A dimensional reduction approach to the Teukolsky equation in higher dimensions.

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Outline

1. Description of a simplified problem for the Teukolsky equation
2. Ricci and Weyl scalars components of a higher dimensional Tangherlini space-time
3. Decoupling the equations in higher dimensions
4. Conclusions

The Teukolsky equation in 4D

- Two Bianchi identities and one Ricci identity

$$
\begin{aligned}
(\delta^* - 4\alpha + \pi)\,\Psi_0 - (D - 4\rho - 2\epsilon)\,\Psi_1 - 3\kappa\Psi_2 &= 0,\\
(\Delta - 4\gamma + \mu)\,\Psi_0 - (\delta - 4\tau - 2\beta)\,\Psi_1 - 3\sigma\Psi_2 &= 0,\\
(D - \rho - \rho^* - 3\epsilon + \epsilon^*)\,\sigma - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\,\kappa - \Psi_0 &= 0,
\end{aligned}
$$

- One applies the operator $(D - 4\rho - \rho^* - 3\epsilon + \epsilon^*)$ to the first identity and the operator $(\delta - 4\tau + \pi^* - \alpha^* - 3\beta)$ to the second one.
- The term with Ψ_1 factors out because of some identities obtained from the Bianchi identities.
- The term with Ψ_2 factors out using the Ricci identity.

The Teukolsky equation in higher dimensions: an open problem

- The generalization of the Teukolsky equation to higher dimensions would be of extreme importance for several studies of black hole perturbation theory.
- So far there have been attempts that are not very encouraging: Durkee and Reall (Class.Quant.Grav.28:035011,2011) have shown that an analogous type of decoupling can happen in higher dimensions only if the space-time has a specific property (Kundt space-time). Unfortunately not even the Schwarzschild generalization to higher dimensions (Tangherlini) satisfies this condition.
- Their result is obtained by simply trying to apply the same mechanism that is valid in four dimensions to the higher dimensional case, obtaining a very complex system.

A simple application

- We try to analyze a simpler problem, with a more direct connection to the 4D case.
- Our group has been working on the numerical study of higher dimensional space-times that possess symmetries.
- It would be highly computationally expensive to simulate numerically space-times with $D > 4$.
- The key idea is that, by means of the symmetry, the dimensionality of the space-time can be reduced, going back to the original four dimensional system, only adding some additional fields that reflect the original higher dimensionality.

The physical scenario

- We want to study a physical system whose metric goes under the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \lambda(x^\mu) d\Omega_{\mathcal{D}-4}$$

- For example the Tangherlini solution for a higher dimensional non spinning black hole is given by

$$ds^2 = -\left[1 - \left(\frac{2M}{r}\right)^{\mathcal{D}-3}\right] dt^2 + \left[1 - \left(\frac{2M}{r}\right)^{\mathcal{D}-3}\right]^{-1} dr^2 + r^2 d\Omega_{\mathcal{D}-2}$$

- In this case $\lambda = r^2 \sin^2\theta \sin^2\phi$.
  In the perturbed case we consider an axisymmetric perturbation such that
- $\lambda = g_{\phi\phi} \sin^2\phi$.

Numerical applications

- Our group has produced over the last couple of years a number of important papers on the numerical application of this approach.
- The simulation of head-on collisions of equal and unequal mass ratio binary black holes has been performed within this formalism, allowing to estimate the gravitational wave content of these space-times.
- The evolution equations for the numerical system have been readapted to the standard BSSN equation, adding the "quasi-matter" terms, plus the evolution equation for the field λ.
- Some references (Phys.Rev.D83:044017,2011) (Phys.Rev.D82:104014,2010) (Phys. Rev. D81 (2010) 084052)

The dynamical equations

- The presence of the λ field can be seen as equivalent to a "quasi-matter" field in the four dimensional space-time. One can derive the expression for the Ricci tensor related to this field given by

$$R_{\mu\nu} = \left(\frac{\mathcal{D}-4}{2\lambda}\right)\left[\nabla_\mu \nabla_\nu \lambda - \frac{1}{2\lambda}\nabla_\mu \lambda \nabla_\nu \lambda\right],$$

- Also the dynamical equation for λ can be derived and reads

$$\nabla^\mu \nabla_\mu \lambda + \left(\frac{\mathcal{D}-6}{2\lambda}\right)\nabla^\mu \lambda \nabla_\mu \lambda = 2\left(\mathcal{D}-5\right).$$

- The fundamental difference from "real" matter is that this "quasi-matter" fills the whole space-time and is not compact, therefore we really have to take care of it in our computation for the Teukolsky equation.

How do we translate matter in the NP formalism?

- Just like the Weyl components of the Riemann tensor lead to the Weyl scalars, the Ricci components lead to the Ricci scalars

$$\Phi_{00} = -\frac{1}{2} R_{ab} \ell^a \ell^b, \quad \Phi_{22} = -\tfrac{1}{2} R_{ab} n^a n^b,$$
$$\Phi_{02} = -\frac{1}{2} R_{ab} m^a m^b, \quad \Phi_{20} = -\tfrac{1}{2} R_{ab} \bar{m}^a \bar{m}^b,$$
$$\Phi_{01} = -\frac{1}{2} R_{ab} \ell^a m^b, \quad \Phi_{10} = -\tfrac{1}{2} R_{ab} \ell^a \bar{m}^b,$$
$$\Phi_{12} = -\frac{1}{2} R_{ab} n^a m^b, \quad \Phi_{21} = -\tfrac{1}{2} R_{ab} n^a \bar{m}^b,$$

and

$$\Phi_{11} = -\frac{1}{4} \left(R_{ab} \ell^a n^b + R_{ab} m^a \bar{m}^b \right),$$
$$\Lambda = \frac{1}{12} \left(R_{ab} \ell^a n^b - R_{ab} m^a \bar{m}^b \right).$$

How do the Bianchi identities change?

- When considering this whole system, we now have to take into account also the Ricci part coming from the "quasi-matter". For example, one of the Bianchi identities that we considered

$$\left(\delta^* - 4\alpha + \pi\right)\Psi_0 + \left(D - 4\rho - 2\epsilon\right)\Psi_1 + 3\kappa\Psi_2 = 0.$$

becomes in this case

$$\begin{aligned}\left(\delta^* - 4\alpha + \pi\right)\Psi_0 + \left(D - 4\rho - 2\epsilon\right)\Psi_1 + 3\kappa\Psi_2 &\quad + \\ \left(D + 2\epsilon + 2\rho^*\right)\Phi_{01} - \left(\delta - \pi^* + 2\alpha^* + 2\beta\right)\Phi_{00} &\quad - \\ 2\sigma\Phi_{10} + 2\kappa\Phi_{11} - \kappa^*\Phi_{02} &\quad = \quad 0.\end{aligned}$$

- The equations thus become much more complicated. This complication would be solved in the case of real matter where we know that the matter part is compact and present only in a limited region of the space-time, therefore we can still use the vacuum equations outside this region.

Relating Ricci to Weyl scalars (I)

- As mentioned before the λ fields, within the symmetry of the system, is completely determined by the four dimensional metric, in particular by the component $g_{\phi\phi}$. It is therefore reasonable to believe that there must be a relation between the Ricci part coming from the λ field and the Weyl part which is connected also to $g_{\phi\phi}$.
  In order to obtain these relations we we proved the following identities

$$\begin{aligned} \rho - \sigma &= -D\mathcal{C}, \\ \mu - \lambda &= \Delta\mathcal{C}, \\ \alpha - \beta &= -(\delta + \delta^*)\,\mathcal{C}. \end{aligned}$$

where

$$\mathcal{C} = \frac{1}{2} \ln \lambda.$$

Relating Ricci to Weyl scalars (II)

- Let us consider the Ricci scalar Φ_{00}

$$\Phi_{00} = -\frac{1}{2}\ell^{\mu}\ell^{\nu}R_{\mu\nu} = \frac{\mathcal{D}-4}{2}\ell^{\mu}\ell^{\nu}\left[\nabla_{\mu}\nabla_{\nu}\mathcal{C} + \nabla_{\mu}\mathcal{C}\nabla_{\nu}\mathcal{C}\right].$$

Using integration by parts and the equations of the directional derivative of the spin coefficients we obtain

$$\Phi_{00} = \frac{\mathcal{D}-4}{2}\left[-D\left(\rho-\sigma\right) + 2\epsilon\left(\rho-\sigma\right) - 2\kappa\left(\alpha-\beta\right) + \left(\rho-\sigma\right)^2\right].$$

However, one Ricci identity tells us that

$$D\left(\rho-\sigma\right) = \left(\rho-\sigma\right)^2 + 2\epsilon\left(\rho-\sigma\right) - 2\kappa\left(\alpha-\beta\right) + \Phi_{00} - \Psi_0.$$

Combining the two equations we obtain

$$\Phi_{00} = \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_0.$$

Relating Ricci to Weyl scalars (III)

- Analogous calculations for the other Ricci scalars gives the following relations

$$
\begin{aligned}
\Phi_{00} &= \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_0, \\
\Phi_{01} &= \Phi_{10} = \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_1, \\
\Phi_{11} &= \left(\frac{\mathcal{D}-4}{2}\right)[\Psi_2 - \Phi_{02}] + (\mathcal{D}-1)\Lambda, \\
\Phi_{11} &= \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_2 - \left(\frac{\mathcal{D}-1}{\mathcal{D}-5}\right)\Lambda, \\
\Phi_{12} &= \Phi_{21} = \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_3, \\
\Phi_{22} &= \left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\Psi_4.
\end{aligned}
$$

Modifying the Bianchi identities (I)

- Let us consider, for example, one of the two identities used to calculate the Teukolsky equation

$$\begin{aligned}
\left(-\delta^* + 4\alpha - \pi\right)\Psi_0 + \left(D - 4\rho - 2\epsilon\right)\Psi_1 + 3\kappa\Psi_2 &\quad - \\
\left(D - 2\epsilon - 2\rho^*\right)\Phi_{01} + \left(\delta + \pi^* - 2\alpha^* - 2\beta\right)\Phi_{00} &\quad + \\
2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02} &\quad = \quad 0.
\end{aligned}$$

Using the properties we just found this can be reduced to

$$\left[\delta - 2\left(\mathcal{D} - 2\right)\alpha\right]\Psi_0 - \left[D - \mathcal{D}\rho\right]\Psi_1 - 3\kappa\Psi_2 = 0.$$

To be compared with the original four dimensional equation in vacuum

$$\left(\delta^* - 4\alpha\right)\Psi_0 - \left(D - 4\rho\right)\Psi_1 - 3\kappa\Psi_2 = 0.$$

Modifying the Bianchi identities (II)

- To summarize, we obtain the following set of two equations

$$
\begin{aligned}
-\left(\Delta+\mu-4\gamma\right)\Psi_0-\left(D-\rho\right)\Phi_{02} &\quad + \\
2\left(\frac{\mathcal{D}-3}{\mathcal{D}-2}\right)\left[\left(\delta-2\beta\right)\Psi_1+3\sigma\Psi_2\right]=0. & \\
\left[\delta-2\left(\mathcal{D}-2\right)\alpha\right]\Psi_0-\left[D-\mathcal{D}\rho\right]\Psi_1-3\kappa\Psi_2 &\quad = \quad 0.
\end{aligned}
$$

- Unfortunately we haven't managed to fully eliminate all the Ricci scalars, as we still obtain, in one of the equations, the presence of Φ_{02} which apparently cannot be eliminated. Φ_{02} goes to zero in the limit of single black hole so it can also be thought as a perturbative scalar.

Decoupling the equations

- However, we do have an independent differential equation for Φ_{02} that can be used for the calculation

$$
\begin{aligned}
-\left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)(\Delta-4\gamma)\,\Psi_0-(D-2\rho)\,\Phi_{02} &\quad + \\
2\left(\frac{\mathcal{D}-4}{\mathcal{D}-2}\right)\left[(\delta-2\beta)\,\Psi_1+3\sigma\Psi_2\right] &= 0.
\end{aligned}
$$

- It turns out that applying the Teukolsky procedure two times it is possible to decouple the equations and eliminate Φ_{02} as well as Ψ_1 and Ψ_2. The final result is

$$
\begin{aligned}
\left[D-(\mathcal{D}+1)\,\rho\right](\Delta+\mu-4\gamma)\,\Psi_0 &\quad - \\
(\delta-2\beta)\left[\delta-2\,(\mathcal{D}-2)\,\alpha\right]\Psi_0-3\Psi_2\Psi_0+\rho^2\Phi_{02} &= 0.
\end{aligned}
$$

- However it is possible to apply the operator $(D-(\mathcal{D}-2)\,\rho)$ once more to eliminate the Φ_{02} term.

Conclusions

- We analyzed a simplified picture for the Teukolsky equations in higher dimensions, namely axisymmetric perturbations of Tangherlini space-time.
- The key difference to the four dimensional problem is the appearance of a new perturbative scalar, namely Φ_{02}, which is a spin 0 field.
- We have shown that in this case it is possible to "almost" decouple the equations, that in fact decouple if we go at one higher order in derivation.
- Work is still progress to understand whether this result can be improved and whether it is possible to understand better the nature of this new field, relating to the other well known perturbative fields (Ψ_0 and Ψ_4).