Aq^{15}}\right).$$

Where the time derivative of tachyon field changes as $\dot{\phi} \propto a^{42}$. Hence, in the region very close to the center (at the limit of classical singularity), the energy density of the system can be determined totally by the potential of the system, that is:

$$\dot{\phi} \propto a^{42} \to 0, \qquad \rho_{\phi}^{loop} \simeq V(\phi); \;\; a \to 0.$$

• The effective mass function can be then easily obtained as

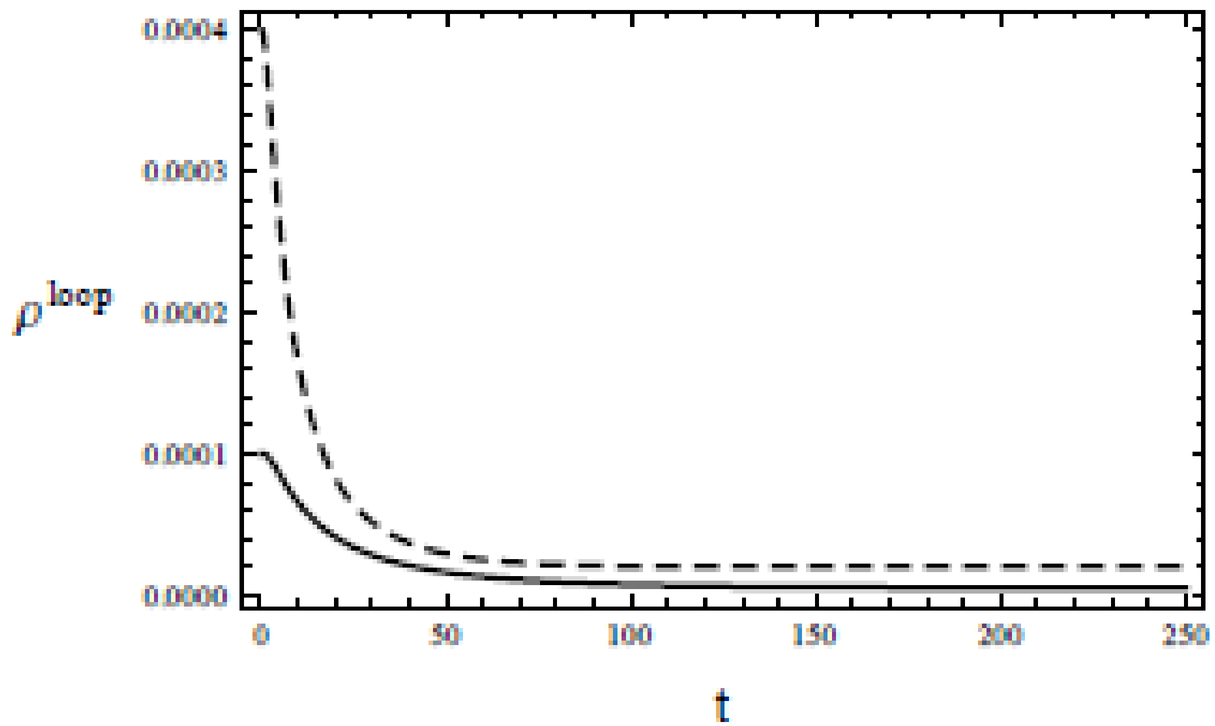
$$a^{30}(\phi) \propto \ln\phi, \qquad \rho_{\phi}^{loop} = H(\phi) \simeq \sqrt{\beta/3}\phi^{-1} \qquad \Rightarrow \quad F^{loop} \propto a^2 .$$

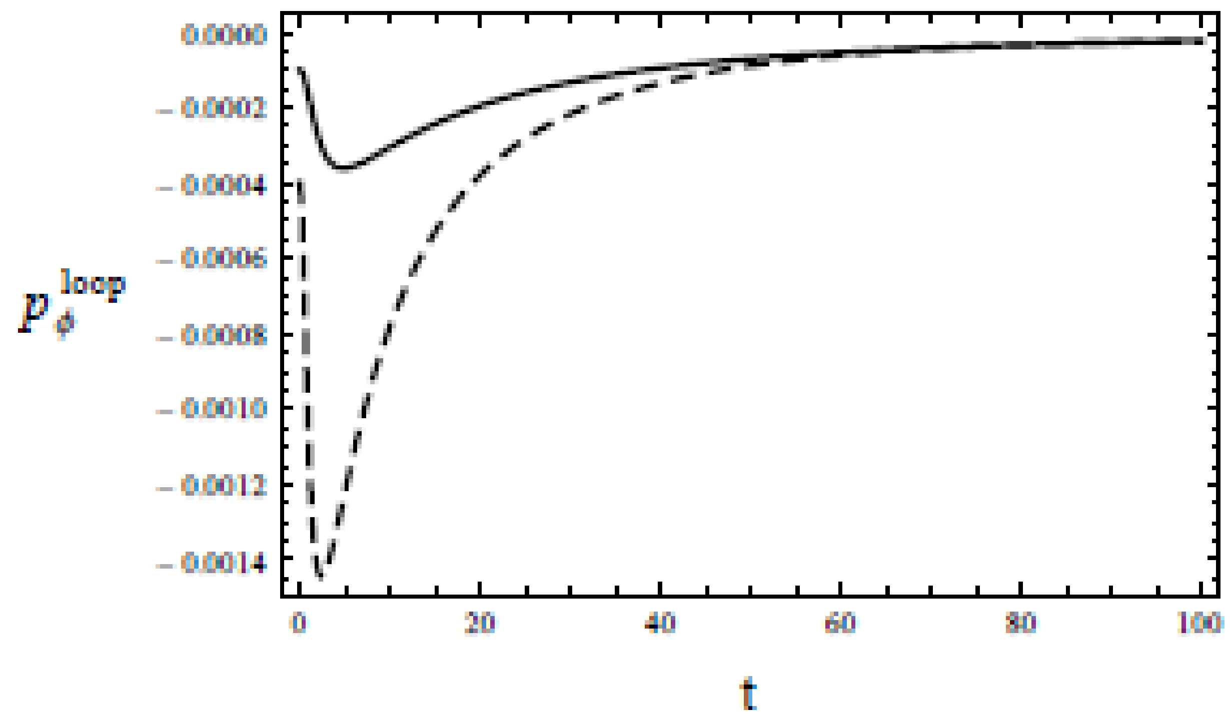
The collapse endstate:

These solution shows that, as central (classical) singularity approaches, the scale factor vanishes as the tachyon field approaches the unity: $a^{30}(\phi) \propto \ln\phi \to 0;$. $\phi \to 1$. Then, the effective energy density at the center remains finite and tends to a constant value:

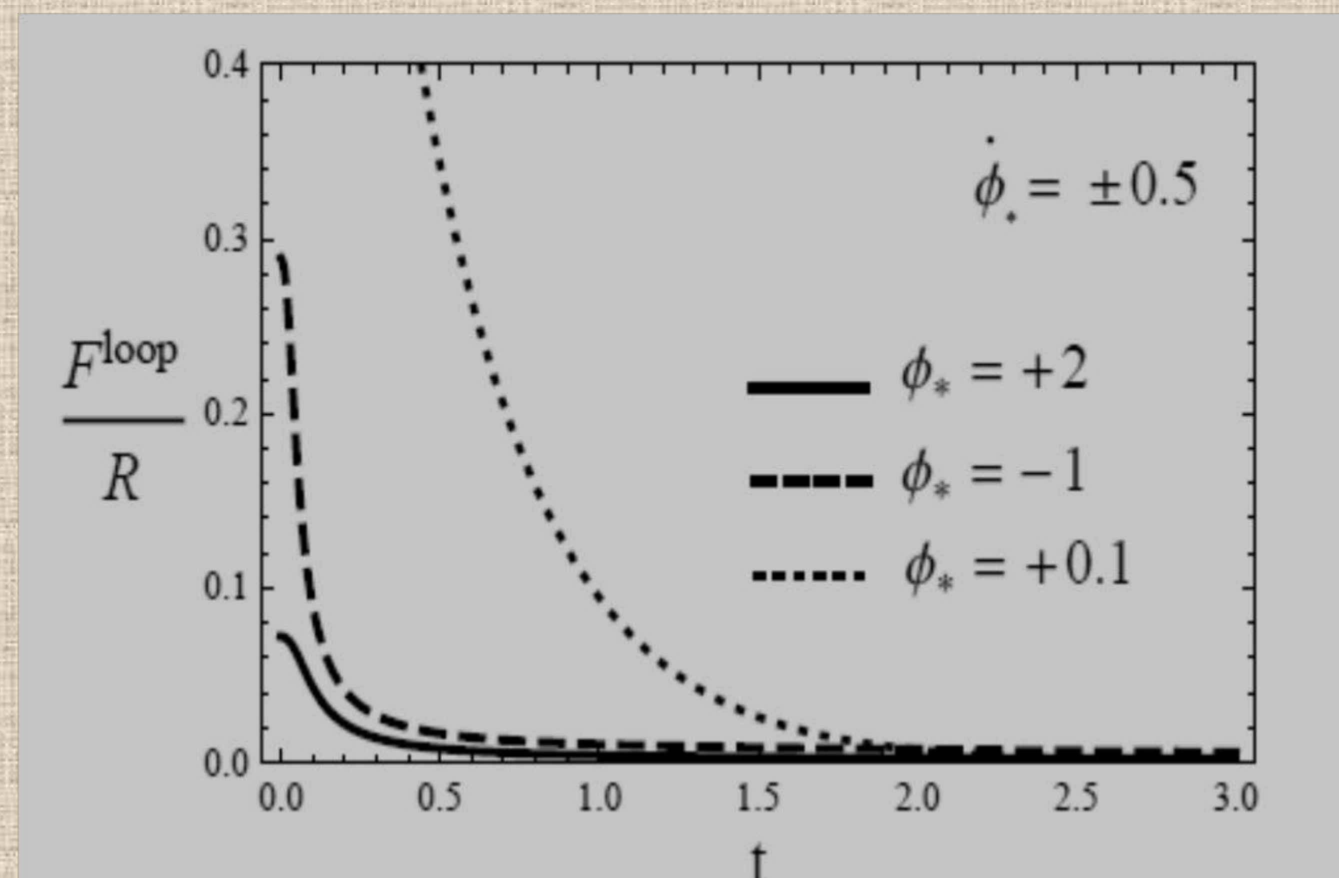
$$\rho_{\phi}^{loop} = \beta\phi^{-2} \to \beta = const$$

During this process, the effective mass function decreases from its initial condition and vanishes at the center as: $F^{loop} \propto a^2 \to 0$





The semiclassical effect on tachyon field collapse gives rise to the classical black hole singularity resolution at this critical point. Furthermore, as classical singularity is approached, the effective pressure tends to unity with the negative sign.

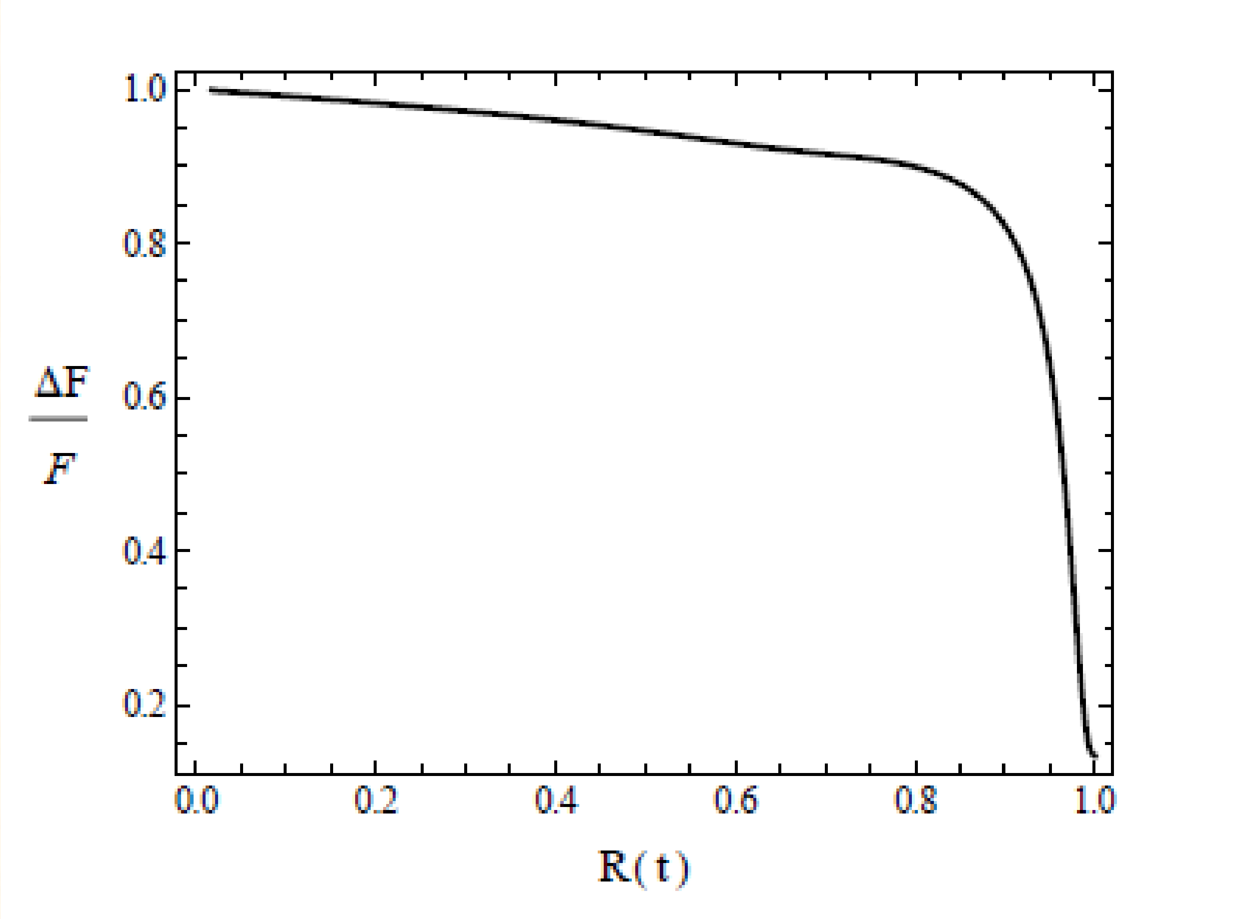


The mass loss, and outward flux of energy:

- The effective pressure is negative which implies the existence of an outward energy flux as the collapse advances.
- If the mass function at scales $a \gg a_*$ in the classical regime is F, whereas for $a < a_*$ (in the semiclassical regime) is F^{loop}, we then get the following expression for the mass loss, as

$$\frac{\Delta F}{F} \equiv \frac{F - F^{loop}}{F} = 1 - \frac{\rho^{loop}}{\rho}$$

- Then as $\Delta F / F \to 1$, implying that the mass function in the semiclassical regime tends to zero and the output is an outward energy flux during the dynamical evolution of collapse scenario. Since no trapped surfaces form up to the singularity, this outward energy flux would be observable to the outside observers.



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Discussion

The singularity problem is a more complicated issue to be addressed in a more general theory of gravity, extending general relativity in a well-defined form:

- No general classification of (classical) singularities is avaliavle and many different types exist: (*open issue in classical GR*)

- In the lack of such definition in quantum theory, there is not exisit a direct way to find out whether a quantum model of gravity is singular or not: (*open issue in canonical quantum gravity*)

- **LQC** was a successful (reduced phase space) cosmological version based on LQG, on avoidance of singularities. However, there is no a general extension of such results to the “full theory of **LQG**”**:** *(open problem in LQG: Is it a singularity-free model for quantum GR?)*

Thanks for your attention