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Massive vector fields on the Schwarzschild spacetime:

quasi-normal modes and bound states

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Motivation

- Black hole perturbation theory
  - Massive vector fields less well understood

    [Gal'tsov, Pomerantseva & Chizhov (1984); Konoplya (2006); Konoplya, Zhidenko & Molina (2007); Herdeiro, Sampaio & Wang (2011)]
  - Challenging problem (separability, coupled dof)
  - Rich perturbation spectrum
- Particle Phenomenology
  - Light U(1)'s in string theory [e.g. Jaeckel & Ringwald (2010)]

    (RR-, NS-forms, D-branes w/ hyperweak couplings)
  - Kerr superradiance → Non-separable equations!
  - Schwarzschild case simpler and useful guide

Proca field on the Schwarzschild spacetime

Massive vector field equations

- Schwarzschild spacetime

$$ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \;, \qquad f(r) = 1 - 2M/r$$

- Proca equations

$$F^{\mu\nu}{}_{;\mu} = \mu^2 A^\nu \;, \qquad F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$$

  – Lorenz condition:

$$A^\mu_{;\mu} = 0$$

  – Field equations (Ricci-flat):

$$A^{\nu;\mu}{}_{\mu} - \mu^2 A^\nu = 0$$

Separation of variables

- Vector spherical harmonics [see e.g. Barack & Lousto (2005)]

$$A_\mu(t,r,\theta,\phi) = \frac{1}{r}\sum_{i=1}^{4}\sum_{lm} c_i\, u^{lm}_{(i)}(t,r) Z^{(i)lm}_\mu(\theta,\phi)$$

$$\begin{aligned}
Z^{(1)lm}_\mu &= [1,0,0,0]\, Y^{lm} \\
Z^{(2)lm}_\mu &= \left[0, f^{-1}, 0, 0\right] Y^{lm} \\
Z^{(3)lm}_\mu &= \frac{r}{\sqrt{l(l+1)}}\,[0,0,\partial_\theta,\partial_\phi]\, Y^{lm} \\
Z^{(4)lm}_\mu &= \frac{r}{\sqrt{l(l+1)}}\left[0,0,\frac{1}{s_\theta}\partial_\phi, -s_\theta\partial_\theta\right] Y^{lm}
\end{aligned}$$

Separation of variables

Using the Lorenz condition:

$$\begin{aligned} \hat{\mathcal{D}}_2 u_{(2)} &- \frac{2f}{r^2}\left(1-\frac{3}{r}\right)\left(u_{(2)}-u_{(3)}\right)=0 \\ \hat{\mathcal{D}}_2 u_{(3)} &+ \left[\frac{2fl(l+1)}{r^2}u_{(2)}\right]=0 \end{aligned} \quad \text{even-parity}$$

$$\hat{\mathcal{D}}_2 u_{(4)} = 0 \quad \text{odd-parity}$$

where

$$\hat{\mathcal{D}}_2 \equiv -\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-f\left[\frac{l(l+1)}{r^2}+\mu^2\right]$$

Electromagnetic limit

- Monopole [Konoplya (2006)]

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-f\left(\frac{2(r-3)}{r^3}+\mu^2\right)\right]u_{(2)}=0$$

$$\mu \to 0$$

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-\frac{2f}{r^3}\right]\dot{u}_{(1)}=0$$

$$A_\mu = \chi_{,\mu}+\frac{q}{r}\delta^0_\mu$$

No monopole radiation

- Odd-parity modes [Price (1972)]

$$\hat{\mathcal{D}}_2(r^2\phi_1)=0$$

where

$$\phi_1 \equiv F_{\mu\nu}(l^\mu n^\nu+\bar{m}^\mu m^\nu)$$

$$\phi_1^{lm}=i\,\frac{l(l+1)}{r^2}u^{lm}_{(4)}(t,r)Y_{lm}(\theta,\phi)$$

Electromagnetic limit

- Even-parity modes

$$\frac{1}{rf^3}\hat{\mathcal{D}}_{RW}^{l,s=1}\left[f^{-1}\hat{\mathcal{D}}_{RW}^{l,s=0}\left(ru_{(3)}\right)\right]=0$$

$$\hat{\mathcal{D}}_{RW}^{l,s}\equiv-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_*^2}-f\left(\frac{l(l+1)}{r^2}+\frac{2(1-s^2)}{r^3}\right)$$

Scalar solution

$$\hat{\mathcal{D}}_{RW}^{l,s=0}(ru_{(3)})=0$$

$$A_\mu=\chi_{,\mu}\ ,\qquad \chi=\frac{u_{(3)}}{l(l+1)}Y_{lm}(\theta,\phi)$$

(unphysical gauge mode)

Vector solution

$$\hat{\mathcal{D}}_{RW}^{l,s=1}\psi=\hat{\mathcal{D}}_2\psi=0$$

$$\psi=u'_{(3)}-\frac{l(l+1)}{r}u_{(2)}$$

(same for odd-parity modes)

Numerical methods

Frequency spectra

- Frequency-domain representation:

$$u_{(i)}^{lm}(t,r) = u_{(i)}^{lm}(\omega,r)e^{-i\omega t}$$

  - 'Ingoing' boundary conditions: $u_{(i)}(\omega,r) \sim e^{-i\omega r_*}$

  - Asymptotic solution:

$$u_{(i)}(\omega,r) \sim B_{(i)}(\omega)e^{-qr_*} + C_{(i)}(\omega)e^{qr_*}$$

$B_{(i)}(\omega)e^{-qr_*}$ term → (quasi-) bound states

$C_{(i)}(\omega)e^{qr_*}$ term → quasi-normal modes

$q = \sqrt{\mu^2 - \omega^2}$

Continued-fraction

- Ansatz: [Leaver (1985), see also Dolan (2007)]

$$u_{(i)}(\omega, r) = f^{-2i\omega} r^{-\nu} e^{qr} \sum_n a_n^{(i)} [f(r)]^n \qquad \nu = (\omega^2 - q^2)/q$$

- Three-term recurrence relations:

$$\begin{aligned} \alpha_0 a_1 + \beta_0 a_0 &= 0 \\ \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0, \quad n > 0 \end{aligned}$$

⬇

$$\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1} - \frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2} - \alpha_{n-3}\gamma_{n-2}/\ldots}} = \frac{\alpha_n \gamma_{n+1}}{\beta_{n+1} - \frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \alpha_{n+2}\gamma_{n+3}/\ldots}}$$

(monopole and odd-parity modes)

Continued-fraction

- Even-parity modes

$$\begin{aligned} \underline{\alpha}_0 \mathbf{U}_1 + \underline{\beta}_0 \mathbf{U}_0 &= 0 \,, \\ \underline{\alpha}_n \mathbf{U}_{n+1} + \underline{\beta}_n \mathbf{U}_n + \underline{\gamma}_n \mathbf{U}_{n-1} &= 0, \quad n > 0 \end{aligned} \qquad \mathbf{U}_n = \begin{pmatrix} a_n^{(2)} \\ a_n^{(3)} \end{pmatrix}$$

- Matrix continued fraction [Simmendinger *et al.* (1999)]

$$\mathbf{U}_{n+1} = \mathbf{R}_n^+ \mathbf{U}_n \qquad \mathbf{R}_n^+ = -\left(\underline{\beta}_{n+1} + \underline{\alpha}_{n+1} \mathbf{R}_{n+1}^+\right)^{-1} \underline{\gamma}_{n+1}$$

$$\mathbf{M} \equiv \underline{\beta}_0 - \underline{\alpha}_0 \left[\underline{\beta}_1 - \underline{\alpha}_1 \left(\underline{\beta}_2 + \underline{\alpha}_2 \mathbf{R}_2^+\right) \underline{\gamma}_2\right]^{-1} \underline{\gamma}_1$$

$$\mathbf{M}\mathbf{U}_0 = 0 \quad \Rightarrow \quad \det |\mathbf{M}| = 0$$

– "Polarization":

$$\mathcal{P} \equiv \lim_{r_* \to \infty} \frac{u_{(3)}(\omega, r)}{u_{(2)}(\omega, r)} = \sum_{n=0}^{\infty} a_n^{(3)} \Big/ \sum_{n=0}^{\infty} a_n^{(2)}$$

Forward-integration (bound states)

- "Ingoing" *ansatz*:

$$u_{(i)}(\omega, r) = (r - r_H)^{-2i\omega} \sum_{n=0}^{\infty} b_{(i)n} (r - r_H)^n$$

  - Bound state solutions converge exponentially
  - Integrate eqs. up to large distance and minimize in complex frequency-plane
  - For even-parity modes, choose basis of coefficients and integrate:

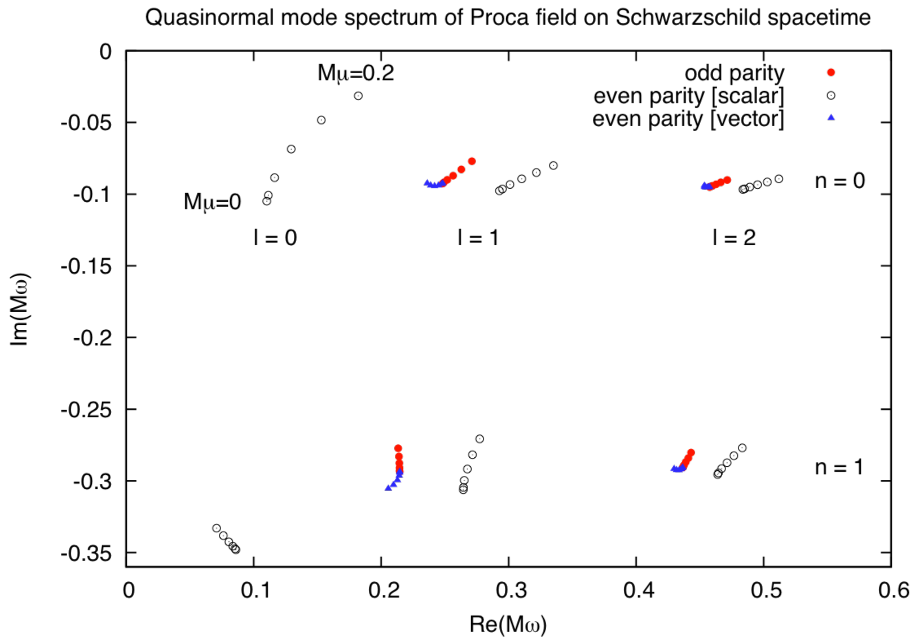
$$\mathbf{S}(\omega, r) = \begin{pmatrix} u_{(2)}^{(1,0)}(\omega, r) & u_{(2)}^{(0,1)}(\omega, r) \\ u_{(3)}^{(1,0)}(\omega, r) & u_{(3)}^{(0,1)}(\omega, r) \end{pmatrix}$$

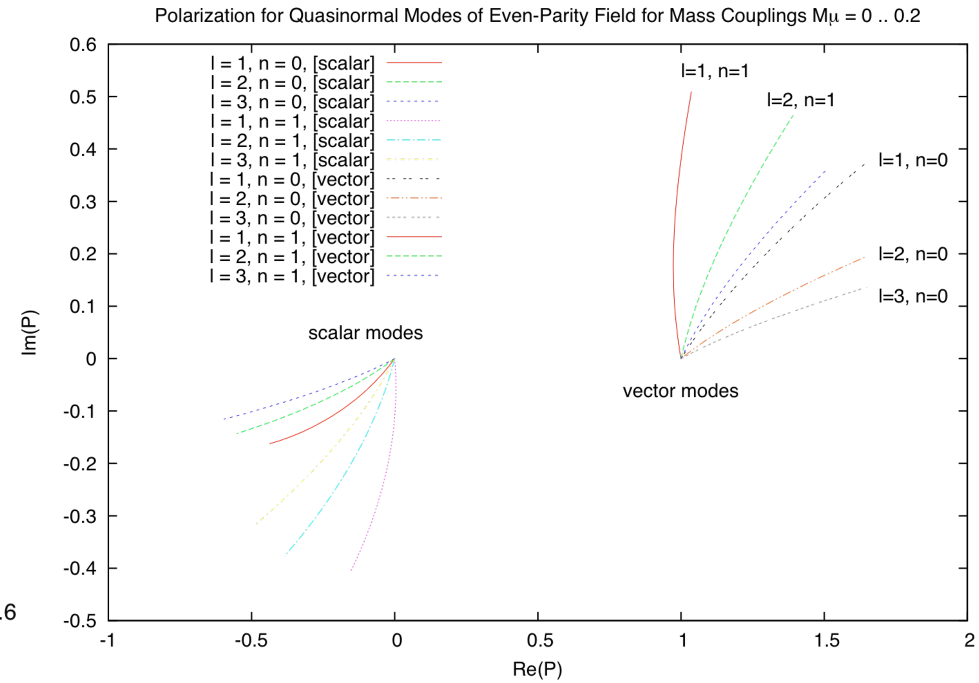
  - Bound states minimize

$$\det |\mathbf{S}(\omega, r \to +\infty)|$$

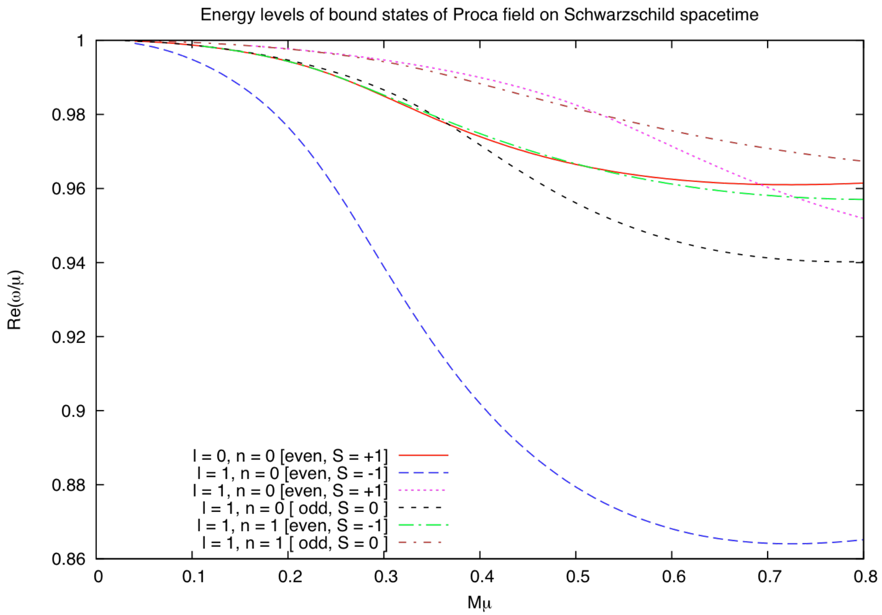
Results

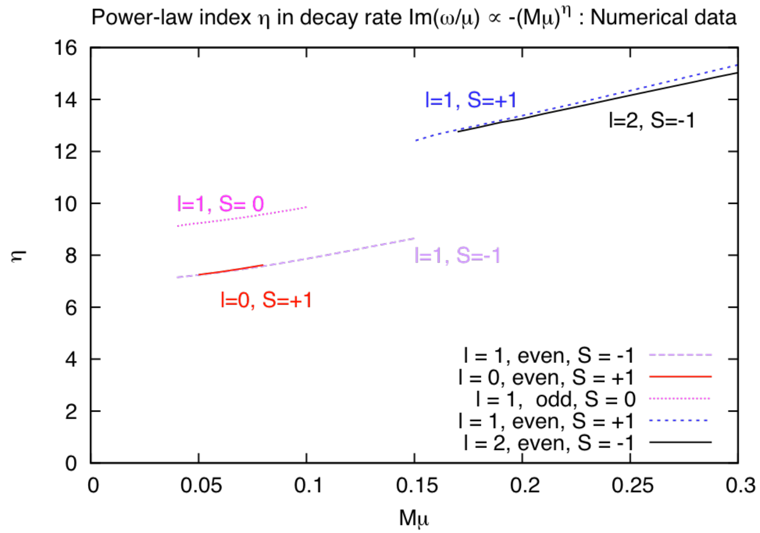
Quasi-normal modes

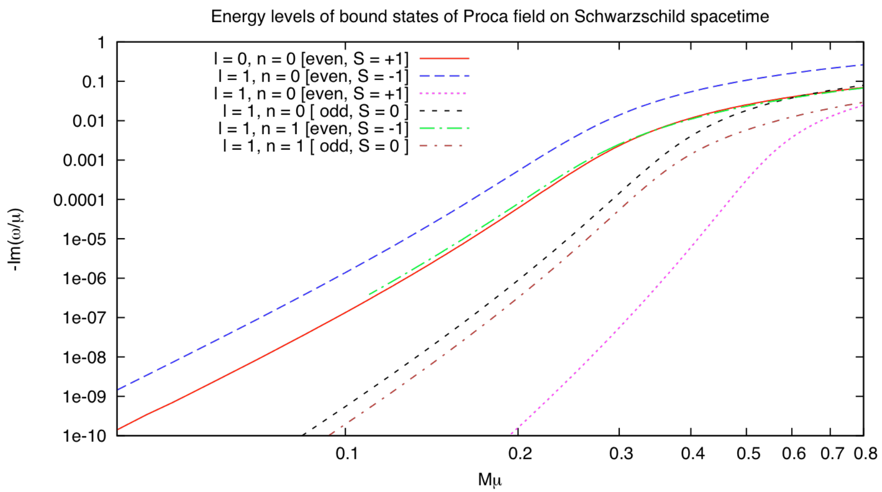




Bound states: frequency spectrum





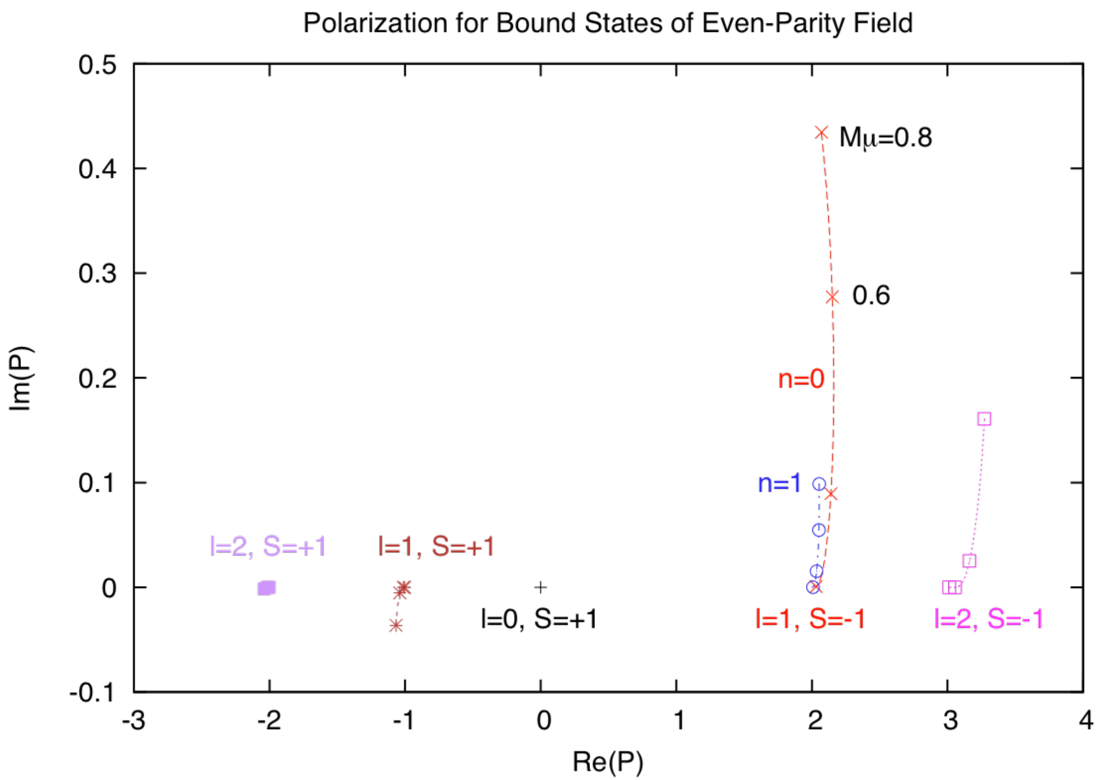


$$\mathrm{Re}\,(\omega/\mu) \approx 1 - \frac{(M\mu)^2}{2(l+S+1+n)^2}$$

[Gal'tsov, Pomerantseva & Chizhov (1984)]

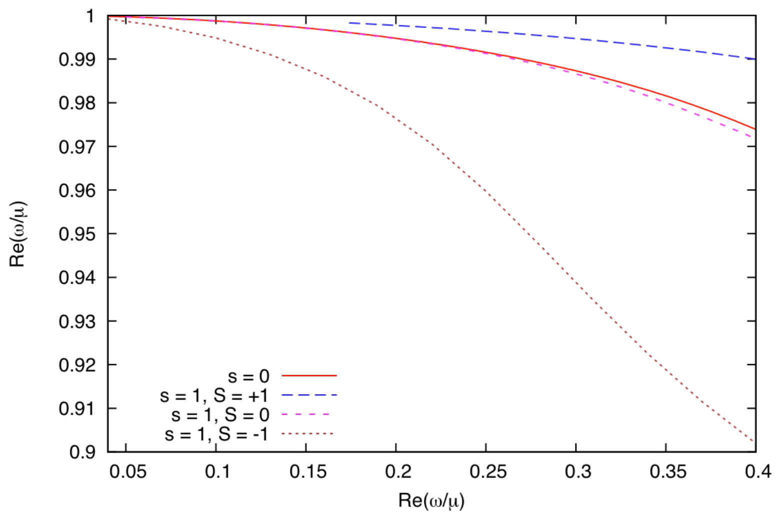
$$\mathrm{Im}\,(\omega/\mu) \approx (M\mu)^\eta$$
$$\eta = 4l + 2S + 5$$

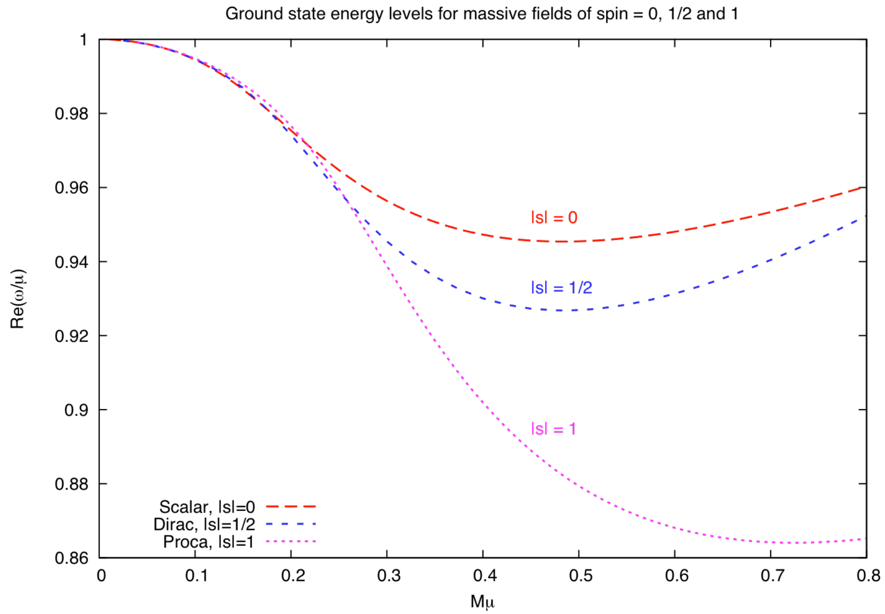
Bound states: polarization

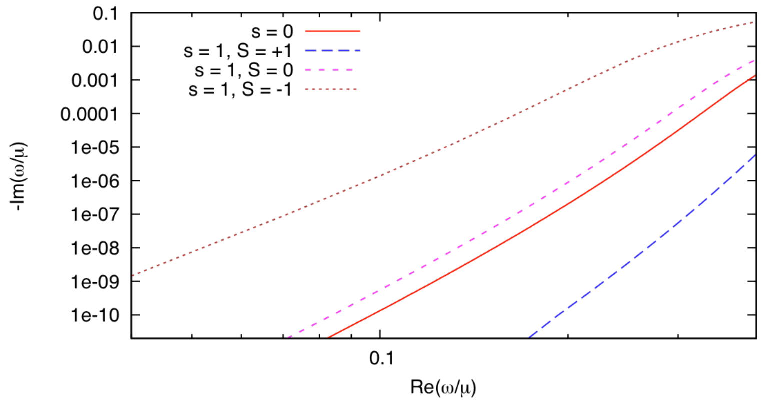


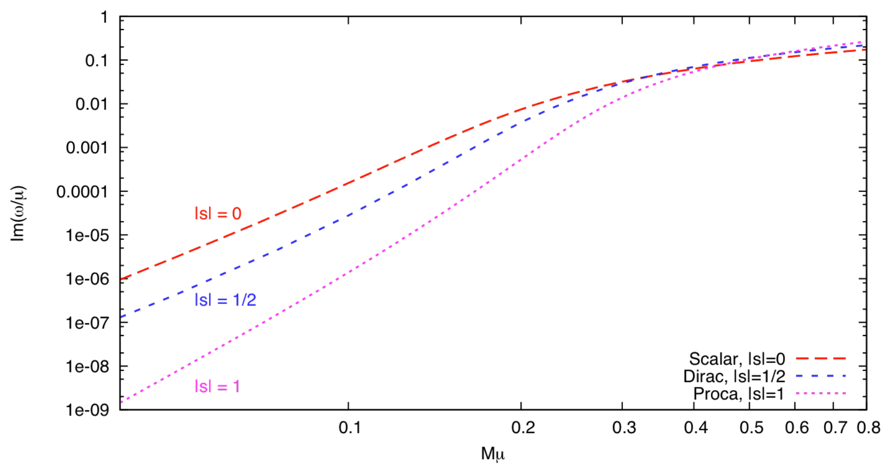
$$\mathcal{P} \to -l,\ l+1\ , \qquad S = \pm 1$$

Bound states: comparison with other spins









l=1, n=0 modes

Analytical results

(small mass coupling)

Odd-parity modes

$$\left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]u_{(4)} = 0 \qquad x = r/2M - 1$$

$$V(x) = 4\omega^2(x+1)^4 - 4\mu^2 x(x+1)^3 - l(l+1)x(x+1)$$

- Near region: $\mu x\ , \omega x \ll l$

$$u_{(4)}^{near} = A_{(4)} x^{-2i\omega}(x+1)^{1+\delta}{}_2F_1(-l-2i\omega+\delta, l+1-2i\omega+\delta, 1-4i\omega, -x)$$

$$\delta = \sqrt{1-4\omega^2}$$

- Far region: $x \gg 1$

$$u_{(4)}^{far} = e^{-z/2}\left[C_{(4)} z^{l+1} M(l+1-\nu, 2l+2, z) + D_{(4)} z^{-l} M(-l-\nu, -2l, z)\right]$$

$$z = 4qx$$

Odd-parity modes

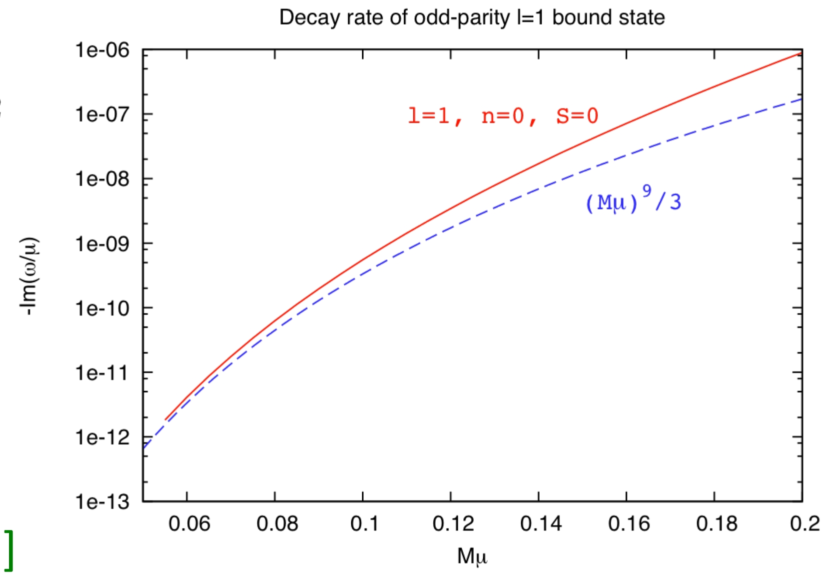
- Matching for bound-states: $\omega \simeq \mu\left(1 - \frac{\mu^2}{2(l+1+n)^2}\right)$

$$\delta\omega \simeq \frac{4^{2l+1}\mu^{4l+5}}{(l+1+n)^{2l+4}} \frac{(2l+1+n)!}{n!} \frac{(l+1)!(l-1)!}{[(2l)!(2l+1)!]^2}(1+2i\omega)\prod_{k=1}^{l}(k^2-1-4i\omega)$$

- Matching for QN modes:

$$1 = i(4q)^{2l+1}(l+1)!(l-1)!\left(\frac{l!}{(2l)!(2l+1)!}\right)^2 \times (1+2i\omega)\prod_{k=1}^{l}(k^2-1-4i\omega)$$

  – Bad approximation for QN modes
  [see also Panchapakesan & Majumdar (1987)]
  – Correct mass dependence



Even-parity modes

$$\begin{aligned}\left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]u_{(2)} - x(2x-1)\left(u_{(2)} - u_{(3)}\right) &= 0\\ \left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]u_{(3)} + 2\lambda^2 x(x+1)u_{(2)} &= 0\end{aligned}$$

- Diagonalization in far region:

$$u_{(2,3)}(z) = \sum_{S=\pm1} c^S_{(2,3)} u_{(S)}(z)$$

$$u^{far}_{(S)} = e^{-z/2}\left[C_{(S)} z^{j+1} M(j+1-\nu, 2j+2, z) + D_{(S)} z^{-j} M(-j-\nu, -2j, z)\right]$$

- No decoupling in near region...

$$\left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]\psi = 4\mu^2 x(x+1)u_{(3)} \ ,$$

$$\left[x^2(x+1)^2\partial_x^2 + x(x+1)(2x+1)\partial_x + V(x)\right]u_{(3)} = 2x(x+1)^2\psi$$

Even-parity modes

$$\begin{aligned}\left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]u_{(2)} - x(2x-1)\left(u_{(2)} - u_{(3)}\right) &= 0 \\ \left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]u_{(3)} + 2\lambda^2 x(x+1)u_{(2)} &= 0\end{aligned}$$

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$$u_{(2,3)}(z) = \sum_{S=\pm 1} c^{S}_{(2,3)} u_{(S)}(z)$$

$$u^{far}_{(S)} = e^{-z/2}\left[C_{(S)} z^{j+1} M(j+1-\nu, 2j+2, z) + D_{(S)} z^{-j} M(-j-\nu, -2j, z)\right]$$

- No decoupling in near region...

$$\left[x^2(x+1)^2\partial_x^2 + x(x+1)\partial_x + V(x)\right]\psi = 4\mu^2 x(x+1)u_{(3)} \ ,$$

$$\left[x^2(x+1)^2\partial_x^2 + x(x+1)(2x+1)\partial_x + V(x)\right]u_{(3)} = 2x(x+1)^2\psi$$

Even-parity modes

What can we conclude?

- Hydrogenic spectrum for bound states from far region solutions

$$\omega \simeq \mu\left(1 - \frac{\mu^2}{2(j+1+n)^2}\right) , \quad j = l \pm 1$$

- Asymptotic solutions are linear combinations of 'scalar' and 'vector' solutions for QN modes

- 'scalar-vector' mixing in far region yields non-trivial spin-dependence for imaginary bound state frequency:

$$\mathrm{Im}\,(\omega/\mu) \approx (M\mu)^\eta \qquad \eta = 4l + 2S + 5$$

Summary and future prospects

- Proca fields have a rich spectrum of perturbations
- Decoupling of odd- and even-parity modes
- Broken degeneracy for ‘vector’ QN modes + physical ‘scalar’ mode
- Hydrogen-like bound states with spin-dependent decay rates
- Larger decay rates for S=-1,0 modes than for scalar fields

⬇ ?

- Larger superradiant instabilities in Kerr spacetimes?
- Encouraging preliminary results for slowly-rotating BH
- Interesting phenomenology for light string U(1) fields!