The purpose of this short course is to present the canonical programme developed by Dirac to deal with the quantization of constrained systems and systems with gauge degrees of freedom. We will, however, restrict our study to the classical formalism, and in particular its application to field theories like Maxwell's electromagnetism and Einstein's General Relativity.

We start with an action integral

$$S = \int L\,dt\,, \tag{1}$$

where the Lagrangian $L(q,\dot{q})$ is a function of N coordinates q_n and their velocities $\dot{q}_n = dq_n/dt$. Extremizing the action (1) yields the Euler-Lagrange equations of motion

$$\delta S = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n}\,. \tag{2}$$

Defining the canonical momentum variable p_n by

$$p_n = \frac{\partial L}{\partial \dot{q}_n}\,, \tag{3}$$

we can write the action integral as

$$S = \int dt\,p_n\dot{q}_n - H\,, \tag{4}$$

where

$$H = p_n\dot{q}_n - L \tag{5}$$

is the Hamiltonian. In many cases, we assume that the momenta are independent functions of the velocities. However, in many cases of interest, this is not true. There exist M relations connecting the momentum variables, of the form

$$\phi_m(q,p) = 0\,, \qquad m = 1,\cdots,M, \tag{6}$$

called the *primary constraints* of the Hamiltonian formalism.

Variation of the Hamiltonian (5) gives

$$\begin{aligned} \delta H &= \delta p_n\dot{q}_n + p_n\delta\dot{q}_n - \frac{\partial L}{\partial q_n}\delta q_n - \frac{\partial L}{\partial \dot{q}_n}\delta\dot{q}_n && (7)\\ &= \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n\,, && (8)\end{aligned}$$

which means $H = H(q,p)$ is a function of the q's and p's only. Eq. (8) holds for variations δq, δp subject to the constraints (6), i.e. the q's and the p's cannot be varied independently. But we know how to deal with constraints of this type. We add to the Lagrangian some linear combination of the constraints

$$S = \int p_n\dot{q}_n - H(q,p) + u_m\phi_m\,, \tag{9}$$

where u_m are unknown coefficients. Then $\delta S = 0$ yields

$$\begin{aligned} \dot{q}_n &= \frac{\partial H}{\partial p_n} + u_m\frac{\partial \phi_m}{\partial p_n} && (10)\\ \dot{p}_n &= -\frac{\partial H}{\partial q_n} - u_m\frac{\partial \phi_m}{\partial q_n}\,. && (11)\end{aligned}$$

These are a generalization of the usual Hamilton's equations of motions, a set of first order differential equations describing how the variables q and p vary in time. But now they involve unknown coefficients.

Before proceeding, we should define the Poisson bracket $\{\,,\,\}$, which acts on functions of the canonical variables (q, p):

$$\{f, g\} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}\,. \qquad \text{(implicit sum)} \tag{12}$$

You should check for yourself the following properties of the Poisson bracket:

- antisymmetry;
- linearity;
- product law (Leibniz rule);
- Jacobi identity.

For any $g(q, p)$, we have

$$\dot{g} = \{g, H\} + u_m\{g, \phi_m\}\,. \tag{13}$$

The equations of motion are all written concisely in the Poisson bracket formalism. Defined in this way, the Poisson bracket is only applicable to functions of (q, p). Let's extend this definition somewhat. Suppose a Poisson bracket exists for any two quantities, and has all the above properties. Then we may write

$$\dot{g} = \{g, H + u_m\phi_m\} = \{g, H\} + \{g, u_m\}\phi_m + u_m\{g, \phi_m\} \tag{14}$$

$$= \{g, H\} + u_m\{g, \phi_m\}\,. \tag{15}$$

The term $\{g, u_m\}$ is not defined, but it is multiplied by $\phi_m = 0$.

It is of uttermost importance that the Poisson brackets are all worked out *before* making use of the constraints, otherwise we will get a wrong result. To remind us of this rule, we will use weak equality signs for the constraints

$$\phi_m \approx 0\,. \tag{16}$$

Hence

$$\dot{g} \approx \{g, H_T\}\,, \qquad H_T = H + u_m\phi_m\,. \tag{17}$$

Now let's examine some consequences of these equations of motion. In the first place, there will be some consistency conditions. The constraints must be satisfied at all times

$$\dot{\phi}_m = \{\phi_m, H\} + u_{m'}\{\phi_m, \phi_{m'}\} \approx 0\,. \tag{18}$$

We have here a number of consistency conditions, one for each value of m. Supposing they don't lead to inconsistencies like $1 = 0$ (which would mean the Lagrangian was ill-defined), they can be divided into three kinds:

1. it reduces to $0 = 0$, i.e. it is identically satisfied with the help of the primary constraints;
2. it reduces to an equation independent of the u's, i.e. $\chi(q, p) = 0$;
3. it does not reduce to any of the previous cases, hence imposes a conditions on u_m.

The second kind means we have yet another constraint on the Hamiltonian variables. Such constraints are called *secondary constraints*. They differ from the primary ones in that those are a consequence of the definition of p_n, whereas secondary constraints make use of the Lagrangian equations of motion.

Each secondary constraint gives rise to another consistency condition of type (18), which can be of any of the three kinds. We carry on like this until we have exhausted all the consistency conditions. We will be left with a number of secondary constraints of type 2 and a number of conditions on the u's of type 3.

Secondary constraints will for many purposes be treated in the same footing as primary constraints. We write

$$\phi_k \approx 0\,, k = M+1, \cdots, M+K\,, \qquad \text{where } K \text{ is the number of secondary constraints,} \tag{19}$$

$$\phi_j \approx 0\,, j = 1, \cdots, M+K \equiv J\,, \qquad \text{the total number of constraints.} \tag{20}$$

The remaining equations are

$$\{\phi_j, H\} + u_{m'}\{\phi_j, \phi_m\} \approx 0\,. \tag{21}$$

We look for a solution $u_m = U_m(q,p)$, which is not unique since we can add any solution of the homogeneous equation

$$u_m\{\phi_j, \phi_m\} \approx 0\,, \tag{22}$$

say $V_m(q,p)$. Then the most general solution is

$$u_m = U_m + v_a V_{am}\,, \qquad a = 1, \cdots, A\,, \tag{23}$$

where A is the number of solutions of (22) and the v's are arbitrary.

The total Hamiltonian of the theory can be written as

$$H_T = H + U_m\phi_m + v_a V_{am}\phi_m \tag{24}$$

$$= H' + v_a\phi_a\,, \tag{25}$$

$$H' = H + U_m\phi_m\,, \tag{26}$$

$$\phi_a = V_{am}\phi_m\,. \tag{27}$$

We still have the equation of motion $\dot{g} \approx \{g, H_T\}$.

We have satisfied all the consistency conditions of the theory and we still have arbitrary coefficients v_a, which can be further allowed to depend on time. These arbitrary functions of time mean we are using a mathematical framework containing arbitrary features. The dynamical variables at future times are not completely determined by the initial dynamical variables: the general solution will contain arbitrary functions.

First- and second-class quantities

A dynamical variable $R(q,p)$ is said to be *first-class* if it has zero Poisson bracket with all the ϕ's,

$$\{R, \phi_j\} \approx 0\,, \qquad j = 1, \cdots, J\,. \tag{28}$$

Otherwise, R is *second-class*. If R is first-class, it must be strongly equal to some linear combination of ϕ's, as anything that is weakly zero must be strongly equal to some linear function of the ϕ's (the ϕ's are, by definition, the only quantities which are weakly zero). So

$$\{R, \phi_j\} = r_{jj'}\phi_{j'}. \tag{29}$$

Theorem: The Poisson bracket of two first-class quantities is also first-class.
Proof: exercise.

The division of constraints into primary/secondary is independent from the division into first-/second-class.

Note that H' and ϕ_a are both first class (*proof:* exercise). The final situation is that the total Hamiltonian is the sum of a first-class Hamiltonian plus a linear combination of the primary, first-class constraints.

Gauge degrees of freedom

The number of independent arbitrary functions of time occurring in the general solution is equal to the number of values which the suffix a takes on, which is equal to the number of independent primary first-class constraints, since all of them are included in the sum $H_T = H' + v_a\phi_a$. The initial physical state of a system is completely specified by the q's and p's . We don't need the coefficients v_a.

For a general dynamical variable g with initial state g_0, its value at time δt is

$$g(\delta t) = g_0 + \dot{g}\delta t = g_0 + \{g, H_T\}\delta t = g_0 + \delta_t \left(\{g, H'\} + v_a\{g, \phi_a\}\right) . \tag{30}$$

Since the v's are arbitrary, we could have chosen a different set. The difference would be

$$\begin{aligned} \Delta g(\delta t) &= \delta t(v_a - v'_a)\{g, \phi_a\} && (31) \\ &= \varepsilon_a\{g, \phi_a\} , \qquad \varepsilon_a = \delta t(v_a - v'_a) \text{ is a small arbitrary number.} && (32) \end{aligned}$$

We can change all our Hamiltonian variables according to this rule, and the new Hamiltonian variables will describe the same state. This amounts to an infinitesimal contact transformation with generating function $\varepsilon_a\phi_a$.

Primary first-class constraints, as generating functions of infinitesimal contact transformations, lead to changes in the q's and p's that do not affect the physical state.

If we apply a second contact transformation with generating function $\gamma_{a'}\phi_{a'}$,

$$g' = g_0 + \varepsilon_a\{g, \phi_a\} + \gamma_{a'}\{g + \varepsilon_a\{g, \phi_a\}, \phi_{a'}\} . \tag{33}$$

In reverse order,

$$g' = g_0 + \gamma_{a'}\{g, \phi_{a'}\} + \varepsilon_a\{g + \gamma_{a'}\{g, \phi_{a'}\}, \phi_a\} . \tag{34}$$

Using the Jacobi identity, the difference is

$$\Delta g = \varepsilon_a\gamma_{a'}\{g, \{\phi_a, \phi_{a'}\}\} . \tag{35}$$

This Δg must also correspond to a change in the q's and p's that doesn't affect the physical state. Thus $\{\phi_a, \phi_{a'}\}$ is a generating function of an infinitesimal contact transformation that still causes no change to the physical state.

Now note that since the ϕ_a are first class, their Poisson brackets are weakly zero, hence strongly equal to some linear combination of the ϕ's, which must be first-class by the Theorem (but can be secondary). Hence the final result is that those transformations of the dynamical variables which do not change the physical state are infinitesimal contact transformations in which the generating function is a primary first-class or possibly secondary first-class constraint.

Dirac bracket

Consider a simple example of two second-class constraints: $q_1 \approx 0$, $p_1 \approx 0$. They are second-class because $\{q_1, p_1\} = 1 \neq 0$. It is clear that the degree of freedom 1 is not of any importance. We can just discard it and work with the other degrees of freedom. That means a different definition for the Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}\,, \qquad r = 2, \cdots, N\,. \tag{36}$$

Let's generalize a bit. Suppose we have $p_1 \approx 0$, $q_1 = f(q_r, p_r)$. We could drop the number 1 degree of freedom if we substitute $f(q_r, p_r)$ for q_1 in the Hamiltonian and all other constraints. Again we could work with (36).

The existence of second-class constraints means that there are some degrees of freedom which are not physically relevant. We have to set up a new Poisson bracket which picks out only those degrees of freedom which are physically important. So let's go back to the general theory.

We have a number of constraints $\phi_j \approx 0$, some first-class, some second-class. We try to take linear combinations of them so has to have as many constraints as possible brought into the first-class. Those that remain in the second-class will be denoted by

$$\chi_s\,, \qquad s = 1, \cdots, S\,. \tag{37}$$

Consider the antisymmetric matrix

$$\Delta_{ss'} = \{\chi_s, \chi_{s'}\}\,. \tag{38}$$

It is non-singular[1] and since any antisymmetric matrix with an odd number of rows and columns has zero determinant, we conclude that the number S of second-class constraints must be even. We can define its inverse by

$$\Delta_{ss'}\{\chi_{s'}, \chi_{s''}\} = \delta_{ss''}\,. \tag{39}$$

We now define the *Dirac bracket*

$$\{f, g\}^* = \{f, g\} - \{f, \chi_s\}\Lambda_{ss'}\{\chi_{s'}, g\}\,, \tag{40}$$

which has the same properties of the usual Poisson bracket. Then notice that

$$\{g, H_T\}^* = \{g, H_T\} - \{g, \chi_s\}\Lambda_{ss'}\{\chi_{s'}, H_T\} \approx \{g, H_T\}\,. \tag{41}$$

Hence

$$\dot{g} \approx \{g, H_T\}^*\,. \tag{42}$$

Also, for any $f(q, p)$,

$$\{f, \chi_{s''}\}^* = \{f, \chi_{s''}\} - \{f, \chi_s\}\Delta_{ss'}\{\chi_{s'}, \chi_{s''}\} = \{f, \chi_{s''}\} - \{f, \chi_s\}\delta_{ss''} = 0\,. \tag{43}$$

Thus we can put the χ's equal to zero before working new Dirac brackets, i.e.

$$\chi_s = 0 \qquad \text{(strong equality)}\,. \tag{44}$$

[1] See [1] for a proof.

Observables

For unconstrained systems, *observables* are defined as phase space functions which correspond to physical quantities. For constrained systems, a measurable quantity should be a function on the constraint surface only. But, nevertheless, we can deal with functions defined on the whole phase space as *representations* of observables. We say that two such functions represent the same observable if they are weakly equal. However, only quantities invariant under gauge transformations are measurable. As gauge transformations are generated by first-class constraints, a function is invariant if its Poisson brackets with all first-class constraints vanish weakly or, equivalently, if its Dirac brackets with *all* constraints vanish weakly.

Functions which are constant in time are called *conserved charges*, and must obey

$$\{Q, H_0\}^* \approx 0\,. \tag{45}$$

They generate symmetry transformations

$$\delta F = \{F, Q\}^*\,, \tag{46}$$

which map solutions onto new solutions (using Jacobi's identity, one can show that if $q(t)$ is a solution, so is $q(t) + \delta q(t)$). In this formalism, all constraints are representations of the trivial conserved charge $Q \approx 0$.

Counting Degrees of Freedom

We conclude with an observation about the number of degrees of freedom. For unconstrained systems, it is just half the dimension of the phase space (equivalently, one usually says that a system with N degrees of freedom has a $2N$-dimensional phase space). For constrained systems, the phase space is restricted by the primary and secondary constraints, and by the number of gauge transformations which represent unphysical degrees of freedom. Since they are nothing but the primary constraints,

$$\#\ \text{DoF} = \frac{1}{2}\Big(\#\ \text{phase space variables} - 2\ \#\ 1^{st}\text{-class constraints} - \#\ 2^{nd}\text{-class constraints}\Big)\,. \tag{47}$$

References

[1] P. A. M. Dirac: *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, New York (1964).

[2] H. J. Matschull: *Dirac's Canonical Quantization Programme*, arxiv:quant-ph/9606031v1 (1996).