Asymptotically AdS spacetimes and isometric embeddings

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Hyperboloid model of H^n

Hyperbolic space H^n s a Riemannian manifold. All of the geometry can be understood in terms of intrinsic properties (metric, curvature..). However, to manifest more symmetries consider: ($\mathbb{M}^{1,n}$ is Minkowski space-time) [fig. 1]

$$\mathcal{H}^n := \{X^\mu \in \mathbb{M}^{1,n} | X^0 > 0,\ X^\mu X^\nu \eta_{\mu\nu} = -1\}. \tag{1}$$

The intrinsic geometry of this hyperboloid is equivalent (globally isometric) to H^n.

- Every timelike straight line through the origin intersects a unique point in $\mathcal{H}$.
- Geodesics: $\mathcal{H} \cap$ timelike 2-planes through the origin.
- Totally geodesic p-surfaces: $\mathcal{H} \cap$ timelike $(p+1)$-planes through the origin.
- Global isometry group: $PSO(n, 1)$.

Hyperboloid model of $\widetilde{\text{AdS}}^n$

Submanifold of $\mathbb{M}^{2,n-1}$ (flat spacetime with two time coordinates) . Define $\eta^{(2,n-1)}_{\mu\nu} := \text{diag}(-1, +1, ..., +1, -1)$. [fig. 2]

$$\mathcal{A} := \{X^\mu \in \mathbb{M}^{2,n-1} | X^\mu X^\nu \eta^{(2,n-1)}_{\mu\nu} = -1\}. \tag{2}$$

The intrinsic geometry of this hyperboloid is Anti de Stter space $\widetilde{\text{AdS}}^n$. This has closed timelike curves.

- Every timelike straight line through the origin intersects two unique points in $\mathcal{H}$.
- $\mathcal{A} \cap$ [2-plane through the origin] are geodesics (timelike) or a pair of disconnected geodesics (spacelike)
- Global isometry group: $PSO(n, 1)$.

The cosmological AdS space is the universal cover of $\widetilde{\text{AdS}}$. Later we present an embedding of AdS^n into $\mathbb{M}^{2,n}$.

Definition (Embedding of differentiable manifold)

A differentiable embedding $\phi : M \rightarrow N$ is an injective map (for x, y distinct $\phi(x) \neq \phi(y)$) so that the image $\phi(M)$ is homeomorphic to M and furthermore the induced tangent space map is injective.

Note $dim(N) \geq dim(N)$. In terms of coordinates (σ^a) on M and (x^μ) on N, we write $\phi : \sigma \rightarrow x(\sigma)$ and $\phi_* : v^a \rightarrow v^a \partial x^\mu / \partial \sigma^a$.

Definition (isometric embedding)

Let (M, h) and (N, g) be (pseudo-)Riemannian manifolds. A smooth embedding $\phi : M \rightarrow N$ is isometric if $\phi_* g = h$.

Straightforward examples

- Clifford torus (flat) $S^1 \times S^1 \to \mathbb{E}^4$;
  $(\theta, \phi) \to (a \cos\theta, a \sin\theta, b \cos\phi, b \sin\phi)$.
- A cone may be regarded an injection $\mathbb{R}_2 \to \mathbb{E}_3$;
  $(x, y) \to (x, y, \alpha\sqrt{x^2 + y^2})$. Locally isometric embedding of $\mathbb{E}^2$ in any neighbourhood not including the origin. Fails to be an embedding at the origin (the tangent space of the image is zero dimensional).

Sometimes convenient to represent embedded submanifold as set of points satisfying constraint equation(s). i.e.

- Clifford torus: $\{\vec{X} \in \mathbb{E}^4 | X^2 + Y^2 = a^2,\ Z^2 + W^2 = b^2\}$
- Cone: $\{\vec{x} \in \mathbb{E}^3 | X^2 + Y^2 = Z^2/\alpha^2\}$

Isometric embedding of the Schwarzschild solution

Embedding found by Kasner (1921) can not be extended beyond the horizon.
An embedding $M \to \mathbb{M}^{5,1}$ was found by [Fronsdal 1959]:

$$X^0 = 2R\sqrt{1-\frac{R}{r}}\cosh(\frac{t}{2R}),$$
$$X^1 = 2R\sqrt{1-\frac{R}{r}}\sinh(\frac{t}{2R}),$$
$$X^2 = \int dr\sqrt{\frac{R}{r}+\frac{R^2}{r^2}+\frac{R^3}{r^3}},$$

and $(X^3)^2+(X^4)^2+(X^5)^2 = r^2$. This can be continued across the horizon.
More recent examples see e.g. [Paston, Sheykin 2012].

Theory of isometric embeddings: some history

- Riemann (1850's) introduced manifolds as intrinsically defined objects; modern abstract definition due to Weyl.
- Natural question: are they more general than surfaces in Euclidean space? Schläfli (1873) conjectured that locally an isometric embedding exists in $d = n(n+1)/2$.
- Janet and Cartan (1920's) proved this for the analytic case.
- (Whitney (1936) global embedding theorem in diff. topology.)
- Nash (1956) resolved the global isometric embedding problem for any C^k Riemannian manifold into Euclidean space of large enough dimension.
- Generalised to pseudo-Riemannian manifolds [Clarke 1970, Greene 1970]. In particular, any globally hyperbolic spacetime admits G.I.E. into $\mathbb{M}^{d,1}$.
- Many other results...

Some applications

- Strings and branes (minimal surfaces)
- Illustrating the geometry and global features of exact solutions of einstein equations.
- Gravity a la string [Regge, Teitelboim]
- Alternative method of studying perturbations/stability

Regarding the last two points, a problem is that there is some degeneracy in the functional derivatives with respect to the embedding coordinate functions. This degeneracy is a result of *isometric bending*. A possible resolution is to consider the function space only in some neighbourhoods of classical solutions which are embedded *freely* in the sense of [Nash 56].

The BTZ black hole ($J = 0$)

In 2+1 dimensional gravity with negative cosmological constant, the solutions satisfy $R^{\mu\nu}{}_{\kappa\lambda} = -\frac{1}{l^2}\delta^{\mu\nu}_{\kappa\lambda}$. We set $l = 1$. The spherically symmetric solution has the static form

$$ds^2 = (r^2 - a^2)d\tau^2 + \frac{dr^2}{r^2 - a^2} + r^2 d\phi^2$$

outside of the event horizon ($r = a$). Kruskal type coordinate system:

$$ds^2 = 4\frac{-dt^2 + dx^2}{(1 + t^2 - x^2)^2} + a^2\frac{(1 - t^2 + x^2)^2}{(1 + t^2 - x^2)^2}d\phi^2 .$$

The domain of the coordinates is $-1 < -t^2 + x^2 < 1$, $\phi \sim \phi + 2\pi$. Singularities at $t^2 - x^2 = 1$, conformal infinity at $x^2 - t^2 = 1$, event horizons $x = \pm t$, bifurcation surface at $x = t = 0$. This covers the maximally extended space-time.

Lemma (S.W. arXiv:1011.3883 gr-qc)

The nonrotating BTZ black hole spacetime can be globally isometrically embedded into the region $X^0 > 0$ of $\mathbb{M}^{2,3}$. The image is the intersection of quadric hypersurfaces:

$$X^\mu X^\nu \eta^{(2,3)}_{\mu\nu} = -1, \qquad (X^1)^2 + (X^2)^2 = \frac{a^2}{1+a^2}(X^0)^2 \,. \tag{3}$$

The past and future singularities are located at the intersection of the two constraint surfaces with the hyperplane $X^0 = 0$.

Proof: It can be verified that the following is an embedding

$$X^0(x,t) = \sqrt{1+a^2}\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right),$$

$$X^1(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\cos\phi\,,\ X^2(x,t,\phi) = a\left(\frac{1-t^2+x^2}{1+t^2-x^2}\right)\sin\phi\,,$$

$$X^3(x,t) = \frac{2x}{1+t^2-x^2}\,,\ X^4(x,y) = \frac{2t}{1+t^2-x^2}\,.$$

- By lifting the restriction $X^0 > 0$ we obtain two copies of BTZ joined at the singularity, but it is not a true embedding at $X^0 = 0$: the tangent space map is not injective (the BTZ central singularity is a conical singularity).
- From the first constraint equation we conclude that an embedding into $\widetilde{\mathrm{AdS}}_4$ exists.

By the analytic continuation $X^4 \to iX^4$ we obtain:

Lemma

The Euclidean nonrotating BTZ space can be G.I.E. into $\mathbb{M}^{1,4}$. Image is given by constraints
$(X^\mu X^\nu \eta_{\mu\nu} = -1,\ (X^1)^2 + (X^2)^2 = \frac{a^2}{1+a^2}(X^0)^2,\ X^0 > 0)$.

This coincides asymptotically with $\{X^\mu \in \mathbb{M}^{1,4} | X^\mu X^\nu \eta_{\mu\nu} = -1, (X^3)^2 + (X^4)^2 = \frac{1}{1+a^2}(X^0)^2, X^0 > 0\}$ which is “thermal AdS", i.e. Euclidean BTZ solution of mass $1/a$ with Euclidean time and ϕ- coordinate reversing roles.

Analytically continuing back to $\mathbb{M}^{2,3}$ by $X^4 \to iX^4$ we find:

Lemma

AdS_3 admits a 1-parameter family of global isometrically embeddings into the region $X^0, X^3 > 0\}$ of $\mathbb{M}^{2,3}$. The image is

$$X^\mu X^\nu \eta^{(2,3)}_{\mu\nu} = -1, (X^3)^2 - (X^4)^2 = \frac{1}{1+a^2}(X^0)^2, \tag{4}$$

- The isometric bending parameter a is an artefact of the finite temperature of the Euclidean continuation.
- The embedded BTZ spacetime is asymptotic to two disconnected copies ($X^3 > 0$ and $X^3 < 0$) of the embedded AdS spacetime of the same "Euclidean temerature" a.
- The above lemma generalises to arbitrary dimension.

Conformal ball model of hyperbolic space

Unit ball ($x \cdot x < 1$) with metric:

$$g = \frac{dx \cdot dx}{1 - x \cdot x} + \frac{(x \cdot dx)^2}{(1 - x \cdot x)^2}$$

Geodesics, totally geodesic surfaces and geodesic spheres coincide with Euclidean counterparts.

- The Euclidean BTZ is given by $x^2 + y^2 = a^2/(1 + a^2)$. This is a hypercylinder
- More generally [Nomizu 73] a hypercylinder over any plane curve in the Klein ball is locally isometric to hyperbolic space.
- It is fairly straightforward to check that a hypercylinder over any closed plane curve is globally isometric to the nonrotating BTZ! (the conformal metric at infinity is an untwisted torus. BTZ is unique hyperbolic manifold with this conformal boundary.)

Asymptotic conditions

We consider manifolds which tend to constant negative curvature at infinity.

- Relevant in the study of holographic principle
- Semiclassical partition functions.
- Asymptotically globally AdS spacetimes
  - Allow the definition of a generalised ADM total mass and total momentum. Positive mass theorems etc.
  - Arise as natural objects of study in the embedding picture in the guise of submanifolds asymptotic to a totally geodesic planes.

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Definition (Conformally compactifiable)

Let (M, g) be a complete noncompact Riemannian manifold. If the we can find:

1. A compact Riemannian manifold $(\hat{M}, \hat{g})$ such that the interior $\hat{M} \setminus \partial\hat{M}$ is diffeomorphic to M;
2. A defining function Ω which has a simple zero at $\partial\hat{M}$ and vanishes nowhere in the interior of $\hat{M}$;
3. A diffeomorphism $\phi : \hat{M} \setminus \partial\hat{M} \to M$ such that $\Omega^2\phi^* g = \hat{g}$,

then we say that M is conformally compactifiable.

Definition (Locally asymptotically hyperbolic)

If furthermore:

4. the defining function satisfies $\hat{g}^{ab}\hat{\nabla}_a\Omega\hat{\nabla}_b\Omega = \kappa^2$ everywhere on $\partial\hat{M}$

then M is asymptotically locally hyperbolic.

(under Weyl transformation, the Riemann tensor transforms:

$$R^{ab}{}_{cd} = \Omega^2\tilde{R}^{ac}{}_{cd} - 4\Omega\delta^{[a}_{[c}\tilde{\nabla}_{d]}\tilde{\nabla}^{b]}\Omega - 2\delta^{[a}_{[c}\delta^{b]}_{d]}\tilde{\nabla}^e\Omega\tilde{\nabla}_e\Omega\,.$$

Condition 4) $\Rightarrow R^{ab}{}_{cd} \to -2\delta^{[a}_{[c}\delta^{b]}_{d]}\kappa^2$.)

A preliminary result:

Lemma

Let (M_n, g) be some noncompact Riemannian manifold and $K_m = (\mathbb{B}_m, g_K)$ be the Klein ball model for H^m for some $m > n$. If there exists a smooth isometric immersion $\psi : M_n \to \mathbb{B}_m$ such that:

1. *The closure $\overline{\psi(M)}$ is a smooth submanifold with boundary $\partial\psi(M) \subset \partial\mathbb{B}$;*
2. *The tangent space of $\overline{\psi(M)}$ is not a subspace of that of $\partial\mathbb{B}$ at the boundary,*

then (M_n, g) is asymptotically hyperbolic in the conformally compactifiable sense.

Outlook

- Classification of embeddings of locally AdS 3-manifolds.
- Free embeddings. Applications to perturbations, Regge-Teitelboim model etc.
- Embedding of asymptotic region. Applications in terms of boundary-bulk holography.