Lie Groups and Lie Algebras in Particle Physics

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These notes are part of a 12 hour lecture course given at the University of Porto, Portugal, to Physics Masters students. The course is part of a larger curricular unit on Mathematical Methods in Physics, which includes a previous part on discrete groups and applications to Condensed Matter Physics. I will nevertheless try to make these notes as self-contained as possible, recalling where necessary some of the concepts and results that were taught in the previous part of the course. These notes are not meant as an extensive review of Lie groups and Lie algebras, and for those interested in learning more about the mathematical aspects of group theory and applications to particle physics I recommend the following textbooks:

- Howard Georgi, *Lie algebras in particle physics* (Westview Press, 1999).
- J. Fuchs and C. Schweigert, Symmetries, *Lie Algebras and Representations* (Cambridge University Press, 2003).

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Contents

1 General properties of Lie Groups and Lie Algebras

Let us start by recalling the mathematical properties of a **group**:

DEFINITION: A **group** G is a set with a map $G \times G \longrightarrow G$ known as group multiplication satisfying the following properties:

- Associativity: $(g.h)\,.l = g.\,(h.l), \quad \forall_{g,h,l\in G}$,
- Identity: $\exists_{e\in G} \quad e.g = g, \quad \forall_{g\in G}$,
- Inverse: $\forall_{g\in G}\ \exists_{g'\in G} \quad g'.g = g.g' = e$.

In particle physics we are mostly interested in representations of a group, which define the concrete realization of group transformations. A **group representation** R is a map that associates to each group element a linear transformation acting on a particular (real or complex) vector space, V:

$$R: \quad G \to GL(V)\ . \tag{1}$$

The dimension of the representation corresponds to the dimension of the associated vector space. A representation is **reducible** if and only if there is a subspace $U \subset V$ left invariant by group transformations, i.e. $R(g)u \in U$ for all $u \in U$ and $g \in G$. It is otherwise called **irreducible** and these are the representations that will mostly interest us in particle physics applications.

A **Lie Group** is a continuous group, i.e. in which all elements $g \in G$ depend continuously on a continuous set of parameters

$$g = g\left(\alpha\right), \quad \alpha = \left\{\alpha_a\right\},\ a = 1, ..., N\ . \tag{2}$$

We can assume, without loss of generality, that the identity element corresponds to the origin in the space of parameters:

$$e = g\left(\alpha\right)|_{\alpha=0}\ , \tag{3}$$

such that for any representation of the group:

$$R\left(\alpha\right)|_{\alpha=0} = \mathbb{I}\ . \tag{4}$$

In a neighbourhood of the identity element, we may then expand $R\left(\alpha\right)$ in a Taylor series:

$$R\left(d\alpha\right) = \mathbb{I} + iX_a d\alpha_a + \dots\ , \tag{5}$$

where we consider Einstein's summation rule that repeated indices should be summed over. In the above expression we have defined:

$$X_a \equiv -i\frac{\partial}{\partial \alpha_a} D\left(\alpha\right)|_{\alpha=0}\ , \tag{6}$$

and used the notation $d\alpha_a$ to denote an infinitesimal change in the $\{\alpha_a\}$ parameters, thus obtaining a representation of a group element arbitrarily close to the identity.

The $\{X_a\}$ vectors are known as **group generators**. We include a factor i in their definition such that, for unitary representations, $R^\dagger\left(\alpha\right)R\left(\alpha\right) = \mathbb{I}$, the group generators are hermitian operators $X_a^\dagger = X_a$ as we will see below.

Group multiplication then allows us to obtain any other finite element of the group by multiplying (5) by an infinitesimal transformation an arbitrary number of times:

$$R\left(\alpha\right)=\lim_{k\to\infty}\left(\mathbb{I}+i\frac{\alpha_a X_a}{k}\right)^k=\exp\left(i\alpha_a X_a\right)\ , \tag{7}$$

where $d\alpha_a=\lim_{k\to\infty}\frac{\alpha_a}{k}$. This defines the **exponential map** or **exponencial parametrization** of the group elements. We may thus write the group elements in terms of its generators, at least in a neighbourhood of the identity. For a unitary representation, $e^{i\alpha_a X_a}e^{-i\alpha_a X_a^\dagger}=\mathbb{I}$, which implies $X_a=X_a^\dagger$ as anticipated. As opposed to the group elements, the generators form a linear vector space and any linear combination of the generators is itself a group generator.

Let us consider a one-parameter family of group elements given by:

$$U\left(\lambda\right)=\exp\left(i\lambda\alpha_a X_a\right)\ . \tag{8}$$

In this case, group multiplication is quite simple and yields:

$$\begin{aligned} U\left(\lambda_1\right)U\left(\lambda_2\right) &= \exp\left(i\lambda_1\alpha_a X_a\right)\exp\left(i\lambda_2\alpha_a X_a\right)\\ &= \exp\left(i\left(\lambda_1+\lambda_2\right)\alpha_a X_a\right)\\ &= U\left(\lambda_1+\lambda_2\right). \end{aligned} \tag{9}$$

However, for group elements that are generated by different linear combinations of the generators this is not so simple, since in general:

$$\exp\left(i\alpha_a X_a\right)\exp\left(i\beta_b X_b\right)\neq\exp\left(i\left(\alpha_a+\beta_b\right)X_a\right)\ . \tag{10}$$

Since any group element admits a parametrization in terms of the exponential map, we must have that:

$$\exp\left(i\alpha_a X_a\right)\exp\left(i\beta_b X_b\right)=\exp\left(i\delta_a X_a\right), \tag{11}$$

for some set of parameters $\{\delta_a\}$, $a=1,...,N$. Continuity and differentiability of the group elements then allows us to find the $\{\delta_a\}$ parameters by expanding both sides of Eq. (11) in a Taylor series. We first note that:

$$i\delta_a X_a = \ln\left[1+\exp\left(i\alpha_a X_a\right)\exp\left(i\beta_b X_b\right)-1\right]\equiv\ln\left(1+K\right)\ , \tag{12}$$

and that, for small K,

$$\ln\left(1+K\right)=K-\frac{K^2}{2}+\ldots \tag{13}$$

Thus, expanding up to to quadratic order in the α_a and β_a parameters:

$$\begin{aligned} K &= \exp\left(i\alpha_a X_a\right)\exp\left(i\beta_b X_b\right)-1\\ &= \left(1+i\alpha_a X_a-\frac{1}{2}\left(\alpha_a X_a\right)^2+\ldots\right)\left(1+i\beta_b X_b-\frac{1}{2}\left(\beta_b X_b\right)^2+\ldots\right)-1\\ &= i\alpha_a X_a+i\beta_a X_a-\alpha_a X_a\beta_b X_b-\frac{1}{2}\left(\alpha_a X_a\right)^2-\frac{1}{2}\left(\beta_a X_a\right)^2+\ldots \end{aligned} \tag{14}$$

Hence, we have that:

$$\begin{aligned} i\delta_a X_a &= i\alpha_a X_a+i\beta_a X_a-\alpha_a X_a\beta_b X_b-\frac{1}{2}\left(\alpha_a X_a\right)^2-\frac{1}{2}\left(\beta_a X_a\right)^2+\frac{1}{2}\left(\alpha_a X_a+\beta_a X_a\right)^2+\ldots\\ &= i\left(\alpha_a+\beta_a\right)X_a-\alpha_a X_a\beta_b X_b+\frac{1}{2}\alpha_a X_a\beta_b X_b+\frac{1}{2}\beta_b X_b\alpha_a X_a+\ldots\\ &= i\left(\alpha_a+\beta_a\right)X_a-\frac{1}{2}\left[\alpha_a X_a,\beta_b X_b\right]+\ldots\ , \end{aligned} \tag{15}$$

where we have taken into account that the generators are linear operators that do not, in general, commute with each other. We thus find that:

$$[\alpha_a X_a, \beta_b X_b] = -2i\left(\delta_c - \alpha_c - \beta_c\right) X_c + \ldots \equiv i\gamma_c X_c + \ldots \tag{16}$$

Since this must hold for any choice of parameters, we conclude that:

$$\gamma_c = f_{abc}\alpha_a\beta_b \tag{17}$$

and that, hence,

$$[X_a, X_b] = i f_{abc} X_c \ , \tag{18}$$

where the constants satisfy:

$$f_{abc} = -f_{bac}, \tag{19}$$

since the commutator is antisymmetric, $[A, B] = -[B, A]$. These are known as the group's **structure constants** and define the **Lie algebra** of the the group G, $\mathcal{L}(G)$, i.e. the set of generators with the closed commutation properties above.

The commutator (18) defines the fundamental properties of the Lie algebra and thus plays a similar role to the group multiplication. We thus conclude that:

$$\delta_a = \alpha_a + \beta_a - \frac{1}{2}\gamma_a + \ldots \tag{20}$$

which implies:

$$\exp\left(i\alpha_a X_a\right)\exp\left(i\beta_b X_b\right) = \exp\left(i\left(\alpha_a + \beta_a\right)X_a - \frac{1}{2}\left[\alpha_a X_a, \beta_b X_b\right] + \ldots\right) \ , \tag{21}$$

which is known as the **Baker-Campbell-Hausdorff (BCH) relation**. The higher-order terms that we have discarded above correspond to commutators of commutators, e.g. $[\alpha_a X_a, [\alpha_c X_c, \beta_b X_b]]$ and are thus determined by the structure constants. Hence, the Lie algebra (18) is sufficient to completely define the group multiplication in a finite neighbourhood of the identity.

The structure constants are an intrinsic property of the Lie algebra and are independent of its representation, being determined solely by the group multiplication rule and by continuity. Each representation of the group then defines a representation of the associated Lie algebra. A useful property to note is that the structure constants are real if there is a unitary group representation.

DEM: Since in a unitary group representation the generators are hermitian operators we have, on the one hand, that:

$$[X_a, X_b]^\dagger = -i f^*_{abc} X^\dagger_c = -i f^*_{abc} X_c \ ,$$

and, on the other hand, that:

$$\begin{aligned} [X_a, X_b]^\dagger &= (X_a X_b)^\dagger - (X_a X_b)^\dagger \\ &= X_b X_a - X_a X_b \\ &= -[X_a, X_b] \\ &= -i f_{abc} X_c. \end{aligned}$$

so that $f_{abc} = f^*_{abc}$. *Q.E.D.*

The group generators satisfy the **Jacobi identity**:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0 \ . \tag{22}$$

This can be shown in a straightforward way by expanding all the commutators, so we leave it as an exercise.

The structure constants themselves can be used to define an important representation of the Lie algebra known as the **adjoint representation**. This can be done by defining the $N \times N$ matrices

$$[T_a]_{bc} \equiv -if_{abc} \ , \tag{23}$$

the commutator of which is given by:

$$\begin{aligned} [T_a, T_b]_{cd} &= (T_a T_b - T_b T_a)_{cd} \\ &= (T_a)_{ce} (T_b)_{ed} - (T_b)_{ce} (T_a)_{ed} \\ &= -f_{ace} f_{bed} + f_{bce} f_{acd} \ . \end{aligned} \tag{24}$$

From the Jacobi identity we have that:

$$\begin{aligned} [X_a, [X_b, X_c]] &= [X_a, if_{bcd} X_d] \\ &= if_{bcd} [X_a, X_d] \\ &= -f_{bcd} f_{ade} X_e \ , \end{aligned} \tag{25}$$

and so

$$\begin{aligned} (f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde}) X_e &= 0 \quad \forall_{X_e \in \mathcal{L}(G)} \\ \Rightarrow \quad f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} &= 0 \ . \end{aligned} \tag{26}$$

If we now interchange the indices d and e, we obtain:

$$-f_{ace} f_{bed} + f_{bce} f_{aed} = -f_{abe} f_{ced} \ , \tag{27}$$

from which we conclude that:

$$\begin{aligned} [T_a, T_b]_{cd} &= -f_{abe} f_{ced} \\ &= f_{abe} f_{ecd} \\ &= if_{abe} (T_e)_{cd} \ , \end{aligned}$$

which can be written as:

$$[T_a, T_b] = if_{abc} T_c \ , \tag{28}$$

such that the T_a matrices satisfy the commutation relation of the Lie algebra in Eq. (18). The dimension of the adjoint representation corresponds to the number of independent generators, i.e. to the number of (real) parameters required to specify a group element. Note that the generators are pure imaginary matrices in the adjoint representation if the structure constants are real.

A more formal way of defining the adjoint representation is given by the map:

$$\begin{aligned} \mathrm{ad}: \qquad & \mathcal{L}(G) \to \mathcal{M}(\mathcal{L}(G)) \\ & \mathrm{ad}(X)(Y) = [X, Y] \ , \quad X, Y \in \mathcal{L}(G). \end{aligned} \tag{29}$$

We may write this in components by choosing a basis of generators for the Lie algebra $\{T_a\}$, $a = 1, ..., N$:

$$\begin{aligned} & \mathrm{ad}\,(T_a)\,(T_b) = [T_a, T_b] = i f_{abc} T_c \\ \Rightarrow \quad & \mathrm{ad}\,(T_a)_{cb} = i f_{abc} = -\,[T_a]_{bc} = [T_a]_{cb} \ , \end{aligned}$$

in agreement with the definition above. Note that we have used that $f_{abc} = -f_{acb}$ as we will show explicitly below.

The adjoint representation of the Lie algebra naturally induces a representation of the group, also known as the *adjoint representation of the group*:

$$\begin{aligned} \mathrm{Ad}: \qquad & G \to GL\,(\mathcal{L}\,(G)) \\ & \mathrm{Ad}\,(g)\,T_a = g T_a g^{-1} \ , \quad g \in G, T_a \in \mathcal{L}(G) \ . \end{aligned} \tag{30}$$

We can easily check that these two representations are related through the exponential map. Writing a group element as $g = \exp\,(i\alpha_a T_a)$, we have that:

$$\begin{aligned} \mathrm{Ad}\,(g)\,(T_a) &= \exp\,(i\alpha_b T_b)\,T_a \exp\,(-i\alpha_b T_b) \\ &= (1 + i\alpha_b T_b + \ldots)\,T_a\,(1 - i\alpha_b T_b + \ldots) \\ &= T_a + i\alpha_b\,[T_b, T_a] + \ldots \\ &= T_a + i\alpha_b \mathrm{ad}\,(T_b)\,(T_a) + \ldots \\ &= \exp\,(i\alpha_b \mathrm{ad}\,(T_b)\,(T_a)) \ . \end{aligned} \tag{31}$$

Let us now introduce a series of definitions and theorems that can be used to classify different Lie algebras.

DEFINITION: A **sub-algebra** $\mathcal{A} \subset \mathcal{L}\,(G)$ is a linear space such that:

$$\forall_{X,Y \in A} \quad [X, Y] \in \mathcal{A} \ . \tag{32}$$

We can highlight the following special cases of sub-algebras:

- A sub-algebra is **abelian** if:
$$\forall_{X,Y \in A}, \quad [X, Y] = 0 \ , \tag{33}$$
and the same applies naturally to the full Lie algebra $\mathcal{L}(G)$.
- A sub-algebra is an **ideal** if:
$$\forall_{X \in A} \forall_{Z \in \mathcal{L}(G)} \quad [X, Z] \in \mathcal{A} \ , \tag{34}$$
and a **proper ideal** if, in addition, $\mathcal{A} \neq \mathcal{L}\,(G)\,, \{0\}$.

DEFINITION: A Lie algebra is **simple** if it does not contain any proper ideals and **semi-simple** if it does not contain abelian ideals except $\{0\}$.

THEOREM: A Lie algebra is **semi-simple** if and only if:

$$\mathcal{L} = \mathcal{L}_1 \oplus ... \oplus \mathcal{L}_N \tag{35}$$

where $\mathcal{L}_i$, $i = 1, ..., N$, are simple algebras.

A rigorous proof of this theorem is out of the scope of these lectures, but let us consider a generic example that illustrates these general definitions and properties. Consider the product of two groups $G = G_1 \times G_2$, where the Lie algebras $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ are simple, i.e. have no proper ideals as defined above. We can write a generic element of G in the matrix form:

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in G \ ,$$

for $g_i \in G_i$, $i = 1, 2$, such that the elements of the Lie algebra, i.e. the generators, can be written as:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \quad \in \mathcal{L}\left(G\right) \ .$$

Let us then consider the sub-algebra $\mathcal{L}(G_1)$ with elements of the form:

$$T' = \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} \ .$$

Then, we have for the commutator between a generic element of $\mathcal{L}(G)$ and an element of this sub-algebra:

$$\begin{aligned} [T, T'] &= \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \\ &= \begin{pmatrix} T_1 T_1' & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T_1' T_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T_1'' & 0 \\ 0 & 0 \end{pmatrix} \ , \end{aligned}$$

where $T_1'' = [T_1, T_1'] \in \mathcal{L}\left(G_1\right)$. Hence, the elements of the sub-algebra $\mathcal{L}(G_1)$ constitute a proper ideal and $\mathcal{L}(G)$ cannot be simple. It is easy to see, by a similar reasoning, that the elements of $\mathcal{L}(G_2)$ also form a proper ideal, and that there are no other ideals since $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ are simple algebras. Thus, the full Lie algebra $\mathcal{L}(G) = \mathcal{L}(G_1) \oplus \mathcal{L}(G_2)$ will be semi-simple if these ideals are not abelian.

DEFINITION: The **Killing form** or **Killing metric** associated with a Lie algebra $\mathcal{L}\left(G\right)$ is a symmetric bilinear map:

$$\Gamma : \quad \mathcal{L}\left(G\right) \times \mathcal{L}\left(G\right) \quad \rightarrow \quad \mathbb{R}$$

defined by:

$$\Gamma\left(X, Y\right) = \mathrm{Tr}\left[\mathrm{ad}\left(X\right) \cdot \mathrm{ad}\left(Y\right)\right] , \quad X, Y \in \mathcal{L}\left(G\right) \ . \tag{36}$$

This defines an inner product within the Lie algebra in the adjoint representation. Considering a basis of generators

$\{T_a\}$ for the Lie algebra we obtain for the Killing metric components:

$$\begin{aligned}\gamma_{ab} &= \Gamma\left(T_a, T_b\right) = \mathrm{Tr}\left[T_a T_b\right] = \left(T_a\right)_{cd}\left(T_b\right)_{dc} = \left(-i f_{acd}\right)\left(-i f_{bdc}\right) \\ &= -f_{acd} f_{bdc}\ . \end{aligned} \tag{37}$$

<u>THEOREM</u>: A Lie algebra is semi-simple if the Killing form is non-degenerate:

$$\Gamma\left(X, Y\right) = 0 \quad \forall_{X \in \mathcal{L}(G)} \qquad \Rightarrow \qquad Y = 0\ ,$$

or equivalently that $\det(\gamma) \neq 0$.

DEM: Let us suppose that Γ is non-degenerate and that $\mathcal{A} \subset \mathcal{L}\left(G\right)$ is an abelian ideal. We can then choose a basis $\{T_a, T_\alpha\}$ for the Lie algebra where T_a generate the elements in $\mathcal{A}$ and T_α correspond to the remaining generators of $\mathcal{L}\left(G\right)$. Then, for $X \in \mathcal{L}\left(G\right)$ and $Y \in \mathcal{A}$ we have:

$$\mathrm{ad}\left(X\right) \cdot \mathrm{ad}\left(Y\right)\left(T_a\right) = \left[X, \left[Y, T_a\right]\right] = 0\ ,$$

since $\left[Y, T_a\right] = 0$. Also:

$$\mathrm{ad}\left(X\right) \cdot \mathrm{ad}\left(Y\right)\left(T_\alpha\right) = \left[X, \left[Y, T_\alpha\right]\right] = \sum_a \alpha_a T_a\ , \tag{38}$$

since $\left[Y, T_\alpha\right] \in \mathcal{A}$ and so $\left[X, \left[Y, T_\alpha\right]\right] \in \mathcal{A}$ as well. We thus conclude that the Killing metric has the form:

$$\begin{aligned}\Gamma\left(X, Y\right) &= \mathrm{Tr}\left[\mathrm{ad}\left(X\right) \cdot \mathrm{ad}\left(Y\right)\right] \\ &= \mathrm{Tr}\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \\ &= 0\ . \end{aligned}$$

Since by assumption Γ is non-degenerate, we must have $Y = 0$, and so there cannot exist any non-trivial abelian ideals and the algebra is semi-simple. The proof in the opposite direction follow an analogous reasoning. *Q.E.D.*

<u>THEOREM</u>: If $\mathcal{L}\left(G\right)$ is a compact Lie algebra, i.e. if the underlying Lie group is compact, being defined in term of a compact manifold of parameters, then the Killing form is positive semi-definite:

$$\Gamma\left(X, X\right) \geq 0, \quad \forall_{X \in \mathcal{L}(G)}\ , \tag{39}$$

and if the algebra is simple, we have:

$$\Gamma\left(X, X\right) > 0, \quad \forall_{X \in \mathcal{L}(G)}\ . \tag{40}$$

Although we will not prove this theorem in this course, we can use it to show an important result that we have anticipated above. If the Killing form is positive definite, then we can choose an orthonormal basis for the Lie algebra such that:

$$\Gamma\left(T_a, T_b\right) = \delta_{ab}\ , \tag{41}$$

which just means that we may diagonalize the Killing metric and choose an appropriate normalization for the generators.

In this basis, the structure constants are **completely antisymmetric.**

DEM:

$$\begin{aligned}\Gamma\left(\left[T_c,T_a\right],T_b\right)+\Gamma\left(T_a,\left[T_c,T_b\right]\right) &= \mathrm{Tr}\left[\left(T_cT_a-T_aT_c\right)T_b\right]+\mathrm{Tr}\left[T_a\left(T_cT_b-T_bT_c\right)\right]\\ &= \mathrm{Tr}\left[T_cT_aT_b\right]-\mathrm{Tr}\left[T_aT_cT_b\right]+\mathrm{Tr}\left[T_aT_cT_b\right]-\mathrm{Tr}\left[T_aT_bT_c\right]\\ &= 0,\end{aligned}$$

using the cyclic property of the trace. Thus,

$$\Gamma\left(if_{cad}T_d,T_b\right)+\Gamma\left(T_a,if_{cbd}T_d\right)=if_{cad}\mathrm{Tr}\left[T_dT_b\right]+if_{cbd}\mathrm{Tr}\left[T_aT_d\right]=if_{cad}\delta_{db}+if_{cbd}\delta_{ad}=f_{cab}+f_{cba}=0\ , \tag{42}$$

so that $f_{cab}=-f_{cba}$. Since by the definition the structure constants are antisymmetric in the first two indices, this implies that they must be completely antisymmetric. *Q.E.D.*

THEOREM: For a semi-simple Lie algebra the (quadratic) **Casimir operator**:

$$C\equiv\gamma^{ij}T_iT_j, \tag{43}$$

where $\gamma^{ij}=\gamma_{ij}^{-1}$, commutes with all the generators:

$$\left[C,T_k\right]=0\quad\forall_{T_k\in\mathcal{L}(G)}\ . \tag{44}$$

DEM:

$$\begin{aligned}\left[C,T_l\right] &= \gamma^{ij}\left[T_iT_j,T_l\right]\\ &= \gamma^{ij}T_i\left[T_j,T_l\right]+\gamma^{ij}\left[T_i,T_l\right]T_j\\ &= i\gamma^{ij}T_if_{jlm}T_m+i\gamma^{ij}f_{ilm}T_mT_j\\ &= i\gamma^{ij}f_{jlm}T_iT_m+i\gamma^{ij}f_{jlm}T_mT_i\\ &= i\gamma^{ij}f_{jlm}\left(T_iT_m+T_mT_i\right)\\ &= if_{jlm}\gamma^{ij}\left(T_iT_m+T_mT_i\right)\\ &= 0\ ,\end{aligned}$$

since f_{jlm} is antisymmetric while $\gamma^{ij}\left(T_iT_m+T_mT_i\right)$ is symmetric under the interchange of the indices j and m, noting that the Killing metric is symmetric. *Q.E.D.*

An important result in the theory of group representations is Schur's Lemma, which can be cast in the following form:

SCHUR'S LEMMA: For an irreducible representation R of a group G over a complex vector space V, if there exists a linear transformation $P\in GL(V)$ that commutes with the action of all group elements:

$$\left[P,R(g)\right]=0\quad\forall_{g\in G}\ , \tag{45}$$

then this operator must be proportional to the identity, $P=\lambda\mathbb{I}$ for some complex constant λ.

DEM: Consider the space of eigenvectors of the operator P with eigenvalue λ:

$$\mathrm{Eig}(\lambda) = \{v \in V : \; Pv = \lambda v\} \; . \tag{46}$$

This must be a non-empty set for some λ since $\det(P - \lambda \mathbb{I}) = 0$ has at least one solution in $\mathbb{C}$. Then, for $v \in \mathrm{Eig}(\lambda)$ we have that:

$$P\left(R(g)v\right) = R(g)(Pv) = \lambda R(g)v \; , \tag{47}$$

which implies that $R(g)v \in \mathrm{Eig}(\lambda)$, i.e. that the eigenspace is left invariant by the action of the group. Since, by assumption, the representation is irreducible, than $\mathrm{Eig}(\lambda) = V$, which implies that $P = \lambda \mathbb{I}$. *Q.E.D.*

It is easy to see that Schur's Lemma applies to the irreducible representations of both a Lie group and its Lie algebra. This then leads us to the conclusion that, in any irreducible representation r of $\mathcal{L}(G)$, the Casimir operator is proportional to the identity:

$$C = C\left(r\right) \mathbb{I}_{\dim(r)} \; , \tag{48}$$

where $C\left(r\right)$ is a number that is characteristic of the representation. In the orthonormal basis where $\gamma_{ij} = \delta_{ij}$ (for a simple and compact Lie algebra), we have:

$$C = \delta^{ij} T_i^{(r)} T_j^{(r)} = \sum_i \left(T_i^{(r)}\right)^2 = C\left(r\right) \mathbb{I}_{\dim(r)} \; . \tag{49}$$

We may also consider the quadratic operator:

$$M_{ij} = \mathrm{Tr}\left[T_i^{(r)}, T_j^{(r)}\right] \; , \tag{50}$$

for the Lie algebra generators in a given representation r. The commutator of this operator with a generator in the adjoint representation is then given by:

$$\begin{aligned} \left([T_i, M]\right)_{jk} &= \left(T_i\right)_{jl} M_{lk} - M_{jl} \left(T_i\right)_{lk} \\ &= -i f_{ijl} \mathrm{Tr}\left[T_l^{(r)}, T_k^{(r)}\right] + i f_{ilk} \mathrm{Tr}\left[T_j^{(r)}, T_l^{(r)}\right] \\ &= -\mathrm{Tr}\left[\left[T_i^{(r)}, T_j^{(r)}\right], T_k^{(r)}\right] - \mathrm{Tr}\left[T_j^{(r)}\left[T_i^{(r)}, T_k^{(r)}\right]\right] \\ &= -\mathrm{Tr}\left[T_i^{(r)} T_j^{(r)} T_k^{(r)} - T_j^{(r)} T_i^{(r)} T_k^{(r)} + T_j^{(r)} T_i^{(r)} T_k^{(r)} - T_j^{(r)} T_k^{(r)} T_i^{(r)}\right] \\ &= 0 \; , \end{aligned} \tag{51}$$

using the cyclic property of the trace. Schur's Lemma than implies that

$$M = \widetilde{C}\left(r\right) \mathbb{I}_{\dim(\mathrm{ad})} \; . \tag{52}$$

The two Casimir constants $C\left(r\right)$ and $\widetilde{C}\left(r\right)$ are thus related via:

$$\begin{aligned} \mathrm{Tr}_{(r)}\left(C\right) &= C\left(r\right) \dim\left(r\right) \\ &= \sum_i \mathrm{Tr}\left(T_i^{(r)}\right)^2 \\ &= \mathrm{Tr}_{(r)}\left(M\right) \\ &= \widetilde{C}\left(r\right) \dim\left(\mathrm{ad}\right) \; , \end{aligned} \tag{53}$$

such that

$$\widetilde{C}\left(r\right)=\frac{\dim\left(r\right)}{\dim\left(\text{ad}\right)}C\left(r\right)\ . \tag{54}$$

We note, as an aside, that the Casimir operators play a very important role in particle physics, since they characterize the irreducible representations in which each type of field transforms under symmetry operations associated with particular Lie groups, as we will discuss in more detail later on.

1.1 Example: the Lie group $SU\left(2\right)$

To better understand the general concepts and definitions introduced in this section, let us look in detail into a particular example, that of the group of 2×2 unitary matrices with unit determinant, denoted as $SU(2)$. This will also serve as a warm up exercise to the more detailed study of $SU(N)$ representations and applications to particle physics that we will do later on. The group is formally defined as

$$SU(2)=\left\{U\in GL\left(\mathbb{C}^2\right):\ U^\dagger U=\mathbb{I},\ \det\left(U\right)=1\right\}\ .$$

For a generic matrix in this group:

$$U=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\qquad\Rightarrow\qquad U^\dagger=\begin{pmatrix}a^* & b^*\\ c^* & d^*\end{pmatrix}\ .$$

The unitarity condition yields for the matrix components:

$$UU^\dagger\ =\ \begin{pmatrix}|a|^2+|b|^2 & ac^*+bd^*\\ ca^*+db^* & |c|^2+|d|^2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\qquad\Rightarrow\qquad\begin{cases}ac^*+bd^*= & 0\\ a^*c+b^*d= & 0\end{cases}\qquad\Leftrightarrow\qquad\begin{cases}a= & d^*\\ b= & -c^*\end{cases}$$

This implies that a generic 2×2 unitary matrix can be written in the form:

$$U=\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}\ ,$$

and the unit determinant condition yields:

$$\det\left(U\right)=|\alpha|^2+|\beta|^2=1\ .$$

The $SU(2)$ group can thus be defined by:

$$SU(2)=\left\{\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}:\quad \alpha,\beta\in\mathbb{C}\ ,\quad |\alpha|^2+|\beta|^2=1\right\}\ ,$$

such that each element of the group is specified by two complex numbers subject to a single real condition, thus having three independent degrees of freedom.

To determine the associated Lie algebra, we consider infinitesimal group transformations (arbitrarily close to the identity), $U=\mathbb{I}+iT+\ldots$ to obtain that:

$$UU^\dagger=\left(\mathbb{I}+iT+....\right)\left(\mathbb{I}-iT^\dagger+....\right)=\mathbb{I}\qquad\Rightarrow\qquad T=T^\dagger\ .$$

and also that:

$$\det(U) = \det(\mathbb{I} + iT + ...) = 1 + i\mathrm{Tr}(T) = 1 \qquad \Rightarrow \qquad \mathrm{Tr}(T) = 0\ .$$

Recall that for a generic matrix with eigenvalues λ_i, $\det(A) = \prod_i \lambda_i$ and $\mathrm{Tr}(A) = \sum_i \lambda_i$. If $A \simeq \mathbb{I} + X + \ldots$ then $\lambda_i = 1 + \epsilon_i + \ldots$, where ϵ_i are the arbitrarily small eigenvalues of X, such that $\det(A) = \prod_i (1 + \epsilon_i) = 1 + \sum_i \epsilon_i + \ldots = 1 + \mathrm{Tr}(X)$, which we used above.

These results then imply that the Lie algebra of $SU(2)$ is given by:

$$\mathcal{L}(SU(2)) = \left\{T \in M\left(\mathbb{C}^2\right) : T = T^\dagger,\ \mathrm{Tr}(T) = 0\right\}\ . \tag{55}$$

A generic matrix in the $SU(2)$ Lie algebra may then be written in the form:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\ ,$$

which, along with the condition $\mathrm{Tr}(T) = a + d = 0$, leads us to the conclusion that the diagonal entries must be real with $a = -d$, while the non-diagonal entries are complex conjugate, leaving only three real degrees of freedom, in agreement to what we found above for the group elements.

The basis of matrices for the $SU(2)$ algebra is conventionally chosen to be given in terms of the three Pauli matrices, $T_i = \sigma_i/2$, $i = 1, 2, 3$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{56}$$

It is easy to check that these matrices satisfy:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\ , \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\ , \qquad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k\ , \tag{57}$$

where the anti-commutator $\{A, B\} = AB + BA$. This thus implies that

$$[T_i, T_j] = i\epsilon_{ijk}T_k\ , \tag{58}$$

such that the $SU(2)$ structure constants are $f_{ijk} = \epsilon_{ijk}$ in terms of the totally antisymmetric Levi-Civita tensor. In this basis we also have that

$$\mathrm{Tr}(T_iT_j) = \frac{1}{4}\mathrm{Tr}(\sigma_i\sigma_j) = \frac{1}{4}\left[\delta_{ij}\mathrm{Tr}(\mathbb{I}) + i\epsilon_{ijk}\mathrm{Tr}(\sigma_k)\right] = \frac{1}{2}\delta_{ij}\ .$$

We note that it is possible to choose the normalization of the generators such that $\mathrm{Tr}(T_iT_j) = \delta_{ij}$ but the above is the most conventionally used normalization. The Killing metric for the $SU(2)$ algebra is given by:

$$\gamma_{ij} = -\epsilon_{ikl}\epsilon_{jlk} = 2\delta_{ij}\ ,$$

as can be checked explicitly for the different components:

$$\begin{aligned}
\gamma_{11} &= -\epsilon_{1kl}\epsilon_{1lk} = -\left(\epsilon_{123}\epsilon_{132} + \epsilon_{132}\epsilon_{123}\right) = 2 \\
\gamma_{22} &= -\epsilon_{2kl}\epsilon_{2lk} = -\left(\epsilon_{213}\epsilon_{231} + \epsilon_{231}\epsilon_{213}\right) = 2 \\
\gamma_{33} &= -\epsilon_{3kl}\epsilon_{3lk} = -\left(\epsilon_{312}\epsilon_{321} + \epsilon_{321}\epsilon_{312}\right) = 2 \\
\gamma_{12} &= -\epsilon_{1kl}\epsilon_{2lk} = 0 \\
\gamma_{13} &= -\epsilon_{1kl}\epsilon_{3lk} = 0 \\
\gamma_{23} &= -\epsilon_{2kl}\epsilon_{3lk} = 0 \ .
\end{aligned}$$

This yields $\det(\gamma) = 8 \neq 0$, so that the algebra is semi-simple. In fact, the $SU(2)$ algebra is simple since there are no proper ideals, as can be inferred from the commutation relations satisfied by the Pauli matrices.

The Pauli matrices define the **fundamental representation** of the $SU(2)$ algebra, also known as the spin-1/2 representation. This representation acts on 2-dimensional vectors known as SU(2) doublets.

The adjoint representation (or spin-1 representation) is given by:

$$\left(T_i^{(\mathrm{ad})}\right)_{jk} = -if_{ijk} = -i\epsilon_{ijk} \ ,$$

with generators:

$$T_1^{(\mathrm{ad})} = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \ , \qquad T_2^{(\mathrm{ad})} = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \qquad T_3^{(\mathrm{ad})} = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{59}$$

Another important representation is the **complex conjugate representation**:

$$R^*(U) = U^* = e^{T^*} \ ,$$

where $T = i\sum_j \alpha_j\sigma_j$. We can use that $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$, as can be explicitly checked:

$$\begin{aligned}
\sigma_2\sigma_1\sigma_2 &= \sigma_2\left(i\epsilon_{123}\sigma_3\right) = i\epsilon_{123}\left(i\epsilon_{231}\right)\sigma_1 = -\sigma_1 \ , \\
\sigma_2\sigma_2\sigma_2 &= \sigma_2 \ , \\
\sigma_2\sigma_3\sigma_2 &= \sigma_2\left(i\epsilon_{321}\sigma_1\right) = i\epsilon_{321}\left(i\epsilon_{213}\right)\sigma_3 = -\sigma_3 \ ,
\end{aligned}$$

to show that:

$$\begin{aligned}
\sigma_2 T\sigma_2 &= i\left(-\alpha_1\sigma_1 + \alpha_2\sigma_2 - \alpha_3\sigma_3\right) \\
&= -i\left(\alpha_1\sigma_1^* + \alpha_2\sigma_2^* + \alpha_3\sigma_3^*\right) \\
&= T^* \ .
\end{aligned}$$

This then implies that:

$$R^*(U) = e^{\sigma_2 T\sigma_2} = e^{\sigma_2 T\sigma_2^{-1}} \ .$$

Now note that, for generic matrices A and B:

$$e^{ABA^{-1}} = \sum_n \frac{\left(ABA^{-1}\right)^n}{n!} \ ,$$

and that

$$\left(ABA^{-1}\right)^n = \left(ABA^{-1}\right)\left(ABA^{-1}\right)...\left(ABA^{-1}\right) = AB^nA^{-1} \ ,$$

so that we have:

$$e^{ABA^{-1}} = A\left(\sum_n \frac{B^n}{n!}\right)A^{-1} = Ae^BA^{-1} \ .$$

Using this result, we may write the complex conjugate representation in the form:

$$R^*\left(U\right) \quad = \quad \sigma_2 e^T \sigma_2^{-1} = \sigma_2 R(U)\sigma_2^{-1} \ .$$

Hence, the fundamental and complex conjugate representations are not really independent and, in fact, are said to be **equivalent**. The fundamental representation is then said to be **pseudo-real**.

1.2 Cartan-Weyl basis

To define the Cartan-Weyl basis, which is extremely useful in determining and classifying representations of Lie groups and associated algebras, we will consider semi-simple Lie algebras with generators $\{T_i\}, i = 1, ..., N$. Let us take a linear combination of the generators:

$$H = a^i T_i \quad \in \quad \mathcal{L} \ , \tag{60}$$

where we use the Killing metric γ_{ij} and its inverse γ^{ij} to lower and raise indices, such that the Einstein summation convention requires repeated upper and lower indices to be summed over. Note that in the previous calculations we worked with lower indices exclusively, which is consistent in the Lie algebra basis where the Killing metric is the identity. However, as we will see in the Cartan-Weyl basis this does not hold and we must use this more correct version of the summation convention.

With the linear combination of generators above, we can study the eigenvalue problem:

$$\mathrm{ad}\left(H\right)\left(T\right) = \left[H, T\right] = \rho T \ . \tag{61}$$

This can be written in components on the given basis as:

$$\begin{aligned} \left[a^i T_i, T_j\right] = \rho T_j \quad &\Leftrightarrow \quad a^i {f_{ij}}^k T_k = \rho T_j \\ &\Leftrightarrow \quad \left(a^i {f_{ij}}^k - \rho {\delta_j}^k\right) T_k = 0 \\ &\Rightarrow \quad \det\left(a^i {f_{ij}}^k - \rho {\delta_j}^k\right) = 0 \ , \end{aligned}$$

where the structure constants are now defined as:

$$\left[T_i, T_j\right] = {f_{ij}}^k T_k \ .$$

CARTAN'S THEOREM

Choosing the linear combination H with the largest possible number of eigenvalues ρ:

1. The eigenvalue $\rho = 0$ can be degenerate with multiplicity $l = \text{rank}(\mathcal{L})$ and the corresponding eigenspace is generated by $\{H_i\}$, $i = 1, \ldots, l$, such that
$$[H_i, H_j] = 0 \ . \tag{62}$$
2. All the non-vanishing eigenvalues $\rho \neq 0$ are non-degenerate.

The proof of Cartan's Theorem is out of the scope of this course, but we can study its implications. First, we can immediately conclude that:

$$\begin{aligned} [H, H_i] &= 0 \ , \\ [H, E_\alpha] &= \alpha E_\alpha, \qquad \alpha \neq 0 \ . \end{aligned} \tag{63}$$

Now, since $[H, H] = 0$, we must have $H = \lambda^i H_i$. In addition, the Jacobi identity implies that:

$$\begin{aligned} [H, [H_i, E_\alpha]] &= -[H_i, [E_\alpha, H]] - [E_\alpha, [H, H_i]] \\ &= \alpha [H_i, E_\alpha] \ , \end{aligned}$$

which means that $[H_i, E_\alpha]$ is an eigenvector of H with eigenvalue $\alpha \neq 0$ which, by Cartan's Theorem, is non-degenerate. Hence:

$$[H_i, E_\alpha] = \alpha_i E_\alpha \ . \tag{64}$$

Therefore, we have that:

$$[H, E_\alpha] = \lambda^i [H_i, E_\alpha] = \lambda^i \alpha_i E_\alpha = \alpha E_\alpha \tag{65}$$

from which we infer the relation $\alpha = \lambda^i \alpha_i$. We may then write the structure constants in the form:

$$[H_i, E_\alpha] = f_{i\alpha}{}^\beta E_\beta \quad \Rightarrow \quad f_{i\alpha}{}^\beta = \alpha_i \delta_\alpha{}^\beta \ . \tag{66}$$

We may further consider the Jacobi identity for the generators H, E_α and E_β:

$$\begin{aligned} [H, [E_\alpha, E_\beta]] + [E_\beta, [H, E_\alpha]] + [E_\alpha, [E_\beta, H]] = 0 \\ [H, [E_\alpha, E_\beta]] + \alpha [E_\beta, E_\alpha] - \beta [E_\alpha, E_\beta] = 0 \\ [H, [E_\alpha, E_\beta]] = (\alpha + \beta) [E_\alpha, E_\beta] \ , \end{aligned} \tag{67}$$

such that $[E_\alpha, E_\beta]$ is an eigenvector of H with eigenvalue $\alpha + \beta$. This leads to two possible commutation relations:

$$\begin{cases} [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \ , & \alpha + \beta \neq 0 \\ [E_\alpha, E_{-\alpha}] = f_{\alpha-\alpha}{}^i H_i \ , & \alpha + \beta = 0 \end{cases} . \tag{68}$$

This yields the structure constants:

$$f_{\alpha\beta}{}^\gamma = N_{\alpha\beta} \delta_{\alpha+\beta}{}^\gamma. \tag{69}$$

Having found the form of the structure constants in the basis $\{H_i, E_\alpha\}$, we may compute the components of the Killing metric in this basis:

$$\begin{aligned}
\gamma_{i\alpha} &= -f_{i\beta}{}^{\gamma} f_{\alpha\gamma}{}^{\beta} \\
&= -\alpha_i \delta_{\beta}{}^{\gamma} N_{\alpha\gamma} \delta_{\alpha+\gamma}{}^{\beta} \\
&= -\alpha_i N_{\alpha\beta} \delta_{\alpha+\beta}{}^{\beta} \\
&= 0 \ ,
\end{aligned} \tag{70}$$

since $\alpha \neq 0$, or also since $N_{\alpha\beta}$ is anti-symmetric and $\delta_{\alpha+\beta}{}^{\beta}$ is symmetric under the exchange of the indices α and β. Similarly, we obtain:

$$\begin{aligned}
\gamma_{\alpha\beta} &= -f_{\alpha\gamma}{}^{\mu} f_{\beta\mu}{}^{\gamma} \\
&= -N_{\alpha\gamma} \delta_{\alpha+\gamma}{}^{\mu} N_{\beta\mu} \delta_{\beta+\mu}{}^{\gamma} \\
&= -N_{\alpha\gamma} N_{\beta,\alpha+\gamma} \delta_{\beta+\alpha+\gamma}{}^{\gamma} \ .
\end{aligned} \tag{71}$$

This implies that $\gamma_{\alpha\beta} \neq 0$ if and only if $\alpha + \beta = 0$. We may then choose to normalize the E_α generators such $\gamma_{\alpha\,-\alpha} = 1$.

Finally, we have:

$$\begin{aligned}
\gamma_{ij} &= -f_{i\alpha}{}^{\beta} f_{j\beta}{}^{\alpha} \\
&= -\alpha_i \delta_{\alpha}{}^{\beta} \alpha_j \delta_{\beta}{}^{\alpha} \\
&= -\alpha_i \alpha_j \delta_{\alpha}{}^{\alpha} \\
&= -\sum_{\alpha} \alpha_i \alpha_j \ .
\end{aligned} \tag{72}$$

This result allows to derive the remaining structure constants:

$$\begin{aligned}
f_{\alpha\,-\alpha}{}^{i} &= \gamma^{ij} f_{\alpha\,-\alpha j} \\
&= \gamma^{ij} f_{j\alpha\,-\alpha} \\
&= \gamma^{ij} f_{j\alpha}{}^{\beta} \gamma_{\beta\,-\alpha} \\
&= \gamma^{ij} f_{j\alpha}{}^{\alpha} \\
&= \gamma^{ij} \alpha_j \\
&= \alpha^i \ .
\end{aligned} \tag{73}$$

Let us summarize the above results. Cartan's Theorem allows us to find a basis for a semi-simple Lie algebra $\{H_i, E_\alpha\}$, $i = 1, ..., l = \mathrm{rank}\,(\mathcal{L})$, known as the **Cartan-Weyl basis**, with the following commutation relations:

$$\begin{aligned}
[H_i, H_j] &= 0 \\
[H_i, E_\alpha] &= \alpha_i E_\alpha \\
[E_\alpha, E_\beta] &= N_{\alpha\beta} E_{\alpha+\beta}, \qquad \alpha + \beta \neq 0 \\
[E_\alpha, E_{-\alpha}] &= \alpha^i H_i \ ,
\end{aligned} \tag{74}$$

The abelian sub-algebra spanned by $\{H_i\}$ is known as **Cartan's sub-algebra**. The vectors $\alpha = (\alpha_i)_{i=1,...,l}$ are denoted as the **root vectors**. In this basis the Killing metric takes the form:

$$\begin{aligned}
\gamma_{ij} &= -\sum_{\alpha} \alpha_i \alpha_j \\
\gamma_{i\alpha} &= 0 \\
\gamma_{\alpha\,-\alpha} &= 1 \\
\gamma_{\alpha\beta} &= 0 \,, \qquad \alpha+\beta \neq 0 \,.
\end{aligned} \tag{75}$$

In the Cartan-Weyl basis, we can find an $SU(2)$ **sub-algebra**. To see this, let us start by choosing $H=\alpha_i H_i$ for a given root vector α. The subset of generators $\{H, E_\alpha, E_{-\alpha}\}$ for each root vector α then satisfies the closed algebra:

$$\begin{aligned}
[H, E_\alpha] &= \alpha^i\,[H_i, E_\alpha] = \alpha^i \alpha_i E_\alpha \\
[H, E_{-\alpha}] &= \alpha^i\,[H_i, E_{-\alpha}] = -\alpha^i \alpha_i E_{-\alpha} \\
[E_\alpha, E_{-\alpha}] &= \alpha^i H_i = H \,.
\end{aligned} \tag{76}$$

To see that this is an $SU(2)$ algebra, note that defining $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and using the commutation relations for the Pauli matrices, we have:

$$\begin{aligned}
[\sigma_\pm, \sigma_3] &= [\sigma_1, \sigma_3] \pm i\,[\sigma_2, \sigma_z] = -\sigma_2 \pm i\sigma_1 \pm i\,(\sigma_1 \pm i\sigma_2) = \pm i\sigma_\pm \\
[\sigma_+, \sigma_-] &= [\sigma_1 + i\sigma_2, \sigma_1 - i\sigma_2] - i\,[\sigma_1, \sigma_2] + i\,[\sigma_2, \sigma_1] = -2i\sigma_3 \,,
\end{aligned} \tag{77}$$

which has the same form as Eq. (76) identifying $H \leftrightarrow \sigma_3$ and $E_{\pm\alpha} \leftrightarrow \sigma_\pm$.

Let us now consider a representation r of the Lie algebra on a vector space V. Let us choose a basis $\{|\lambda\rangle\}$ for V such that:

$$H_i^{(r)}|\lambda\rangle = \lambda_i |\lambda\rangle \,, \tag{78}$$

where we note that since all H_i generators in the Cartan sub-algebra commute they have a common eigenspace. The l-dimensional vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ are then designated as **weight vectors** of a given representation. Now, note that:

$$\begin{aligned}
H_i\,(E_\alpha|\lambda\rangle) &= E_\alpha H_i|\lambda\rangle + [H_i, E_\alpha]|\lambda\rangle \\
&= \lambda_i E_\alpha|\lambda\rangle + \alpha_i E_\alpha|\lambda\rangle \\
&= (\lambda_i + \alpha_i)\,(E_\alpha|\lambda\rangle) \,,
\end{aligned} \tag{79}$$

such that $E_\alpha|\lambda\rangle$ is an eigenvector of H_i with eigenvalue $\lambda_i + \alpha_i$, i.e. with weight vector $\lambda + \alpha$. We thus see that the $E_{\pm\alpha}$ generators act as raising and lowering operators in a given representation and can be used to find all the weights in the representation.

For the adjoint representation, where the vector space V coincides with the Lie algebra itself, we have:

$$\begin{aligned}
\mathrm{ad}(H_i)(H_j) &= [H_i, H_j] = 0 \,, \\
\mathrm{ad}(H_i)(E_\alpha) &= [H_i, E_\alpha] = \alpha_i E_\alpha \,.
\end{aligned} \tag{80}$$

This means that the eigenstates corresponding to the generators H_i have null weights, while for the eigenstates E_α the weights coincide with the roots of the Lie algebra.

For tensor representations of the group $R(g) = R_1(g) \times R_2(g)$ and associated representations of the Lie algebra, we have:

$$H_i^{(1)}|\lambda\rangle = \lambda_i|\lambda\rangle \ , \qquad H_i^{(2)}|\mu\rangle = \mu_i|\mu\rangle \ , \tag{81}$$

such that

$$\begin{aligned} H_i(|\lambda\rangle \otimes |\mu\rangle) &= \left(H_i^{(1)}|\lambda\rangle\right) \otimes |\mu\rangle + |\lambda\rangle \otimes \left(H_i^{(2)}|\mu\rangle\right) \\ &= (\lambda_i + \mu_i)|\lambda\rangle \otimes |\mu\rangle \ , \end{aligned} \tag{82}$$

such that the weights of tensor representations correspond to the sum of the weights of the individual representations.

1.3 Example: the Cartan-Weyl basis for $SU(2)$

To illustrate the advantages of using the Cartan-Weyl basis, let us return to our $SU(2)$ example, which is a rank 1 Lie algebra, i.e. there is only one generator in the Cartan sub-algebra. Recalling that the basis of generators in the fundamental representation is $T_i = \sigma_i/2$, $i = 1, 2, 3$, we can take $H = T_3$ corresponding to the diagonal generator and define $E_\pm = (T_1 \pm iT_2)/2$, such that:

$$[H, E_\pm] = \pm E_\pm \ , \qquad [E_+, E_-] = \frac{1}{2}H \ . \tag{83}$$

Thus, the $SU(2)$ algebra has roots $\alpha_\pm = \pm 1$. For a given $SU(2)$ representation we label the states in the associated vector space basis as $|j, m\rangle$, where m denotes the weights of the representation:

$$H|j, m\rangle = m|j, m\rangle \tag{84}$$

and the spin $j = \max(m)$ is the highest weight in the representation, which is then denoted as a spin-j representation. For the fundamental representation, the eigenvalues of $H = \sigma_3/2$ are $\pm 1/2$, so that the fundamental representation is the spin-1/2 representation as mentioned earlier. The adjoint representation has three weights, $m = 0, \pm 1$ as can be inferred explicitly from the form of the adjoint generators in Eq. (59), thus corresponding to the spin-1 representation.

For a generic spin-j representation, we can obtain all the states in the basis using the raising and lowering operators $E_\pm$. It is conventional to use instead the raising and lowering operators $J_\pm = 2E_\pm$, and from our discussion in the previous sub-section we must have:

$$J_-|j, m\rangle = N_m|j, m-1\rangle \ , \qquad J_+|j, m-1\rangle = N_m|j, m\rangle \ , \tag{85}$$

with constants N_m that we wish to determine. On the one hand, we have that:

$$\langle j, m|J_+J_-|j, m\rangle = ||J_-|j, m\rangle||^2 = |N_m|^2\langle j, m-1|j, m-1\rangle = |N_m|^2 \ , \tag{86}$$

assuming that the states are normalized. On the other hand, we also have that:

$$\begin{aligned} \langle j, m|J_+J_-|j, m\rangle &= \langle j, m|[J_+, J_-]|j, m\rangle + \langle j, m|J_-J_+|j, m\rangle \\ &= \langle j, m|2H|j, m\rangle + |N_{m+1}|^2 \\ &= 2m + |N_{m+1}|^2 \ . \end{aligned} \tag{87}$$

This then yields the recurrence relation:

$$|N_m|^2 = 2m + |N_{m+1}|^2 \ , \tag{88}$$

with boundary condition $N_{j+1} = 0$ since by definition j is the highest weight and $J_+|j,j\rangle = 0$. It is easy to verify that the solution is then:

$$N_m = \sqrt{j(j+1) - m(m-1)}\ , \tag{89}$$

such that

$$\begin{aligned} J_+|j,m\rangle &= \sqrt{j(j+1)-m(m+1)}|j,m+1\rangle\ , \\ J_-|j,m\rangle &= \sqrt{j(j+1)-m(m-1)}|j,m-1\rangle\ . \end{aligned} \tag{90}$$

In particular, $J_-|j,-j\rangle = 0$, such that the weights in each spin-j representation are $m = -j, -j+1, \ldots, j-1, j$, being a $(2j+1)$-dimensional representation.

For each spin-j representation, the Casimir operator is then given by:

$$C = \frac{1}{2}\sum_i T_i^2 = \frac{1}{2}H^2 + \frac{1}{4}J_+J_- + \frac{1}{4}J_-J_+\ . \tag{91}$$

As we have seen, Schur's Lemma implies that this is proportional to the identity, and to obtain the proportionality constant we can take the trace of the Casimir operator in a given representation:

$$\begin{aligned} \mathrm{Tr}_j(C) &= \sum_m \langle j,m| \left(\frac{1}{2}H^2 + \frac{1}{4}J_+J_- + \frac{1}{4}J_-J_+\right)|j,m\rangle \\ &= \frac{1}{2}\sum_m m^2 + \frac{1}{4}\sum_m \sqrt{j(j+1)-m(m-1)}\langle j,m|J_+|j,m-1\rangle + \frac{1}{4}\sum_m \sqrt{j(j+1)-m(m+1)}\langle j,m|J_-|j,m+1\rangle \\ &= \frac{1}{2}\sum_m m^2 + \frac{1}{4}\sum_m \left(\sqrt{j(j+1)-m(m-1)}\right)^2 + \frac{1}{4}\sum_m \left(\sqrt{j(j+1)-m(m+1)}\right)^2 \\ &= \frac{1}{2}\sum_m j(j+1) \\ &= \frac{1}{2}j(j+1)(2j+1)\ , \end{aligned} \tag{92}$$

and since the trace of the $(2j+1) \times (2j+1)$ identity matrix is $(2j+1)$, we conclude that the Casimir of a spin-j representation is:

$$C(j) = \frac{1}{2}j(j+1)\ . \tag{93}$$

We thus see that the $SU(2)$ representations correspond to the familiar spin (angular-momentum) states, with the Casimir operator $C = \mathbf{J}^2/2$.

2 Lorentz and Poincaré groups

The first groups that we will study in detail are the most fundamental groups in relativistic particle physics - the Lorentz group and its extension known as the Poincaré group. We will start by looking at the Lorentz group in detail and then explore its extension to include space-time translations.

2.1 Lorentz group

The Lorentz group is the group of space-time transformations that preserve the relativistic space-time distance, corresponding to coordinate changes between two inertial frames (moving at constant velocity with respect to each other). We can formally define it as:

$$L = O(3,1) = \left\{ \Lambda \in GL(\mathbb{R}^4) : \Lambda^T \eta \Lambda = \eta \right\} \tag{94}$$

where the 4-dimensional Minkowski metric is given by:

$$\eta = \mathrm{diag}(-1, +1, +1, +1) \ . \tag{95}$$

The Lorentz group transformations are at the heart of the theory of Special Relativity, such that coordinate changes of the form:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{96}$$

preserve the infinitesimal line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \ , \tag{97}$$

where we recall that $\mu = 0$ corresponds to the time coordinate and $\mu = i = 1, 2, 3$ correspond to the spatial coordinates. The invariance of the line element follows trivially from the group definition:

$$ds'^2 = \eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha dx^\alpha \Lambda^\nu{}_\beta dx^\beta = (\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta dx^\alpha dx^\beta = \eta_{\alpha\beta} dx^\alpha dx^\beta = ds^2 \ . \tag{98}$$

In components, the Lorentz group matrices satisfy:

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \ , \tag{99}$$

and include the well-known **boosts** and (spatial) **rotations**.

Examples:

- Boost along the x direction:

$$\Lambda = B_x = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{100}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, with v denoting the boost velocity and c the speed of light in vacuum. Writing

$\beta \equiv \tanh\phi$, it is easy to see that the boost matrix can be written in the form:

$$B_x = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{101}$$

such that a boost can be seen as a "hyperbolic rotation" (or rotation by an imaginary angle).

- Rotation about the z axis:

$$\Lambda = R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{102}$$

Generic spatial rotations form the $O(3)$ matrix group:

$$O(3) = \left\{ O \in GL(\mathbb{R}^4) : O = \begin{pmatrix} 1 & 0 \\ 0 & R_3 \end{pmatrix} \ , \ R_3 R_3^T = \mathbb{I}_3 \right\} \in O(3,1) \tag{103}$$

where R_3 are 3×3 orthogonal matrices.

The Lorentz group includes four distinct components. To see this, note that:

$$\det(\Lambda^T \eta \Lambda) = (\det\Lambda)^2 \det(\eta) = \det(\eta) \qquad \Rightarrow \qquad \det\Lambda = \pm 1 \ . \tag{104}$$

In addition, we have that for $\alpha = \beta = 0$ in Eq. (99)

$$\eta_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = -(\Lambda^0{}_0)^2 + \sum_i (\Lambda^i{}_0)^2 = \eta_{00} = -1 \ , \tag{105}$$

such that

$$(\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1 \qquad \Rightarrow \qquad \Lambda^0{}_0 \geq 1 \ \vee \ \Lambda^0{}_0 \leq -1 \ . \tag{106}$$

We may thus split the Lorentz group into the sub-groups:

$$\begin{aligned} L_+^\uparrow &= \left\{ \Lambda \in L : \det\Lambda = +1, \Lambda^0{}_0 \geq 1 \right\} \\ L_+^\downarrow &= \left\{ \Lambda \in L : \det\Lambda = +1, \Lambda^0{}_0 \leq 1 \right\} \\ L_-^\uparrow &= \left\{ \Lambda \in L : \det\Lambda = -1, \Lambda^0{}_0 \geq 1 \right\} \\ L_-^\downarrow &= \left\{ \Lambda \in L : \det\Lambda = -1, \Lambda^0{}_0 \leq 1 \right\} \ . \end{aligned} \tag{107}$$

These four sub-groups are disconnected from each other, since one cannot continuously change the sign of the determinant or of the $\Lambda^0{}_0$ component. It is also conventional to define the **proper Lorentz group** as:

$$L_+ = L_+^\uparrow \cup L_+^\downarrow \tag{108}$$

and the **orthocronous Lorentz group** as:

$$L^{\uparrow} = L^{\uparrow}_{+} \cup L^{\uparrow}_{-} \; . \tag{109}$$

For this reason, the $L^{\uparrow}_{+}$ sub-group is also known as the **proper orthocronous Lorentz group**, which includes the elements continuously related to the identity $\mathbb{I}_4$ as the boosts and rotations mentioned above.

The **parity** and **time-reserval** transformations are also part of the Lorentz group, such that:

$$\begin{aligned} P &= \mathrm{diag}(+1,-1,-1,-1) \in L^{\uparrow}_{-} \; , \\ T &= \mathrm{diag}(-1,+1,+1,+1) \in L^{\downarrow}_{-} \; , \\ PT &= \mathrm{diag}(-1,-1,-1,-1) \in L^{\downarrow}_{+} \; . \end{aligned} \tag{110}$$

Along with the identity matrix, these elements form an abelian sub-group of L, $L_0 = \{\mathbb{I}_4, P, T, PT\}$. Any element of the Lorentz group can be written as a unique product of an element of L_0 and an element of $L^{\uparrow}_{+}$, as illustrated in the diagram below.

$$\begin{tikzcd} L^{\uparrow}_{+} \arrow[r, "PT"] \arrow[d, "P"'] \arrow[dr, "T"] & L^{\downarrow}_{+} \\ L^{\uparrow}_{-} & L^{\downarrow}_{-} \end{tikzcd}$$

2.1.1 Relation with $SL(2,\mathbb{C})$

<u>DEFINITION</u>: The group $SL(2,\mathbb{C})$ of complex 2×2 matrices with unit determinant is defined as:

$$SL(2,\mathbb{C}) = \left\{ M \in \mathcal{M}(\mathbb{C}^2) : \det(M) = 1 \right\} \; . \tag{111}$$

Let us define $\sigma_\mu = (\mathbb{I}_2, \sigma_i)$, where σ_i, $i = 1,2,3$, are the Pauli matrices. We may then define a map:

$$\begin{aligned} \tau : \qquad & \mathbb{R}^4 \to \left\{ S \in \mathcal{M}(\mathbb{C}^2) : S = S^{\dagger} \right\} \\ & \tau(x) = x^{\mu}\sigma_{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \; . \end{aligned} \tag{112}$$

We can use this map to define a map between $SL(2,\mathbb{C})$ and the Lorentz group:

$$\begin{aligned} \Lambda : \qquad & SL(2,\mathbb{C}) \to GL(\mathbb{R}^4) \\ & x' = \Lambda(M)x = \tau^{-1}(M^{\dagger}\tau(x)M) \; , \end{aligned} \tag{113}$$

i.e. $\tau(x') = M^\dagger \tau(x) M$. In components:

$$\sigma_\mu x'^\mu = \sigma_\mu \Lambda(M)^\mu{}_\nu x^\nu = M^\dagger x^\alpha \sigma_\alpha M \ . \tag{114}$$

To show that the matrices $\Lambda(M)$ defined through this map belong to the proper orthocronous Lorentz group, let us first note that:

$$x^T \eta x = \eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + \sum_i (x^i)^2 = -\det(x^\mu \sigma_\mu) = -\det(\tau(x)) \ . \tag{115}$$

Thus,

$$x'^T \eta x' = -\det(\tau(x')) = -\det(\sigma_\mu \Lambda^\mu{}_\nu x^\nu) = -\det(M^\dagger \sigma_\mu x^\mu M) = -|\det(M)|^2 \det(\sigma_\mu x^\mu) = x^T \eta x \ , \tag{116}$$

so that $\Lambda(M)$ preserves space-time distances and hence $\Lambda(M) \in L$. Noting that the map is continuous and that $\Lambda(\pm\mathbb{I}_2) = \mathbb{I}_4$, since:

$$\sigma_\mu x'^\mu = \sigma_\mu \Lambda(\pm\mathbb{I}_2)^\mu{}_\nu x^\nu = x^\alpha \sigma_\alpha \tag{117}$$

for $M = \pm\mathbb{I}_2 \in SL(2,\mathbb{C})$, we conclude that only the elements of L continuously connected to the identity can be obtained through this map, i.e. that $\Lambda(M) \in L_+^\uparrow$. We thus say that $SL(2,\mathbb{C})$ is the double-cover of the proper orthocronous Lorentz group:

$$L_+^\uparrow = SL(2,\mathbb{C})/\mathbb{Z}_2 \ , \tag{118}$$

where the $\mathbb{Z}_2$ group factor identifies the elements continuously connected to $+\mathbb{I}_2$ and $-\mathbb{I}_2$ in $SL(2,\mathbb{C})$, since these yield the same element of $L_+^\uparrow$.

Let us now consider the Lie algebra of $SL(2,\mathbb{C})$, which can be obtained by considering infinitesimal transformations:

$$\det(M) = \det(\mathbb{I}_2 + T + \ldots) = 1 + \mathrm{Tr}(T) = 1 \qquad \Rightarrow \qquad \mathrm{Tr}(T) = 0 \ , \tag{119}$$

which implies the following definition for the Lie algebra:

$$\begin{aligned} \mathcal{L}(SL(2,\mathbb{C})) &= \left\{T \in \mathcal{M}(\mathbb{C}^2) : \mathrm{Tr}(T) = 0\right\} \\ &= \mathrm{span}\left\{J_i, K_i\right\}_{i=1,2,3} \end{aligned} \tag{120}$$

where the six generators of the Lie algebra are defined by:

$$J_i = -\frac{i}{2}\sigma_i \ , \qquad K_i = -\frac{1}{2}\sigma_i \tag{121}$$

and satisfy the following commutation relations:

$$[J_i, J_j] = \epsilon_{ijk} J_k \ , \qquad [K_i, K_j] = -\epsilon_{ijk} J_k \ , \qquad [J_i, K_j] = \epsilon_{ijk} K_k \ , \tag{122}$$

which follow trivially from the commutators of the Pauli matrices. Note that the tracelessness condition corresponds to a complex constraint on the 4 complex components of the Lie algebra matrices, leading to 3 complex degrees of freedom or equivalently 6 real degrees of freedom, which yields the dimension of the $SL(2,\mathbb{C})$ Lie algebra.

We may proceed in a similar fashion to determine the Lie algebra of the (proper orthocronous) Lorentz group, which via the map defined above should coincide the one of $SL(2,\mathbb{C})$. For an infinitesimal Lorentz transformation $\Lambda = \mathbb{I}_4 + \hat{T}$

we then have:

$$\Lambda^T \eta \Lambda = (\mathbb{I}_4 + \hat{T})^T \eta (\mathbb{I}_4 + \hat{T}) = \eta + \hat{T}^T \eta + \eta \hat{T} = \eta \ , \tag{123}$$

such that

$$\mathcal{L}(L_+^\uparrow) = \left\{ \hat{T} \in \mathcal{M}(\mathbb{R})^4 : \hat{T} = -\eta \hat{T}^T \eta \right\} \ . \tag{124}$$

In components, we have:

$$\hat{T}^\mu{}_\nu = -\eta^{\mu\alpha} (\hat{T}^T)_\alpha{}^\beta \eta_{\beta\nu} = -\eta^{\mu\alpha} \eta_{\beta\nu} \hat{T}^\beta{}_\alpha \ . \tag{125}$$

In particular, this yields the conditions:

$$\begin{aligned} \hat{T}^0{}_0 &= -\hat{T}^0{}_0 = 0 \ , \\ \hat{T}^0{}_i &= \hat{T}^i{}_0 \ , \\ \hat{T}^i{}_j &= -\hat{T}^j{}_i \ , \end{aligned} \tag{126}$$

so that the symmetric $\hat{T}^0{}_i$ sector has 3 independent components and the anti-symmetric $\hat{T}^i{}_j$ sector has also 3 independent components. The dimension of the Lorentz and $SL(2,\mathbb{C})$ algebras thus coincides and we can define the Lorentz group generators:

$$\hat{J}_i = \frac{1}{2} \epsilon_{ijk} \sigma_{jk} \ , \qquad \hat{K}_i = \sigma_{0i} \ , \tag{127}$$

where

$$(\sigma_{\mu\nu})^\rho{}_\sigma = \eta^\rho{}_\mu \eta_{\nu\sigma} - \eta^\rho{}_\nu \eta_{\mu\sigma} \tag{128}$$

can be shown to satisfy the commutation relation:

$$[\sigma_{\mu\nu}, \sigma_{\alpha\beta}] = \eta_{\mu\beta} \sigma_{\nu\alpha} + \eta_{\mu\alpha} \sigma_{\beta\nu} + \eta_{\nu\beta} \sigma_{\alpha\mu} + \eta_{\nu\alpha} \sigma_{\mu\beta} \ . \tag{129}$$

The generators $\hat{J}_i$ are associated with spatial rotations and span the anti-symmetric components $\hat{T}^i{}_j$, while the $\hat{K}_i$ generators are associated with boosts and span the symmetric components $T^0{}_i$. One can also use the above commutator to show that these satisfy the same commutation relations as their unhatted $SL(2,\mathbb{C})$ counterparts, which we leave as an exercise.

2.1.2 Representations of the Lorentz group

Using the generators of either $SL(2,\mathbb{C})$ or $L_+^\dagger$, we may define:

$$J_i^\pm = \frac{1}{2} \left(J_i \pm i K_i \right) \ , \tag{130}$$

such that it is easy to show that:

$$[J_i^\pm, J_j^\pm] = \epsilon_{ijk} J_k^\pm \ , \qquad [J_i^+, J_j^-] = 0 \ . \tag{131}$$

This defines two independent, i.e. commuting, $SU(2)$ algebras. Therefore, the irreducible representations of $L_+^\uparrow$ and $SL(2,\mathbb{C})$ correspond to the pairs of representations (j_+, j_-), where $j_\pm$ denotes the spin of each $SU(2)$ representation.

EXAMPLES

- **Left-handed Weyl spinors** $(1/2, 0)$:

  In this representation, $J_i^+ = -i\sigma_i/2$ and $J_i^- = 0$, corresponding to $J_i = -i\sigma_i/2$ and $K_i = -\sigma_i/2$, which is the fundamental representation of $SL(2,\mathbb{C})$ as we have seen above. Using the map between $SL(2,\mathbb{C})$ and the Lorentz group, we can write Lorentz transformations in this representation as:

  $$\Lambda_L(M) = e^{-\frac{1}{2}(s^i+it^i)\sigma_i} , \qquad s^i, t^i \in \mathbb{R} \tag{132}$$

  and the vector space in this representation corresponds to 2-component left-handed spinors transforming as:

  $$\psi_L(x) \to \Lambda_L(M)\psi_L(x) . \tag{133}$$

- **Right-handed Weyl spinors** $(0, 1/2)$:

  In this representation, $J_i^+ = 0$ and $J_i^- = -i\sigma_i/2$, corresponding to $J_i = -i\sigma_i/2$ and $K_i = \sigma_i/2$. We can then write Lorentz transformations in this representation as:

  $$\Lambda_R(M) = e^{\frac{1}{2}(s^i-it^i)\sigma_i} , \qquad s^i, t^i \in \mathbb{R} \tag{134}$$

  and the vector space in this representation corresponds to 2-component right-handed spinors transforming as:

  $$\psi_R(x) \to \Lambda_R(M)\psi_R(x) . \tag{135}$$

  Note that:

  $$\Lambda_L(M)^* = e^{\frac{1}{2}(-s^i+it^i)\sigma_i^*} = e^{-\frac{1}{2}(-s^i+it^i)\sigma_2\sigma_i\sigma_2} = \sigma_2 e^{\frac{1}{2}(s^i-it^i)\sigma_i}\sigma_2^{-1} = \sigma_2\Lambda_R(M)\sigma_2^{-1} \tag{136}$$

  such that the complex conjugate left-handed Weyl representation $\Lambda_L(M)^*$ is equivalent to the right-handed Weyl representation. We then say that a right-handed spinor transforms in the complex conjugate representation of $SL(2,\mathbb{C})$. Similarly, we have that:

  $$\left(\Lambda_L(M)^{-1}\right)^\dagger = \left(e^{\frac{1}{2}(s^i+it^i)\sigma_i}\right)^\dagger = e^{\frac{1}{2}(s^i-it^i)\sigma_i} = \Lambda_R(M) , \tag{137}$$

  such that $\Lambda_R^\dagger = \Lambda_L(M)^{-1}$ and the right-handed representation gives the **contragredient representation** of $\Lambda_L(M)$.

- **Dirac spinors** $(1/2, 0) \oplus (0, 1/2)$:

  In this composite representation Lorentz transformations correspond to 4×4 matrices:

  $$\Lambda_D(M) = \begin{pmatrix} \Lambda_L(M) & 0 \\ 0 & \Lambda_R(M) \end{pmatrix} , \tag{138}$$

  acting on four-component Dirac spinors:

  $$\psi_D(x) \to \Lambda_D(M)\psi_D(x) . \tag{139}$$

  Dirac spinors thus include both a left-handed and a right-handed component.

- **Vectors** $(1/2, 1/2)$:

  This coincides with the fundamental representation of the Lorentz group, with:

$$\Lambda_V(M) = e^{s^i \hat{K}_i + t^i \hat{J}_i} \ , \qquad s^i, t^i \in \mathbb{R} \tag{140}$$

denoting 4-dimensional matrices acting on 4-vectors:

$$A^\mu(x) \to \Lambda_V(M)^\mu{}_\nu A^\nu(x) \ , \tag{141}$$

for which a particular case is $A^\mu = x^\mu$. It is common to denote the left- and right-handed spinor indices in the form:

$$\begin{aligned} \psi_{La} &\to \Lambda_L(M)_a{}^b \psi_{Lb} \ , \\ \psi_{R\dot{a}} &\to \Lambda_R(M)_{\dot{a}}{}^{\dot{b}} \psi_{R\dot{b}} \ . \end{aligned} \tag{142}$$

The vectorial representation thus carries both a left-handed and a right-handed index, and can be equivalently expressed in spinor form as:

$$A_{a\dot{a}} = \sigma^\mu_{a\dot{a}} A_\mu(x) \tag{143}$$

Note that since $\Lambda_L(M) = M$ and $\Lambda_R(M) = M^*$ (up to the equivalency established above), a Lorentz vector in spinor form transforms as:

$$A_{a\dot{a}} \to M_a{}^b (M^*)_{\dot{a}}{}^{\dot{b}} A_{b\dot{b}} = M_a{}^b (M^*)_{\dot{a}}{}^{\dot{b}} \sigma^\mu_{b\dot{b}} A_\mu = (M\sigma_\mu M^\dagger)_{a\dot{a}} A^\mu = (\sigma_\mu)_{a\dot{a}} \Lambda_V(M)^\mu{}_\nu A^\nu \ , \tag{144}$$

using the map between the $SL(2, \mathbb{C})$ and Lorentz groups, so we see that the vector transformation defined above is equivalent to the spinor transformation laws.

- **Tensors:** $\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right)$

  The tensor product of two vector representations of the Lorentz group can be obtained from the tensor product of spin-1/2 representations of $SU(2)$:

$$\begin{aligned} \left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right) &= \left(\frac{1}{2} \otimes \frac{1}{2}, \frac{1}{2} \otimes \frac{1}{2}\right) \\ &= (0 \oplus 1, 0 \oplus 1) \\ &= (0,0) \oplus (1,1) \oplus (0,1) \oplus (1,0) \ , \end{aligned} \tag{145}$$

where the first two terms correspond to the symmetric part of the tensor product and the last two terms correspond to the anti-symmetric part. The latter, in particular, corresponds to an anti-symmetric Lorentz tensor with two vector indices:

$$F_{\mu\nu} = -F_{\nu\mu} \ . \tag{146}$$

We may define the dual tensor:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \ , \tag{147}$$

from which we may define:

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right) \ , \tag{148}$$

such that $F_{\mu\nu} = F^{+}_{\mu\nu} + F^{-}_{\mu\nu}$. It is easy to check that $\tilde{F}^{\pm}_{\mu\nu} = \pm F^{\pm}_{\mu\nu}$. We thus obtain a decomposition of the anti-symmetric tensor in terms of a **self-dual** and an **anti-self dual** component. Taking into account that under a Lorentz transformation:

$$F_{\mu\nu} \to \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}F_{\rho\sigma} \ , \qquad \tilde{F}_{\mu\nu} \to \Lambda_{\mu}{}^{\rho}\Lambda_{\nu}{}^{\sigma}\tilde{F}_{\rho\sigma} \ , \tag{149}$$

we can see that the self-dual or anti-self dual character of the tensor is preserved under Lorentz transformations, so that $F^{+}_{\mu\nu}$ and $F^{-}_{\mu\nu}$ correspond to the $(1,0)$ e $(0,1)$ irreducible representations of the Lorentz group.

We note that the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms in this representation, and its components describe the electric and magnetic fields in terms of the electrostatic potential and the vector potential in terms of the 4-vector potential $A^\mu = (\phi, \mathbf{A})$. In this case, the duality transformation exchanges the components of the electric and magnetic fields $E_i \leftrightarrow B_i$, which constitutes a symmetry of the electromagnetic interactions.

The symmetric part of the tensor product yields a Lorentz scalar in the $(0,0)$ representation, corresponding to scalar fields ϕ that are invariant under Lorentz transformations, and a symmetric and traceless tensor $h_{\mu\nu}$ in the $(1,1)$ representation. In general relativity, perturbations of the metric about flat Minkowski space of this form correspond to gravitational waves and to the putative graviton particles that correspond to the latter in the (yet unknown) quantum formulation of the theory.

2.1.3 Spinor bilinears

Weyl spinors are used to describe spin-1/2 particles and the associated fields that generalize the wavefunction in the relativistic formulation of quantum mechanics. It is thus useful to discuss some of the Lorentz invariant quantities that we may construct from spinor fields and which may thus appear in the Lagrangian function that describes such particles. The most important terms in a Lagrangian are the quadratic terms, which lead to linear terms in the equations of motion via the Euler-Lagrange equations. These then correspond to kinetic and mass terms for the fields, with the former including field derivatives.

- **Majorana mass term**:

$$\begin{aligned}
\psi_L^T\sigma_2\psi_L &\to (\Lambda_L(M)\psi_L)^T\,\sigma_2\,(\Lambda_L(M)\psi_L) \\
&\to \psi_L^T\Lambda_L^T(M)\sigma_2\Lambda_L(M)\psi_L \\
&\to \psi_L^T\left(\sigma_2\Lambda_L^{-1}(M)\sigma_2\right)\sigma_2\Lambda_L(M)\psi_L \\
&\to \psi_L^T\sigma_2\Lambda_L^{-1}(M)\Lambda_L(M)\psi_L \\
&\to \psi_L^T\sigma_2\psi_L \ ,
\end{aligned} \tag{150}$$

where we have used that $\left(\Lambda_L^{-1}\right)^T = \Lambda_R(M)^\dagger = \sigma_2\Lambda_L(M)\sigma_2$ as obtained in Eq. (136) and (137). A term of the form $\psi_R^T\sigma_2\psi_R$ is also invariant under Lorentz transformations. Mass terms for spin-1/2 left-handed or right-handed fields can be constructed with Majorana terms of this form provided that they are allowed by other symmetries.

- **Dirac mass term**:

$$\begin{aligned}\chi_L^\dagger \psi_R &\rightarrow (\Lambda_L(M)\chi_L)^\dagger\,(\Lambda_R(M)\psi_R)\\ &\rightarrow \chi_L^\dagger \Lambda_L^\dagger(M)\Lambda_R(M)\psi_R\\ &\rightarrow \chi_L^\dagger \Lambda_R^{-1}(M)\Lambda_R(M)\psi_L\\ &\rightarrow \chi_L^\dagger\psi_R\ .\end{aligned}\tag{151}$$

A term of the form $\chi_R^\dagger\psi_L$ is also Lorentz invariant for similar reasons. Note that Dirac mass terms combine the left- and right-handed parts of a Dirac spinor, which are given by distinct Weyl spinors, while the Majorana terms combine spinors in the same Weyl representation. In the Standard Model, as we will see later, the masses of the spin-1/2 charged leptons and quarks corresponds exclusively to Dirac mass terms, while the neutrino masses may possibly have a contribution from Majorana terms.

- **Weyl currents:**

$$\begin{aligned}j_{L\mu} = \chi_L^\dagger \sigma_\mu \psi_L &\rightarrow (\Lambda_L(M)\chi_L)^\dagger\,\sigma_\mu\,(\Lambda_L(M)\psi_L)\\ &\rightarrow \chi_L^\dagger\Lambda_L^\dagger(M)\sigma_\mu\Lambda_L(M)\psi_L\\ &\rightarrow \chi_L^\dagger M^\dagger\sigma_\mu M\psi_L\\ &\rightarrow \chi_L^\dagger\sigma_\nu\Lambda^\nu{}_\mu\psi_L\\ &\rightarrow \Lambda^\nu{}_\mu j_{L\nu}\ .\end{aligned}\tag{152}$$

Thus the left-handed Weyl current $j_{L\mu}$ transforms as a Lorentz vector, the same occurring for the analogous right-handed Weyl current $j_{R\mu}$ involving right-handed spinors. To obtain Lorentz invariant quantities, we may contract these currents with other vectors in a Lorentz invariant way. For example, the fermion kinetic term given by:

$$\chi_{L,R}^\dagger\sigma^\mu\partial_\mu\psi_{L,R}\tag{153}$$

is a Lorentz scalar since:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \left(\Lambda^{-1}\right)^\nu{}_\mu\,\partial_\nu\ .\tag{154}$$

In a similar way, we can couple the fermionic current to the electromagnetic potential in a Lorentz invariant way, such that $j_{L,R\mu}$ describes the electric current associated with a charged fermion:

$$\eta_{\mu\nu}j^\mu_{L,R}A^\nu \rightarrow \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta j^\alpha_{L,R}A^\beta = \eta_{\alpha\beta}j^\alpha_{L,R}A^\beta\ .\tag{155}$$

2.1.4 Parity and charge conjugation

The parity operator acts on the Lorentz group generators in the adjoint representation as:

$$P\hat{J}_iP^{-1} = \hat{J}_i\ ,\qquad P\hat{K}_iP^{-1} = -\hat{K}_i\ ,\tag{156}$$

as one can check by explicitly constructing the matrices associated to the generators. Hence, the angular momentum operators $\hat{J}_i$, which generate spatial rotations, are pseudo-vectors that remain invariant under spatial reflections, while the boost generators $\hat{K}_i$ are normal vectors that change direction under parity transformations. From this we may also conclude that:

$$PJ_i^{\pm}P^{-1} = \frac{1}{2}P\left(\hat{J}_i \pm i\hat{K}_i\right)P^{-1} = J_i^{\mp} \ , \tag{157}$$

such that a parity transformation exchanges the left- and right-handed representations.

Charge conjugation, which exchanges particles and the corresponding anti-particles, has similar consequences. For example, for a right-handed Weyl spinor ψ_R we may define the conjugate spinor as:

$$\psi_R^c \equiv \sigma_2 \psi_R^* \ , \tag{158}$$

such that under a Lorentz transformation:

$$\begin{aligned} \psi_R^c &\to \sigma_2 \Lambda_R(M)^* \psi_R^* \\ &\to \sigma_2\sigma_2\Lambda_L(M)\sigma_2\psi_R^* \\ &\to \Lambda_L(M)\psi_R^c \ . \end{aligned} \tag{159}$$

Hence, the conjugate of a right-handed spinor is a left-handed spinor and, analogously, the conjugate of a left-handed spinor $\chi_L^c = \sigma_2\chi_L^*$ transforms in the right-handed representation.

2.2 Poincaré group

The transformations of the Poincaré group include both Lorentz transformations and space-time translations, forming the group:

$$\mathcal{P} = \left\{(\Lambda, a) : \Lambda \in L, a \in \mathbb{R}^4\right\} \ , \tag{160}$$

such that the action of the group elements in a coordinate system is given by:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \ . \tag{161}$$

The group multiplication law is given by:

$$\begin{aligned} (\Lambda_1, a_1).(\Lambda_2, a_2)x &= (\Lambda_1, a_1).(\Lambda_2 x + a_2) = \Lambda_1(\Lambda_2 x + a_2) + a_1 = \Lambda_1\Lambda_2 x + \Lambda_1 a_2 + a_1 \\ &= (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)x \ . \end{aligned} \tag{162}$$

The identity element is naturally $e = (\mathbb{I}_4, 0)$ and for each group element we have the inverse element:

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a) \ , \tag{163}$$

since

$$(\Lambda, a)^{-1}.(\Lambda, a) = (\Lambda^{-1}, -\Lambda^{-1}a).(\Lambda, a) = (\Lambda^{-1}\Lambda, \Lambda^{-1}a - \Lambda^{-1}a) = (\mathbb{I}_4, 0) = e \ . \tag{164}$$

As should be familiar in quantum mechanics, spatial translations are generated by the linear momentum operator, while the Hamiltonian generates time translations:

$$\begin{aligned}\Psi(\mathbf{x}+\epsilon, t+\tau) &= \Psi(\mathbf{x}, t) + i\epsilon \cdot (-i\nabla\Psi(\mathbf{x}, t)) - i\tau\left(i\frac{\partial\Psi(\mathbf{x}, t)}{\partial t}\right) \\ &= \Psi(\mathbf{x}, t) + i\epsilon \cdot \hat{\mathbf{P}}\Psi(\mathbf{x}, t) - i\tau\hat{H}\Psi(\mathbf{x}, t) \ , \qquad (165)\end{aligned}$$

for infinitesimal translations ϵ and τ in space and time, respectively. We can assemble these two operators in the 4-momentum relativistic operator generating space-time translations:

$$\hat{P}_\mu = -i\partial_\mu \ . \qquad (166)$$

The Lorentz group generators that we have determined earlier also admit a representation in terms of differential operators:

$$\hat{M}_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ , \qquad (167)$$

satisfying the commutation relations:

$$[\hat{M}_{\mu\nu}, \hat{M}_{\mu\nu}] = i\left(\eta_{\mu\beta}\hat{M}_{\nu\alpha} + \eta_{\mu\alpha}\hat{M}_{\beta\nu} + \eta_{\nu\beta}\hat{M}_{\alpha\mu} + \eta_{\nu\alpha}\hat{M}_{\mu\beta}\right) \ , \qquad (168)$$

which, apart from a factor of i, are the same commutators as for the $\sigma_{\mu\nu}$ generators in Eq. (129). We can also easily check that:

$$[\hat{M}_{\mu\nu}, \hat{P}_\rho] = i\left(\eta_{\nu\rho}\hat{P}_\mu - \eta_{\mu\rho}\hat{P}_\nu\right) = i\left(\sigma_{\mu\nu}\right)^\sigma_{\ \rho}\hat{P}_\sigma \ , \qquad [\hat{P}_\mu, \hat{P}_\nu] = 0 \ . \qquad (169)$$

The Poincaré algebra is thus given by:

$$\mathcal{L}(\mathcal{P}) = \text{span}(\hat{P}_\mu, \hat{M}_{\mu\nu}) \ . \qquad (170)$$

We note that, as previously defined:

$$\hat{J}_i = \frac{1}{2}\epsilon_{ijk}\hat{M}_{jk} = \epsilon_{ijk}\hat{x}_j\hat{P}_k \qquad (171)$$

are the angular momentum operators generating spatial rotations, while $\hat{K}_i = \hat{M}^{0i}$ are the boost generators.

An important quantity to define is the **Pauli-Lubanski vector**:

$$\hat{W}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\rho}\hat{P}^\sigma \ , \qquad (172)$$

with the following properties:

$$\begin{aligned}\hat{W}_\mu\hat{P}^\mu &= 0 \ , \qquad (173) \\ \left[\hat{W}_\mu, \hat{P}_\nu\right] &= 0 \ , \\ \left[\hat{W}_\mu, \hat{M}_{\alpha\beta}\right] &= -i\left(\eta_{\mu\beta}\hat{W}_\alpha - \eta_{\mu\alpha}\hat{W}_\beta\right) \ , \\ \left[\hat{W}_\mu, \hat{W}_\alpha\right] &= -i\epsilon_{\mu\alpha\beta\nu}\hat{W}^\beta\hat{P}^\nu \ , \qquad (174)\end{aligned}$$

The Lorentz invariant quadratic operators:

$$\hat{P}^2 = \hat{P}_\mu \hat{P}^\mu \ , \qquad \hat{W}^2 = \hat{W}_\mu \hat{W}^\mu \ , \tag{175}$$

commute with all the generators of the Poincaré group, thus constituting the Casimir operators of the Poincaré group and from which we can label the different irreducible representations. We may write, in particular:

$$\begin{aligned}
\hat{W}^2 &= \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu}{}_{\alpha\beta\gamma}\hat{M}^{\nu\rho}\hat{P}^\sigma\hat{M}^{\alpha\beta}\hat{P}^\gamma \\
&= \frac{1}{4}\left[-\eta_{\nu\alpha}(\eta_{\rho\beta}\eta_{\sigma\gamma}-\eta_{\rho\gamma}\eta_{\sigma\beta})-\eta_{\nu\beta}(\eta_{\rho\gamma}\eta_{\sigma\alpha}-\eta_{\rho\alpha}\eta_{\sigma\gamma})-\eta_{\nu\gamma}(\eta_{\rho\alpha}\eta_{\sigma\beta}-\eta_{\rho\beta}\eta_{\sigma\alpha})\right]\hat{M}^{\nu\rho}\hat{P}^\sigma\hat{M}^{\alpha\beta}\hat{P}^\gamma \\
&= -\frac{1}{2}\hat{M}_{\alpha\beta}\hat{P}_\gamma\hat{M}^{\alpha\beta}\hat{P}^\gamma + \hat{M}_{\alpha\gamma}\hat{P}_\beta\hat{M}^{\alpha\beta}\hat{P}^\gamma \\
&= -\frac{1}{2}\hat{M}_{\alpha\beta}\hat{M}^{\alpha\beta}\hat{P}^2 + \hat{M}_{\alpha\gamma}\hat{M}^{\alpha\beta}\hat{P}_\beta\hat{P}^\gamma \ ,
\end{aligned} \tag{176}$$

where we used the form of the contraction of two Levi-Civita tensors and the commutation relations (169).

As we know from the theory of Special Relativity, the Casimir opertor $\hat{P}^2 = -m^2$ for a particle of mass m, so that different irreducible representations of the Poincaré group will correspond to particles with different mass. We also have that the state of a particle with mass m and 4-momentum p is an eigenstate of the $\hat{P}_\mu$ generator:

$$\hat{P}_\mu|m,p,\sigma\rangle = p_\mu|m,p,\sigma\rangle \tag{177}$$

where we have denoted by σ any other quantities that characterize the state in a given representation and that we wish to determine. Under a Lorentz transformation in a given representation $R(\Lambda,0)$ we obtain a state with 4-momentum Λp:

$$R(\Lambda,0)|m,p,\sigma\rangle = |m,\Lambda p,\sigma\rangle \ . \tag{178}$$

Let us check that this is also an eigenstate of $\hat{P}_\mu$. First, let us note that $\hat{P}_\mu$, being a group generator, transforms in the adjoint representation and is a Lorentz vector:

$$R(\Lambda,0)\hat{P}_\mu R^{-1}(\Lambda,0) = \Lambda_\mu{}^\nu\hat{P}_\nu \ . \tag{179}$$

Thus,

$$\begin{aligned}
\hat{P}_\mu\left(R(\Lambda,0)|m,p,\sigma\rangle\right) &= R(\Lambda,0)R^{-1}(\Lambda,0)\hat{P}_\mu R(\Lambda,0)|m,p,\sigma\rangle \\
&= R(\Lambda,0)\Lambda_\mu{}^\nu\hat{P}_\nu|m,p,\sigma\rangle \\
&= \Lambda_\mu{}^\nu p_\nu R(\Lambda,0)|m,p,\sigma\rangle \\
&= (\Lambda p)_\mu|m,\Lambda p,\sigma\rangle \ .
\end{aligned} \tag{180}$$

This means that, in constructing the states in the different representations of the Poincaré group, we may take as reference state a state with a given momentum, such that states with an arbitrary momentum can be obtained from this reference state by performing a Lorentz transformation. The choice of this reference momentum will be different for particles with and without mass, such that we must analyze these two cases separately.

2.2.1 Massive representations

For states of a massive particle, $m \neq 0$, we may take as reference the 4-momentum in the particle's rest frame, $p^\mu = (m, \mathbf{0})$. This choice is obviously invariant under spatial rotations, i.e the states:

$$R(O,0)|m,(m,\mathbf{0}),\sigma\rangle \ , \qquad O \in SO(3) \ , \tag{181}$$

are also eigenstates of $\hat{P}_\mu$ with momentum eigenvalue $p^\mu = (m, \mathbf{0})$. We may thus obtain the irreducible representations for massive particles from the irreducible representations of $SO(3) \cong SU(2)$, since the generators of the two groups satisfy the same algebra. Note, in particular, that the fundamental representation of the $SO(3)$ Lie algebra, corresponding to the anti-symmetric 3×3 matrices, coincides with the adjoint representation of the $SU(2)$ algebra obtained in Eq. (59). As we have seen, $SU(2)$ irreducible representations are labelled by their spin and its component along the z axis, which to avoid confusion with the particle's mass we may write as $j_3 = -j, \ldots, j$. We thus have $\sigma = (j, j_3)$ and:

$$R(O,0)|m,(m,\mathbf{0}),j,j_3\rangle = R^{(j)}_{j_3,j_3'}(O)|m,(m,\mathbf{0}),j,j_3'\rangle \ , \tag{182}$$

where $R^{(j)}_{j_3,j_3'}(O)$ are the rotation matrices in a spin-j representation. The $SO(3)$ group is designated as the **Little Group** for massive representations of the Poincaré group, i.e. the sub-group that preserves the form of the states with 4-momentum $p^\mu = (m, \mathbf{0})$.

For these states, the components of the Pauli-Lubanski vector are given by:

$$\begin{aligned}
\hat{W}_0|m,(m,\mathbf{0}),j,j_3\rangle &= \frac{1}{2}\epsilon_{0ijk}\hat{M}^{ij}\hat{P}^k|m,(m,\mathbf{0}),j,j_3\rangle = 0 \ , \\
\hat{W}_i|m,(m,\mathbf{0}),j,j_3\rangle &= \frac{1}{2}\epsilon_{ijk0}\hat{M}^{jk}\hat{P}^0|m,(m,\mathbf{0}),j,j_3\rangle = -m\hat{J}_i|m,(m,\mathbf{0}),j,j_3\rangle \ ,
\end{aligned} \tag{183}$$

and thus the Casimir operator $\hat{W}^2$ is given by:

$$\hat{W}^2|m,(m,\mathbf{0}),j,j_3\rangle = m^2\hat{J}^2|m,(m,\mathbf{0}),j,j_3\rangle = m^2 j(j+1)|m,(m,\mathbf{0}),j,j_3\rangle \ . \tag{184}$$

Hence, we see that the irreducible representations of the Poincaré group correspond to particles with different mass and spin for the case $m \neq 0$.

2.2.2 Massless representations

In the case $m = 0$, there is no rest frame for the particle to use as reference 4-momentum, but we could e.g. choose $k^\mu = E(1,0,0,1)$, where E denotes the particle's energy. This form of the 4-momentum is preserved by the isometries of the Euclidean (x,y) plane, which include rotations about the z axis and translations along x and y. These form the group $ISO(2)$, which thus constitutes the Little Group for $m = 0$.

Let us denote the states with 4-momentum k^μ as $|k,\sigma\rangle$, such that the components of the Pauli-Lubanski vector are:

$$\begin{aligned}
\hat{W}_0|k,\sigma\rangle &= \frac{1}{2}\epsilon_{0ij3}\hat{M}^{ij}\hat{P}^3|k,\sigma\rangle = E\hat{M}_{12}|k,\sigma\rangle = E\hat{J}_3|k,\sigma\rangle \ , \\
\hat{W}_3|k,\sigma\rangle &= \frac{1}{2}\epsilon_{3ij0}\hat{M}^{ij}\hat{P}^0|k,\sigma\rangle = -E\hat{M}_{12}|k,\sigma\rangle = -E\hat{J}_3|k,\sigma\rangle \ , \\
\hat{W}_1|k,\sigma\rangle &= \left(\frac{1}{2}\epsilon_{1ij0}\hat{M}^{ij}\hat{P}^0 + \epsilon_{1023}\hat{M}^{02}\hat{P}^3\right)|k,\sigma\rangle = -E\left(\hat{M}^{23} + \hat{M}^{02}\right)|k,\sigma\rangle = -E\left(\hat{J}_1 + \hat{K}_2\right)|k,\sigma\rangle \ , \\
\hat{W}_2|k,\sigma\rangle &= \left(\frac{1}{2}\epsilon_{2ij0}\hat{M}^{ij}\hat{P}^0 + \epsilon_{2013}\hat{M}^{01}\hat{P}^3\right)|k,\sigma\rangle = -E\left(\hat{M}^{31} - \hat{M}^{01}\right)|k,\sigma\rangle = -E\left(\hat{J}_2 - \hat{K}_1\right)|k,\sigma\rangle \ .
\end{aligned} \tag{185}$$

The relevant operators are, thus, $\left\{\hat{J}_3, \hat{S}_1, \hat{S}_2\right\}$, where $\hat{S}_i \equiv \hat{W}_i$, such that with (122) we obtain the algebra:

$$[\hat{S}_1, \hat{S}_2] = 0 \ , \qquad [\hat{J}_3, \hat{S}_1] = i\hat{S}_2 \ , \qquad [\hat{J}_3, \hat{S}_2] = -i\hat{S}_1 \ . \tag{186}$$

Note that this is also the algebra satisfied by the linear momentum operators on the (x, y)-plane, identifying $\hat{S}_i \leftrightarrow \hat{P}_i$, $i = 1, 2$, along with the z component of the angular momentum, which thus constitutes the algebra of $ISO(2)$. It is easy to check that, defining $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2)$,

$$[\hat{\mathbf{S}}^2, \hat{S}_i] = [\hat{\mathbf{S}}^2, \hat{J}_3] = 0 \ , \tag{187}$$

such that $\hat{\mathbf{S}}^2$ commutes with all the generators of the Little Group and is therefore the non-trivial Casimir operator for massless representations. We may then define the Weyl-Cartan basis for the Little Group $ISO(2)$ with $\hat{J}_3$ yielding the only element in the Cartan sub-algebra and the ladder operators:

$$\hat{S}_\pm = \hat{S}_1 \pm i\hat{S}_2 \tag{188}$$

such that:

$$[\hat{J}_3, \hat{S}_\pm] = \pm\hat{S}_\pm \ . \tag{189}$$

We may then obtain the states in massless particle representations in a similar way to the irreducible representations of $SU(2)$, with:

$$\begin{aligned} \hat{\mathbf{S}}^2|k, s, j_3\rangle &= s^2|k, s, j_3\rangle \ , \\ \hat{J}_3|k, s, j_3\rangle &= j_3|k, s, j_3\rangle \ , \\ \hat{J}_3\left(\hat{S}_\pm|k, s, j_3\rangle\right) &= \left(\hat{S}_\pm\hat{J}_3 \pm \hat{S}_\pm\right)|k, s, j_3\rangle = (j_3 \pm 1)\,\hat{S}_\pm|k, s, j_3\rangle \ , \end{aligned} \tag{190}$$

which means that $\hat{S}_\pm|k, s, j_3\rangle \propto |k, s, j_3 \pm 1\rangle$. However, note that:

$$\langle k, s, j_3|\hat{\mathbf{S}}^2|k, s, j_3\rangle = \langle k, s, j_3|\hat{S}_\pm^\dagger\hat{S}_\pm|k, s, j_3\rangle = ||\hat{S}_\pm|k, s, j_3\rangle||^2 = s^2\,\||k, s, j_3\rangle\|^2 \ . \tag{191}$$

This means that $\hat{S}_\pm|k, s, j_3\rangle = 0$ only for $s = 0$. Therefore, for $s \neq 0$ there is an infinite set of states, which do not find any realization in nature. The only physical states are those with $s = 0$, characterized uniquely by the eigenvalues of $\hat{J}_3$ and which we may thus denote as $|k, j_3\rangle$, such that:

$$\hat{S}_\pm|k, , j_3\rangle = 0 \ . \tag{192}$$

Given our choice for the reference 4-momentum, we see that $\hat{J}_3$ is the component of the particle's spin in the direction of its 3-momentum, and its eigenvalues are denoted as the **helicity** of a particle. Note that we obtain the same result whatever the form of the reference 4-momentum that we had chosen, with $k^\mu k_\mu = 0$, so that helicity is Lorentz invariant for massless particles. For example, massless spin-1/2 particles are described by Weyl spinors, with $h \equiv j_3 = +1/2$ for right-handed spinors and $h = -1/2$ for left-handed spinors, which are thus irreducible representations of the Lorentz and Poincaré groups.

3 $SU(N)$ groups

The **unitary** and **special unitary** groups also play a special role in particle physics, constituting the basis for the global and gauge symmetries upon which the Standard Model of particle physics is built. They are defined, respectively, as:

$$U(N) = \left\{U \in GL(\mathbb{C}^N) : U^\dagger U = \mathbb{I}_N\right\} \ , \qquad SU(N) = \{U \in U(N) : \det(U) = 1\} \ . \tag{193}$$

Note that the unitary condition $U^\dagger U = \mathbb{I}_N$ implies $|\det(U)|^2 = 1$, i.e. $\det(U) = \pm 1$. One can construct a map between the groups $U(1) \times SU(N)$ and $U(N)$ in the following way:

$$\begin{aligned} f : \qquad & U(1) \times SU(N) \to U(N) \\ & f(z, U) = zU \ , \qquad z \in U(1), \ U \in SU(N) \ . \end{aligned} \tag{194}$$

Note that $U(1)$ corresponds to the group of complex numbers with unit modulus:

$$U(1) \quad = \quad \{z \in \mathbb{C} : |z| = 1\} \ . \tag{195}$$

Let us check that this map is a group homomorphism. First, note that for any unitary matrix $U \in U(N)$, we may write:

$$\det U = \zeta^N \qquad \Rightarrow \qquad |\zeta|^N = 1 \qquad \Rightarrow \qquad |\zeta| = 1 \qquad \Rightarrow \qquad \zeta \in U(1) \ . \tag{196}$$

Let us then consider matrices of the form $A = \zeta^{-1} U$, which satisfy:

$$\begin{aligned} A^\dagger A \quad &= \quad \left(\zeta^{-1} U\right)^\dagger \left(\zeta^{-1} U\right) = U^\dagger \zeta \zeta^{-1} U = U^\dagger U = \mathbb{I}_N \ , \\ \det(A) \quad &= \quad \left(\zeta^{-1}\right)^N \det U = (\det(U))^{-1} \det U = 1 \ , \end{aligned} \tag{197}$$

so that $A \in SU(N)$. Hence, we have that:

$$f(\zeta, \zeta^{-1} U) = \zeta \zeta^{-1} U = U \ , \tag{198}$$

such that we can obtain any matrix U in $U(N)$ through the map f.

Second, we need to find how many elements of $U(1) \times SU(N)$ are mapped to the identity matrix $\mathbb{I}_N$:

$$\mathrm{Ker}(f) = \{(z, A) \in U(1) \times SU(N) : f(z, A) = \mathbb{I}_N\} \ . \tag{199}$$

Note that, for these elements:

$$\det(zA) = z^N \det(A) = z^N = 1 \qquad \Rightarrow \qquad z = e^{\frac{2\pi i n}{N}} \ , n \in \mathbb{Z} \ , \tag{200}$$

such that $\mathrm{Ker}(f) = \mathbb{Z}_N$. This then implies the following group isomorphism:

$$U(N) \cong U(1) \times SU(N) / \mathbb{Z}_N \ , \tag{201}$$

i.e. the matrices in $SU(N)$ and $U(N)$ differ only by a complex phase $z = e^{i\theta}$ with period $2\pi/N$.

We will then focus on the $SU(N)$ groups, and analogously to what we have already found for $SU(2)$ the Lie algebra for these groups is given by:

$$\mathcal{L}\left(SU\left(N\right)\right) = \left\{T \in M\left(\mathbb{C}^N\right) : \ T = T^\dagger, \ \mathrm{Tr}\left(T\right) = 0\right\} \ . \tag{202}$$

On the $2N^2$ real degrees of freedom of a complex $N \times N$ matrix, the hermiticity condition imposes N^2 constraints, and a further real constraint is imposed by the tracelessness condition, leaving $N^2 - 1$ degrees of freedom which is the dimension of the Lie algebra and of the adjoint representation.

The Cartan sub-algebra of $SU(N)$, corresponding to the maximum number of commuting generators, is trivially given by the diagonal generators of the Lie algebra. Since there are N distinct real components in a hermitian matrix and the tracelessness condition allows one to write one of these components in terms of the others, we conclude that this yields an $(N-1)$-dimensional Cartan sub-algebra, i.e. that $SU(N)$ groups have rank $N-1$. As we have described in the first section, we may construct a basis for the Lie algebra and determine its roots and weights from the Cartan sub-algebra generators, which we will do explicitly for the case of $SU(3)$.

3.1 Irreducible representations of $SU(N)$

Representations of $SU(N)$ are given in terms of tensor fields that transform under $SU(N)$ in different ways. They are also represented in terms of their dimension. In particular, we have:

- **Fundamental representation**

$$\mathbf{N}: \qquad \psi_a \to U_a{}^b \psi_b \ , \qquad a,b = 1,\ldots,N \tag{203}$$

  where $U \in SU(N)$.

- **Complex conjugate representation**

$$\bar{\mathbf{N}}: \qquad \psi^a \to U^{*a}{}_b \psi^b \ . \tag{204}$$

- **Tensor representations**

$$\mathbf{N}^p \times \bar{\mathbf{N}}^q: \qquad \psi^{a_1\ldots a_q}{}_{b_1\ldots b_p} \to U^{*a_1}{}_{c_1} \ldots U^{*a_q}{}_{c_q} U_{b_1}{}^{d_1} \ldots U_{b_p}{}^{d_p} \psi^{c_1\ldots c_q}{}_{d_1\ldots d_p} \ . \tag{205}$$

There are special tensors that are invariant under $SU(N)$ transformations, in particular the generalized Kronecker delta and Levi-Civita tensors:

$$\delta_a{}^b \quad \to \quad U_a{}^c U^{*b}{}_d \delta_c{}^d = U_a{}^c U^{*b}{}_c = U_a{}^c \left(U^\dagger\right)_c{}^b = \left(UU^\dagger\right)_a{}^b = \delta_a{}^b \ , \tag{206}$$

$$\begin{aligned}
\epsilon_{a_1\ldots a_N} \quad &\to \quad U_{a_1}{}^{b_1} \ldots U_{a_N}{}^{b_N} \epsilon_{b_1\ldots b_N} \\
&\to \quad \left(\delta_{a_1}{}^{b_1} + iT_{a_1}{}^{b_1}\right) \ldots \left(\delta_{a_N}{}^{b_N} + iT_{a_N}{}^{b_N}\right) \epsilon_{b_1\ldots b_N} \\
&\to \quad \epsilon_{a_1\ldots a_N} + iT_{a_1}{}^{b_1} \epsilon_{b_1 a_2\ldots a_N} + \ldots + iT_{a_N}{}^{b_N} \epsilon_{a_1 a_2\ldots b_N} \\
&\to \quad \epsilon_{a_1\ldots a_N} + iT_{a_1}{}^{a_1} \epsilon_{a_1 a_2\ldots a_N} + \ldots + iT_{a_N}{}^{a_N} \epsilon_{a_1 a_2\ldots a_N} \\
&\to \quad \epsilon_{a_1\ldots a_N} \left(1 + i\mathrm{Tr}(T)\right) \\
&\to \quad \epsilon_{a_1\ldots a_N} \ ,
\end{aligned} \tag{207}$$

and analogously for $\epsilon^{a_1\ldots a_N}$. An equivalent way of proving the invariance of the Levi-Civita tensor is to note that:

$$U_{a_1}{}^{b_1} \ldots U_{a_N}{}^{b_N} \epsilon_{b_1\ldots b_N} = \det(U) \epsilon_{a_1\ldots a_N} \ , \tag{208}$$

such that for an $SU(N)$ matrix with unit determinant the invariance of the Levi-Civita tensor follows trivially.

These tensors can be used to construct new tensors, for example:

$$\begin{aligned}
\psi^{a_1\ldots a_p}{}_{a_1\ldots a_p} &\to U^{*a_1}{}_{c_1}\ldots U^{*a_p}{}_{c_p}U_{a_1}{}^{d_1}\ldots U_{a_p}{}^{d_p}\psi^{c_1\ldots c_p}{}_{d_1\ldots d_p} \\
&\to \left(U^{Td_1}{}_{a_1}U^{*a_1}{}_{c_1}\right)\ldots\left(U^{Td_p}{}_{a_p}U^{*a_p}{}_{c_p}\right)\psi^{c_1\ldots c_p}{}_{d_1\ldots d_p} \\
&\to \delta^{d_1}{}_{c_1}\ldots\delta^{d_p}{}_{c_p}\psi^{c_1\ldots c_p}{}_{d_1\ldots d_p} \\
&\to \psi^{a_1\ldots a_p}{}_{a_1\ldots a_p}\ ,
\end{aligned} \tag{209}$$

where we used that $U^TU^* = (U^\dagger U)^* = \mathbb{I}$. Similarly, another invariant under $SU(N)$ is:

$$\epsilon^{a_1\ldots a_N}\psi_{a_1\ldots a_N} \to \epsilon^{a_1\ldots a_N}U_{a_1}{}^{b_1}\ldots U_{a_N}{}^{b_N}\psi_{b_1\ldots b_N} = \epsilon^{b_1\ldots b_N}\psi_{b_1\ldots b_N}\ . \tag{210}$$

The complex conjugate representation can also be obtained as a contraction of the Levi-Civita tensor and a tensor representation with fundamental (lower) indices:

$$\begin{aligned}
\psi^a = \epsilon^{ab_1\ldots b_{N-1}}\chi_{b_1\ldots b_{N-1}} &\to \epsilon^{ab_1\ldots b_{N-1}}U_{b_1}{}^{c_1}\ldots U_{b_{N-1}}{}^{c_{N-1}}\chi_{c_1\ldots c_{N-1}} \\
&\to \epsilon^{ed_1\ldots d_{N-1}}U^{*a}{}_{e}U^{*b_1}{}_{d_1}\ldots U^{*b_{N-1}}{}_{d_{N-1}}U_{b_1}{}^{c_1}\ldots U_{b_{N-1}}{}^{c_{N-1}}\chi_{c_1\ldots c_{N-1}} \\
&\to U^{*a}{}_{e}\epsilon^{ed_1\ldots d_{N-1}}\left(U^\dagger U\right)_{d_1}{}^{c_1}\ldots\left(U^\dagger U\right)_{d_{N-1}}{}^{c_{N-1}}\chi_{c_1\ldots c_{N-1}} \\
&\to U^{*a}{}_{e}\epsilon^{ed_1\ldots d_{N-1}}\chi_{d_1\ldots d_{N-1}} \\
&\to U^{*a}{}_{e}\psi^e\ .
\end{aligned} \tag{211}$$

Hence, we see that the free indices, i.e. those that are not summed over, determine the transformation law for a given tensor. Tensors with no free indices will be invariant under $SU(N)$ transformations.

We would like to determine the irreducible representations within such tensorial products. For this, let us focus on representations with only lower indices, since as we have seen indices can be raised with the Levi-Civita tensor.

Let us consider first the 2-index tensor ψ_{ab}, which we may decompose into its symmetric and anti-symmetric parts:

$$\psi_{ab} = \psi^+_{ab} + \psi^-_{ab}\ , \qquad \psi^\pm_{ab} \equiv \frac{1}{2}\left(\psi_{ab} \pm \psi_{ba}\right)\ . \tag{212}$$

Under $SU(N)$ transformations:

$$\psi^\pm_{ab} \to \tilde{\psi}_{ab} = U_a{}^cU_b{}^d\psi^\pm_{cd} = U_a{}^dU_b{}^c\psi^\pm_{dc} = \pm U_a{}^dU_b{}^c\psi^\pm_{cd} = \pm\tilde{\psi}_{ba}\ , \tag{213}$$

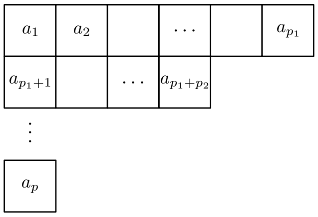
so that (anti-)symmetric tensors are transformed into (anti-)symmetric tensors, and they form invariant sub-spaces. Hence, ψ_{ab} is a reducible representation, while $\psi^\pm_{ab}$ are irreducible representations.

In general, the irreducible representations of $SU(N)$ are in a one-to-one correspondence with the irreducible representations of the permutation group, i.e. a tensor with p indices can be decomposed into irreducible representations of S_p. In the 2-index tensor example above, the permutations of the two indices a and b yield two irreducible representations corresponding to the symmetric and anti-symmetric parts.

The irreducible representations of the permutation group can be mapped into the so-called **Young tableaux**, and this is also one of the most useful descriptions of the $SU(N)$ irreducible representations. Let us consider an $SU(N)$ tensor with p indices $\psi_{a_1\ldots a_p}$ and numbers:

$$p_1 \geq p_2 \geq \ldots \geq p_s\ , \qquad \sum_{i=1}^{s} p_i = p\ . \tag{214}$$

We can then construct the following Young tableau, associated with the tensor $\psi_{a_1 \dots a_{p_1} a_{p_1+1} \dots a_{p_1+p_2} \dots a_p}$, with p_1 boxes in the first line, p_2 boxes in the second, etc:



This tableau is:

- symmetric in the indices appearing in the same line of the tableau;
- anti-symmetric in the indices appearing in the same column of the tableau.

<u>THEOREM</u>: Young tableaux with a number of lines smaller or equal to N are in a one-to-one correspondence with the irreducible representations of $SU(N)$.

To compute the dimension of a representation from the associated Young tableau, we need to consider **standard tableaux**, which correspond to inserting the indices $1, \dots, N$ in the boxes of Young tableaux such that:

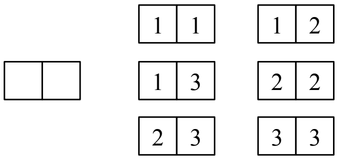
- indices do not decrease from left to right within each row;
- indices increase from top to bottom in each column.

The number of standard tableaux that we can construct then yields the dimension of the representation. The following table illustrates the Young tableaux and the standard tableaux associated with the simplest representations.

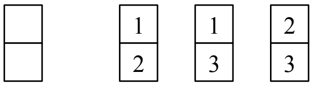
Representation	**Tensor**	**Young tableau**	**Standard tableaux**	**Dimension**
Fundamental $\mathbf{N}$	ψ_a	$\boxed{}$	$\boxed{1} \cdots \boxed{N}$	N
Symmetric $\mathbf{N} \times \mathbf{N}$	$\psi_{(ab)}$	$\boxed{}\boxed{}$	$\boxed{1}\boxed{1} \cdots \boxed{1}\boxed{N}$ $\boxed{2}\boxed{2} \cdots$	$\frac{1}{2}N(N+1)$
Anti-symmetric $\mathbf{N} \times \mathbf{N}$	$\psi_{[ab]}$	$\begin{array}{c}\boxed{}\\\boxed{}\end{array}$	$\begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \cdots \begin{array}{c}\boxed{1}\\\boxed{N}\end{array}$	$\frac{1}{2}N(N-1)$
Symmetric $\mathbf{N}^k$	$\psi_{(a_1 \dots a_k)}$	$\boxed{}\boxed{\cdots}\boxed{}$	$\boxed{1}\boxed{\cdots}\boxed{1}$ $\boxed{1}\boxed{\cdots}\boxed{2}$ $\vdots$	$\binom{N+k+1}{k}$
Anti-symmetric $\mathbf{N}^k$	$\psi_{[a_1 \dots a_k]}$	$\begin{array}{c}\boxed{}\\\vdots\\\boxed{}\end{array}$	$\begin{array}{c}\boxed{1}\\\vdots\\\boxed{k}\end{array} \cdots \begin{array}{c}\boxed{}\\\vdots\\\boxed{}\end{array}$	$\binom{N}{k}$

Consider, for example, the case of $SU(3)$, for which the symmetric and anti-symmetric representations are given by:

$\mathbf{6} =$

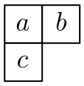


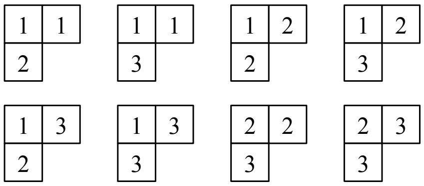
$\bar{\mathbf{3}} =$



Notice that the anti-symmetric representation in $SU(3)$ corresponds to the complex conjugate representation, also denoted as **anti-fundamental representation**, since we can write $\psi^a = \epsilon^{abc}\psi_{bc}$. The adjoint representation of $SU(3)$, which has the dimension of the Lie algebra, $3^2 - 1 = 8$, is given by the following Young tableau:

$\mathbf{8} =$

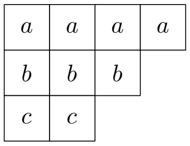




This corresponds to a tensor symmetric in the a and b indices and anti-symmetric in the a and c indices, which can be written as the following combination of permutations of the 3-index tensor:

$$\frac{1}{4}\left(\psi_{abc} + \psi_{bac} - \psi_{cba} - \psi_{bca}\right) \ . \tag{215}$$

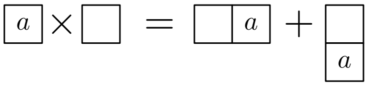
Young tableaux are particularly useful in decomposing tensor products of two $SU(N)$ representations into a sum of irreducible representations. To do this, we start by writing the labels $a, b, c, \ldots$ in the boxes of the Young tableau of the first representation, with a symbols in the first row, b symbols in the second row, etc, as illustrated in the figure below:



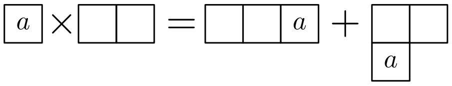
We then attach the boxes of the first diagram to the second diagram, starting with the boxes with a label and then the boxes with b label, etc, such that we get new Young tableaux where equal labels cannot appear in the same column. In addition, for each new Young tableau that we obtain with this procedure, we must read all the labels from *right to left* in the first row and then in the second row, etc. This yields a sequence of labels for each new tableau where, to the left of each label, the number of b labels cannot exceed the number of a labels, the number of c labels cannot exceed the number of b labels, etc. This means, for example, that sequences aab and aba are allowed, while $abba$ is not allowed.

This procedure is best understood by considering a few examples, so let us consider the products of the lowest dimension irreducible representations in $SU(3)$:

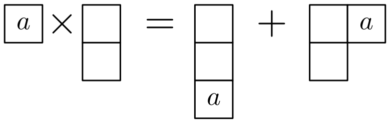
$$\mathbf{3}\times\mathbf{3}=\mathbf{6}+\bar{\mathbf{3}}$$



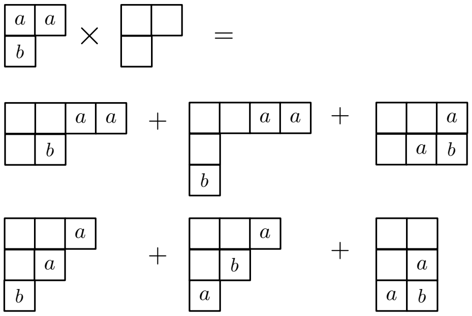
$$\mathbf{3}\times\mathbf{6}=\mathbf{10}+\mathbf{8}$$



$$\mathbf{3}\times\bar{\mathbf{3}}=\mathbf{1}+\mathbf{8}$$

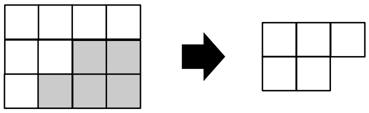


$$\mathbf{8}\times\mathbf{8}=\mathbf{27}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{8}+\mathbf{8}+\mathbf{1}$$



Note that in $SU(3)$ a column with 3 boxes corresponds to the singlet representation $\mathbf{1}$, i.e. an invariant tensor, since there is only one standard tableau with numbers $(1,2,3)$. This means that in determining the dimension of a representation we may ignore all columns with 3 boxes. The same holds for columns with N boxes in a general $SU(N)$ representation.

Also note that in the last example above there are two representations with dimension 10, which we have labeled as $\mathbf{10}$ and $\overline{\mathbf{10}}$, so that they are complex conjugate. Which one is the conjugate representation, denoted with a bar, is a matter of convention, but looking at the corresponding Young tableaux one sees that, putting them together, one obtains a tableau with three rows (ignoring the singlet column as explained above). The same holds for $\mathbf{3}$ and $\bar{\mathbf{3}}$. This is actually a generic feature of complex conjugate representations: for each $SU(N)$ representation, its complex conjugate can be obtained by looking at the completion of its Young tableau that yields N rows and the same number of columns, as illustrated below. For the example of $SU(3)$ above, we can clearly see that the complex conjugate of the adjoint representation is the adjoint representation itself, and this is true for all $SU(N)$ - the adjoint representation is said to be a real representation.



3.2 $SU(3)$ and the quark model

$SU(N)$ groups play an important role in particle physics, being the basis for the gauge theories on which the Standard Model is built. The first and simplest application of these groups, other than $SU(2)$ for the spin of elementary particles as we have already discussed, is the **quark model** to describe the meson and baryon particles in terms of elementary constituents, known as the quarks. The lightest mesons and baryons can be described in terms of three elementary quarks, known as up, down and strange, although now we know that there are actually six quark "flavours" with the addition of the charm, bottom and top quarks. The similarity between the masses of certain groups of mesons and baryons suggests an underlying symmetry between the quark constituents, and for the case of the three lightest quarks this is naturally $SU(3)$ as we will now describe in more detail.

The main idea is that the three lightest elementary quarks are related by $SU(3)$ transformations, forming a vector space that transforms in the fundamental representation of this group. Their corresponding anti-particles then naturally transform in the anti-fundamental (or complex conjugate) representation, and composite states as mesons and baryons transform in tensor products of these representations, which as we have seen above can be decomposed into irreducible representations. Mesons and baryons in the same irreducible representation of $SU(3)$ should have equal masses since they should be related by $SU(3)$ transformations, but actually their masses are slightly different due to the fact that the elementary quarks themselves have different masses and charges. Note that the mass, i.e. the rest energy, of a composite state depends not only on the mass of its constituents but also on their interaction energy, which include in the case of quarks their strong, weak and electromagnetic interactions. Hence, the $SU(3)$ flavour symmetry, and its generalization to $SU(6)$ when including all the known elementary quarks, is only an approximate symmetry, which nevertheless explains very well the spectrum of meson and baryon masses.

Let us start by discussing the $SU(3)$ Lie algebra, which corresponds to the hermitian traceless 3×3 matrices. The most widely used basis for this algebra is given by the **Gell-Mann matrices**, after Murray Gell-Mann who, along with George Zweig, independently proposed the quark model, also known as the "Eightfold way", in 1964. These are given by:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .
\end{aligned} \tag{216}$$

One can naturally see in these matrices the structure found in the Pauli matrices, and in fact there are three $SU(2)$ sub-algebras in $SU3$):

$$SU(2)_I = \text{span}\{\lambda_1, \lambda_2, \lambda_3\} , \qquad SU(2)_V = \text{span}\left\{\lambda_4, \lambda_5, \frac{\lambda_3 + \sqrt{3}\lambda_8}{2}\right\} , \qquad SU(2)_U = \text{span}\left\{\lambda_6, \lambda_7, \frac{\sqrt{3}\lambda_8 - \lambda_3}{2}\right\}. \tag{217}$$

The Gell-Mann matrices satisfy $\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$ and conventionally we choose the basis $T_i = \lambda_i/2$, such that:

$$\text{Tr}(T_i T_j) = \frac{1}{2}\delta_{ij} , \tag{218}$$

as we have used for $SU(2)$ as well. There are two diagonal generators in the rank-2 $SU(3)$ algebra, and the Cartan sub-algebra can be chosen as:

$$\mathcal{H}\left(SU(3)\right) = \text{span}\left\{T_3, Y = \frac{2}{\sqrt{3}}T_8\right\} , \tag{219}$$

where T_3 is known as the **isospin** generator and Y as the **hypercharge** generator. We also define:

$$Q \equiv T_3 + \frac{Y}{2} \ , \tag{220}$$

which as we will see below corresponds to the **electric charge** of the quarks.

To complete the Weyl-Cartan basis for $SU(3)$, we define the ladder operators:

$$T_\pm = T_1 \pm iT_2 \ , \qquad V_\pm = T_4 \pm iT_5 \ , \qquad U_\pm = T_6 \pm iT_7 \ , \tag{221}$$

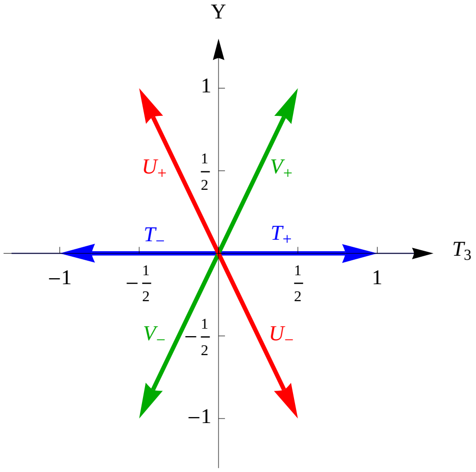
which satisfy the following commutation relations with the diagonal generators of the Cartan sub-algebra:

$$\begin{aligned} [T_3, T_\pm] &= \pm T_\pm \ , & [T_3, V_\pm] &= \pm\tfrac{1}{2} V_\pm \ , & [T_3, U_\pm] &= \mp\frac{1}{2} U_\pm \ , \\ [Y, T_\pm] &= \pm 0 \ , & [Y, V_\pm] &= \pm V_\pm \ , & [Y, U_\pm] &= \pm U_\pm \ . \end{aligned} \tag{222}$$

From these we can immediately infer the roots of the $SU(3)$ Lie algebra:

$$\alpha(T_\pm) = (\pm 1, 0) \ , \qquad \alpha(V_\pm) = (\pm 1/2, \pm 1) \ , \qquad \alpha(U_\pm) = (\mp 1/2, \pm 1) \ . \tag{223}$$

This means that $T_\pm$ raise/lower the isospin of states in a representation by one unit, while $V_\pm$ and $V_\pm$ either raise or lower the isospin by 1/2. While $T_\pm$ do not change the hypercharge of a state, both $V_\pm$ and $U_\pm$ change the hypercharge by ±1. This is illustrated in the following **root diagram**:



Note that the root diagram for $SU(2)$ would be 1-dimensional, corresponding e.g. to the pair (T_+, T_-).

The **fundamental representation** of $SU(3)$, **3**, in the quark model has a basis of states $|u\rangle$, $|d\rangle$ and $|s\rangle$ corresponding to the the **up**, **down** and **strange** quarks, respectively. These correspond to the basis of vectors $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$. The generators in this representation are given by the T_i normalized Gell-Mann matrices defined above, and the Cartan sub-algebra is given by:

$$T_3 = \mathrm{diag}(1/2, -1/2, 0) \ , \qquad Y = \mathrm{diag}(1/3, 1/3, -2/3) \ , \tag{224}$$

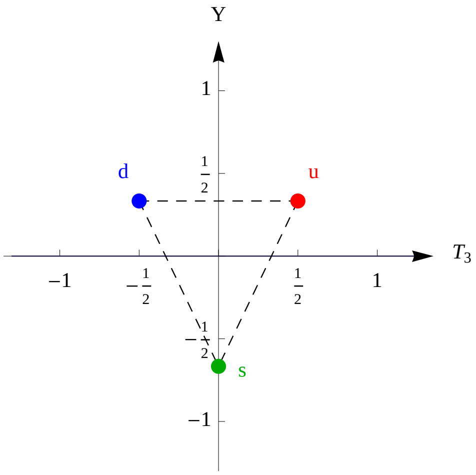
which implies that the up and down quarks form an isospin doublet, while the strange quark is an isospin singlet. While u and s have hypercharge 1/3, the d quark has hypercharge $-2/3$. This yields:

$$Q = \mathrm{diag}(2/3, -1/3, -1/3) \ , \tag{225}$$

so that the up quark has a $+2/3$ electric charge, while the down and strange quarks have charge $-1/3$. The weights of the fundamental representation are then:

$$\lambda(u) = (1/2, 1/3) \ , \qquad \lambda(d) = (-1/2, 1/3) \ , \qquad \lambda(s) = (0, -2/3) \ , \tag{226}$$

as illustrated in the **weight diagram** below.



It should be clear in this figure that the ladder operators can be used to obtain different states in this representation. In particular, the $T_\pm$ generators exchange $u \leftrightarrow d$, the $V_\pm$ generators exchange $u \leftrightarrow s$ and the $U\pm$ generators exchange $d \leftrightarrow s$.

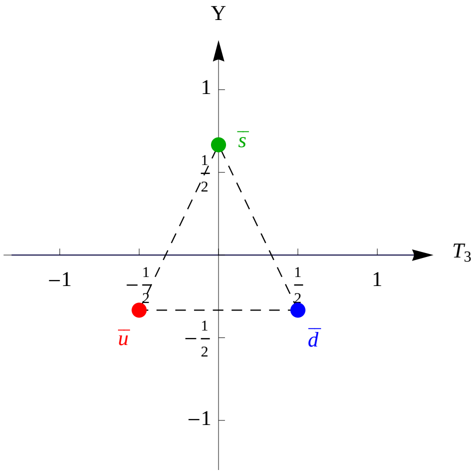
The **complex conjugate representation $\overline{\mathbf{3}}$** has a basis of states $|\bar{u}\rangle$, $|\bar{d}\rangle$ and $|\bar{s}\rangle$, corresponding to the anti-quark states. Recalling that $U = e^{iT}$, we have $U^* = e^{-iT^*}$, so that the generators in the $\overline{\mathbf{3}}$ representation are $\overline{T}_i = -T_i^*$. In particular, in the Cartan sub-algebra we have:

$$\bar{T}_3 = \mathrm{diag}(-1/2, 1/2, 0) \ , \qquad \bar{Y} = \mathrm{diag}(-1/3, -1/3, 2/3) \ , \qquad \bar{Q} = \mathrm{diag}(-2/3, 1/3, 1/3) \ , \tag{227}$$

so that the anti-quarks have opposite electric charge to the corresponding quarks. The weights in this representation are:

$$\lambda(\bar{u}) = (-1/2, -1/3) \ , \qquad \lambda(d) = (1/2, -1/3) \ , \qquad \lambda(s) = (0, 2/3) \ , \tag{228}$$

and the weight diagram is given by:

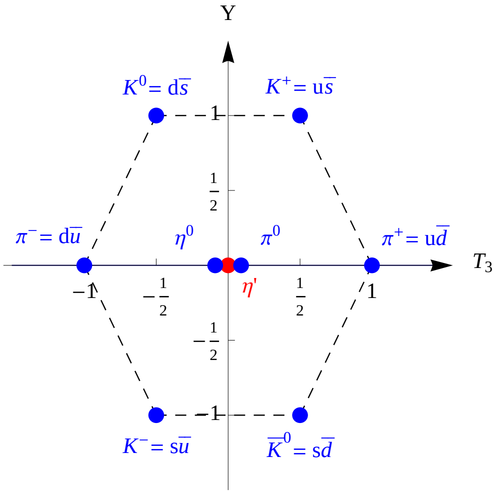


We can now look at composite states, generically known as **hadrons**. The simplest states are the **mesons**, which are spin-0 bound states of a quark and an anti-quark. Note that quarks are spin-1/2 fermions, corresponding to Dirac spinors. This means that the quark and anti-quark in a given meson must have opposite spins. In terms of $SU(3)$, mesons must lie in the tensor product representation $\mathbf{3}\times\overline{\mathbf{3}}=\mathbf{8}+\mathbf{1}$ as we have obtained above. We hence expect to find an octet of mesons with similar masses, transforming in the adjoint representation, and an additional $SU(3)$ singlet meson.

As we have discussed in the first section, the weights in a tensor product representation correspond to the sum of the weights in the individual representations. We then have the following basis of states in the $\mathbf{3}\times\overline{\mathbf{3}}$ representation:

States	Weights
$\lvert u\bar{u}\rangle$, $\lvert d\bar{d}\rangle$, $\lvert s\bar{s}\rangle$	$(0,0)$
$\lvert u\bar{d}\rangle$	$(1,0)$
$\lvert u\bar{s}\rangle$	$(1/2,1)$
$\lvert d\bar{u}\rangle$	$(-1,0)$
$\lvert d\bar{s}\rangle$	$(-1/2,1)$
$\lvert s\bar{u}\rangle$	$(-1/2,-1)$
$\lvert s\bar{d}\rangle$	$(1/2,-1)$

The associated weight diagram is illustrated below, including the names given to the corresponding mesons. Note that the electric charge of each meson, being a linear combination of the weights, is also the sum of the electric charges of its quark and anti-quark constituents.



As one can see, there are three degenerate states with $(0,0)$ weight, corresponding to $\lvert u\bar{u}\rangle$, $\lvert d\bar{d}\rangle$ and $\lvert s\bar{s}\rangle$, but only two linear combinations of these states belong to the meson octet, while the remaining must correspond to the singlet meson. To determine which states are in the octet, we note that such states should be obtained from the remainder states in the octet by applying $SU(3)$ transformations. Since this can be achieved using the ladder operators, we may, for example, determine $T_-\lvert u\bar{d}\rangle$ and $U_-\lvert d\bar{s}\rangle$. Note that these generators take the following form in the fundamental and anti-fundamental representations:

$$\begin{aligned}
T_- &= T_1 - iT_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \bar{T}_- = -T_1 - iT_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
U_- &= T_6 - iT_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \bar{U}_- = -T_6 - iT_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{229}$$

such that:

$$\begin{aligned} T_-|u\bar{d}\rangle &= (T_-|u\rangle)\otimes|\bar{d}\rangle + |u\rangle\otimes(\bar{T}_-|\bar{d}\rangle) = |d\bar{d}\rangle - |u\bar{u}\rangle \ , \\ U_-|d\bar{s}\rangle &= (U_-|d\rangle)\otimes|\bar{s}\rangle + |d\rangle\otimes(\bar{U}_-|\bar{s}\rangle) = |s\bar{s}\rangle - |d\bar{d}\rangle \ . \end{aligned} \tag{230}$$

These states are not, however, orthogonal, but we can choose the following orthogonal linear combinations:

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right) \ , \qquad |\eta^0\rangle = \frac{1}{\sqrt{6}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle\right) \ , \tag{231}$$

which thus yield the two $(0,0)$ mesons in the octet. The singlet meson must be a linear combination of $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ that is orthogonal to both $|\pi^0\rangle$ and $|\eta^0\rangle$, yielding:

$$|\eta'\rangle = \frac{1}{\sqrt{3}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle\right) \ , \tag{232}$$

and it is easy to see that this state remains invariant under $SU(3)$ transformations. i.e. under the exchange of any pair of quarks.

The other quark bound states occurring in Nature are the **baryons**, which are composed of 3 quarks, with the corresponding anti-baryons having 3 anti-quarks. Baryons must then transform in the tensor representation:

$$\mathbf{3}\times\mathbf{3}\times\mathbf{3} = (\mathbf{6}+\overline{\mathbf{3}})\times\mathbf{3} = \mathbf{6}\times\mathbf{3}+\overline{\mathbf{3}}\times\mathbf{3} = \mathbf{10}+\mathbf{8}+\mathbf{8}+\mathbf{1} \ , \tag{233}$$

where we have used the $SU(3)$ tensor products obtained earlier using Young tableaux. Not all of these irreducible representations occur in Nature, since quarks are spin-1/2 fermions that also carry an additional $SU(3)$ “colour” charge, which is distinct from the $SU(3)$ flavour symmetry that we are considering. Due to Pauli’s exclusion principle, the total wavefunction of a composite fermionic state must be anti-symmetric under the exchange of any two identical fermions, so that they cannot simultaneously be in the same state. This singles out the representations **10** and only one of the adjoint **8** representations obtained above. The baryon decuplet includes the spin-3/2 baryons, while the baryon octet contains the states with total spin 1/2.

Although we will not study in detail the states in these representations, one should note that the most important baryons are the proton and the neutron, which have spin-1/2. These are the most stable of the baryons, since the others are short-lived and can decay into other baryons and mesons. The proton is a uud state with weight vector $(1/2, 1)$, and is thought to be completely stable, while the neutron is a udd state with weight vector $(-1/2, 1)$, and is only stable when forming bound states with other neutrons and protons inside atomic nuclei. The electric charge of the proton is then $+1$, while the neutron has zero electric charge according to our definition of Q.

3.3 Branching $SU(3)$ representations into $SU(2)$ representations

A common feature to $SU(N)$ groups and also other Lie groups used in particle physics is that they include lower rank sub-groups, and it is useful to understand how the representations of the sub-group are embedded into the larger $SU(N)$ representations. We will consider the example of $SU(3)$, which as we have seen above includes the group $SU(2)$ in three possible ways. The simplest possibility is $SU(2)_I$, which corresponds to the sub-group:

$$U = \begin{pmatrix} U_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \in SU(3) \ , \tag{234}$$

where the 2×2 matrix $U_2 \in SU(2)$. It is clear that this embedding leaves the strange quark invariant, while performing $SU(2)$ transformations in the sub-space spanned by the $|u\rangle$ and $|d\rangle$ states in the fundamental representation. These then form an $SU(2)$-isospin doublet, so that we have the decomposition:

$$\mathbf{3}_{SU(3)} \to (\mathbf{2}+\mathbf{1})_{SU(2)_I} \tag{235}$$

Similarly, for the complex conjugate representation:

$$\overline{\mathbf{3}}_{SU(3)} \to (\overline{\mathbf{2}}+\mathbf{1})_{SU(2)_I} \tag{236}$$

where we recall that the fundamental $\mathbf{2}$ and complex conjugate $\overline{\mathbf{2}}$ representations of $SU(2)$ are equivalent as we derived in the first section.

Knowing how the fundamental and anti-fundamental representations of $SU(3)$ branch into $SU(2)$-isospin representations then allows us to do the branching of other representations. For the case of mesons, for example, we have:

$$\begin{aligned} \left(\mathbf{3}\times\overline{\mathbf{3}}\right)_{SU(3)} = (\mathbf{8}+\mathbf{1})_{SU(3)} &\to (\mathbf{2}+\mathbf{1})\times(\mathbf{2}+\mathbf{1})_{SU(2)_I} \\ &\to (\mathbf{2}\times\mathbf{2}+\mathbf{2}\times\mathbf{1}+\mathbf{1}\times\mathbf{2}+\mathbf{1}\times\mathbf{1})_{SU(2)_I} \\ &\to (\mathbf{3}+\mathbf{1}+\mathbf{2}+\mathbf{2}+\mathbf{1})_{SU(2)_I} \ . \end{aligned} \tag{237}$$

Since an $SU(3)$ singlet will also be an $SU(2)_I$ singlet, we then conclude that the meson octet in the adjoint representation has the following branching:

$$\mathbf{8}_{SU(3)} \to (\mathbf{3}+\mathbf{2}+\mathbf{2}+\mathbf{1})_{SU(2)_I} \ . \tag{238}$$

Each of these multiplets must have a constant hypercharge Y, since the only diagonal generator leading to matrices of the form (234) is T_3. These then correspond to the mesons along the same horizontal lines in the weight diagram given earlier:

$$\begin{aligned} \mathbf{3} &= (\pi^+,\pi^0,\pi^-)_{Y=0} \ , \\ \mathbf{2} &= (K^+,K^0)_{Y=1} \ , \\ \mathbf{2} &= (K^-,\bar{K}^0)_{Y=-1} \ , \\ \mathbf{1} &= (\eta^0)_{Y=0} \ . \end{aligned} \tag{239}$$

The pions then form an isospin triplet, which corresponds to the isospin-1 adjoint representation of $SU(2)_I$, while the kaons form two isospin doublets with opposite hypercharges. That the η^0 is an isospin singlet should be clear from its explicit form in (231), since it is invariant under the exchange of the u and d quarks and of their anti-particles, the same happening for the $SU(3)$ singlet state η'.

Note that the isospin symmetry is actually a much better symmetry than the full $SU(3)$ flavour symmetry since, while the up and down quarks have similar masses of a few MeV/c^2, the strange quark is significantly heavier, with a mass of around $\sim$ 140 MeV/c^2. The isospin triplet pions then have similar masses of around 140 MeV/c^2, while the remaining octet mesons with a strange quark have masses around 550 MeV/c^2. The η', which is an $SU(3)$ and isospin singlet, has a much larger mass of $\sim$ 956 MeV/c^2.

4 Problems

1. Consider the following set of 4×4 matrices:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad A_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

a) Justify that these matrices form a basis for the Lie algebra of the group $SO(4)$.

b) Verify, for $i = 1$ and $j = 2$, the following commutation relations:

$$[A_i, A_j] = \epsilon_{ijk} A_k \ , \qquad [B_i, B_j] = \epsilon_{ijk} A_k \ , \qquad [A_i, B_j] = \epsilon_{ijk} B_k$$

c) Consider now the following linear combinations of generators:

$$X_i = \frac{1}{2}(A_i + B_i) \ , \qquad Y_i = \frac{1}{2}(A_i - B_i) \ , \qquad i = 1, 2, 3$$

Show that these span two independent sub-algebras, i.e. two commutting algebras, and indicate the Lie group to which each of these algebras is associated. Is the Lie algebra of $SO(4)$ semi-simple?

d) Compute the Killing form and the Casimir operator for each of the sub-algebras determined in the previous question.
Hint: Use the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

2. The sympletic group $Sp(2, \mathbb{R})$ is defined as the group of 2×2 matrices:

$$Sp(2, \mathbb{R}) = \left\{ S \in GL(\mathbb{R}^2) : \ S^T J S = J \right\}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

a) Explain why these matrices form a group under the usual matrix multiplication rule.

b) Given that λ is one of the eigenvalues of a matrix in this group, determine the other eigenvalue.

c) Justify that the following matrices form a basis for the Lie algebra of $Sp(2,\mathbb{R})$ and determine the associated non-vanishing structure constants:

$$T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \qquad T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix} .$$

d) Construct the Weyl-Cartan basis from the matrices given above and plot in the same diagram the weights of the fundamental representation and the roots of the Lie algebra.

e) Determine the components of the Killing metric in the basis $\{T_1, T_2, T_3\}$.

3. Consider the following set of 4×4 matrices:

$$A_a = \begin{pmatrix} \mathbf{0} & \sigma_a \\ \sigma_a & \mathbf{0} \ , \end{pmatrix} , \qquad B_a = \begin{pmatrix} \sigma_a & \mathbf{0} \\ \mathbf{0} & -\sigma_a \ , \end{pmatrix} , \qquad C_a = \begin{pmatrix} \sigma_a & \mathbf{0} \\ \mathbf{0} & \sigma_a \ , \end{pmatrix} , \qquad D = \begin{pmatrix} \mathbf{0} & -i\mathbb{I} \\ i\mathbb{I} & \mathbf{0} \ , \end{pmatrix}$$

where $\mathbf{0}$ and $\mathbb{I}$ denote the zero matrix and the 2×2 identity matrix, respectively, and σ_a, $a = 1, 2, 3$, are the Pauli matrices, given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}$$

a) Show that these matrices form a closed Lie algebra.
Hint: Use the relation $\sigma_a \sigma_b = \mathbb{I}\delta_{ab} + i\epsilon_{abc}\sigma_c$.

b) Taking $H_1 = B_3$ and $H_2 = C_3$ as the generators of the Cartan sub-algebra, determine the weight diagram of the fundamental representation.

c) Show that the matrices $E_\alpha = \{A_\pm, D_\pm, F_\pm, G_\pm\}$ ccomplete the Weyl-Cartan basis for this Lie algebra, where:

$$\begin{aligned} A_\pm &= A_1 \pm iA_2 \ , \qquad D_\pm = A_3 \pm iD \ , \qquad B_\pm = B_1 \pm iB_2 \ , \qquad C_\pm = C_1 \pm iC_2 \\ F_\pm &= B_\pm + C_\pm \ , \qquad G_\pm = B_\pm - C_\pm \end{aligned}$$

and plot the associated root diagram.

4. The Lie group $SL(2,\mathbb{C})$ corresponds to the group of complex 2×2 matrices with unit determinant:

$$SL(2,\mathbb{C}) = \{M \in M(\mathbb{C}) : \det(M) = 1\} \ . \tag{240}$$

a) Justify that the following matrices form a basis for the $SL(2,\mathbb{C})$ Lie algebra:

$$\begin{gathered} \Sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \qquad \Sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \Sigma_3 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} , \\ \sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \qquad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{gathered} \tag{241}$$

Note: You do not need to compute the commutation relations explicitly.

b) From the set of matrices in the previous question, identify the algebra of $SL(2,\mathbb{R})$, where $SL(2,\mathbb{R})$ corresponds to the group of real 2×2 matrices with unit determinant. Compute the associated Killing form.

c) Consider the sub-algebra spanned by the matrices:

$$T_1 = \Sigma_3 \ , \qquad T_2 = \sigma_1 \ , \qquad T_3 = \sigma_2 \ . \tag{242}$$

Show that these matrices satisfy:

$$T_i^\dagger = -\eta T_i \eta^{-1} \ , \tag{243}$$

where $\eta = \mathrm{diag}(1,-1)$. From this result deduce the analogous relation between $U^\dagger$ e U^{-1}, where $U = \exp\left(\sum_{i=1}^3 \alpha_i T_i\right)$ is a matrix in the Lie group $SU(1,1)$ for $\alpha_i \in \mathbb{R}$.

d) Consider now the set of hermitian 2×2 matrices defined as:

$$X = \{2x^\mu \sigma_\mu : \ x^\mu \in \mathbb{R} \ , \mu = 0,1,2,3\} \ ,$$

where $2\sigma_0$ is the 2×2 identity matrix.

Show that $\det X = \eta_{\mu\nu} x^\mu x^\nu$ onde $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric and verify that this relation is invariant under $SL(2,\mathbb{C})$ transformations of the form $X \to X' = MXM^\dagger$ where $M \in SL(2,\mathbb{C})$.

5. A Dirac spinor ψ transforms in the $(1/2,0)\oplus(0,1/2)$ representation of the Lorentz group and can be written in the form:

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} ,$$

where χ_L is a left-handed Weyl spinor and χ_R a right-handed Weyl spinor. The transformation matrices in the Dirac representation are then given by:

$$\Lambda_D(M) = \begin{pmatrix} \Lambda_L(M) & 0 \\ 0 & \Lambda_R(M) \end{pmatrix} ,$$

where $M \in SL(2,\mathbb{C})$ such that:

$$\Lambda_L(M) = e^{-\frac{1}{2}(s^i+it^i)\sigma_i} , \qquad \Lambda_R(M) = \left(\Lambda_L(M)^{-1}\right)^\dagger ,$$

where s^i, t^i are real parameters and σ_i are the Pauli matrices.

Define the Dirac gamma matrices as:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} ,$$

where $\sigma_\mu = (\mathbb{I}_2, \sigma_i)$ and $\bar{\sigma}_\mu = (\mathbb{I}_2, -\sigma_i) = \sigma_2\sigma_\mu^T\sigma_2$.

a) Show that for an infinitesimal Lorentz transformation $\psi \to \psi + \delta\psi$ one has

$$\delta\psi = i\epsilon^{\mu\nu}\sigma_{\mu\nu}\psi ,$$

where $\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and $\epsilon^{\mu\nu}$ is an infinitesimal anti-symmetric tensor.

b) Show that the spinor bilinear

$$\bar{\psi}\psi ,$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$, is invariant under finite Lorentz transformations.

c) Using the relation between $SL(2,\mathbb{C})$ and Lorentz group matrices

$$\sigma_\mu\Lambda^\mu{}_\nu = M^\dagger\sigma_\nu M ,$$

show that:

$$\Lambda_D(M)^{-1}\gamma_\mu\Lambda_D(M) = \Lambda_\mu{}^\nu\gamma_\nu$$

d) Use the previous result to determine how the following spinor bilinear:

$$\bar{\psi}\gamma_\mu\psi$$

transforms under the Lorentz group.

Hint: Use the following relations:

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k , \qquad \epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij} ,$$

$$\Lambda_{L,R}(M)^* = \sigma_2\Lambda_{R,L}\sigma_2 .$$

6. Consider the group $SU(3)$.

a) Verify the following identity for an arbitrary 3×3 matrix:

$$U_a{}^i U_b{}^j U_c{}^k \epsilon_{ijk} = \det(U)\epsilon_{abc}$$

for $a = 1$, $b = 2$ and $c = 3$. Use this result to show that the Levi-Civita tensor is invariant under $SU(3)$.

Assuming the transformation law for an $SU(3)$ tensor ψ^{ab}, verify that $\tilde{\psi}_a = \epsilon_{abc}\psi^{bc}$ transforms in the fundamental representation of the group.

b) Using Young tableaux, decompose the following tensor product into irreducible representations of $SU(3)$:

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3}$$

For each of the irreducible representations obtained, indicate the symmetry properties of the associated tensor.

c) In the $SU(3)$ quark model, the fundamental representation $q_i = (u, d, s)^T$ has weights $u = (1/2, 1/3)$, $d = (-1/2, 1/3)$ and $s = (0, -2/3)$ with respect to the isospin and hypercharge generators (T_3, Y) of the Cartan sub-algebra.

Determine the weight diagram for the baryon decuplet and write the corresponding normalized states in the basis $|q_i q_j q_k\rangle$.

Hint: Construct the Young tableau associated with the **10** representation of $SU(3)$ to determine the symmetry properties of the associated states.

d) Show that the trace of the electric charge generator $Q = T_3 + Y/2$ vanishes for the baryon decuplet.

Note: The trace of an operator O in a vector space spanned by states $\{|\psi_i\rangle\}$ is given by:

$$\mathrm{Tr}(O) = \sum_i \langle\psi_i|O|\psi_i\rangle$$

7. Consider the group $SU(3)$.

a) Using Young tableaux, decompose the following tensor products in terms of irreducible representations:

$$\mathbf{3} \times \bar{\mathbf{3}}$$

$$\mathbf{8} \times \mathbf{3}$$

$$\bar{\mathbf{6}} \times \bar{\mathbf{3}}$$

Note that the **6** representation corresponds to the symmetric part of the tensor product $\mathbf{3} \times \mathbf{3}$.

b) In Quantum Chromodynamics (QCD), each quark transforms in the fundamental representation of the group $SU(3)_c$, forming a “colour” triplet (q_r, q_g, q_b) (for “red”, “green” and “blue”) with weights $q_r = (1/2, 1/3)$,

$q_g = (-1/2, 1/3)$ and $q_b = (0, -2/3)$ relative to the isospin and hypercharge generators (T_3, Y) of the Cartan sub-algebra. Each state q_i, $i = r, g, b$ corresponds to a Dirac spinor composed of two Weyl spinors q_{Li} and q_{Ri} in the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group, respectively.

Gluons, on the other hand, are described by 4-vectors A^μ that transform in the adjoint representation of $SU(3)_c$.

b1) Justify that the following interaction between quarks and gluons

$$g q_L^\dagger \sigma_\mu q_L A^\mu \ ,$$

where we have omitted the $SU(3)_c$ indices for simplicity and g is a coupling constant, is invariant under both Lorentz transformations and $SU(3)_c$ transformations. Justify also that the mass term:

$$m_q q_L^\dagger q_R \ ,$$

where m_q is a constant, is invariant under both types of transformation.

Hint: Recall the relation $M^\dagger \sigma_\mu M = \Lambda^\nu{}_\mu \sigma_\nu$ between $M \in SL(2, \mathbb{C})$ and the matrices Λ in the Lorentz group. Consider also the tensor products of $SU(3)_c$ representations involved in each term.

b2) Justify that gluons can be seen as "colour−anti-colour" composite states and describe them in terms of linearly independent states in the basis $|i\bar{j}\rangle$, $i, j = r, g, b$:

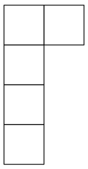
$$|G^a\rangle = \sum_{i,j} A^a_{ij} |i\bar{j}\rangle \ , \qquad a = 1, \ldots, \dim(Ad)$$

Justify, in particular, that the matrices A^a must satisfy the same properties of the generators of the Lie algebra of $SU(3)_c$ and identify the state(s) in this basis that do not describe gluons.

8. a) Indicate the transformation properties of the following tensors under a group $SU(N)$:

$$\psi_{ab}{}^{bc} \ , \qquad \tilde{\psi}_{abc}{}^{abc} \tag{244}$$

b) Compute the dimension of the $SU(N)$ irreducible representation associated with the following Young tableau for $N = 4, 5$ e 6.



Also indicate the symmetry properties of the associated tensor.

c) In the $SU(3)$ quark model, the fundamental representation $q_i = (u, d, s)^T$ has weights $u = (1/2, 1/3)$, $d = (-1/2, 1/3)$ and $s = (0, -2/3)$ relative to the isospin and hypercharge generators (T_3, Y) of the Cartan sub-algebra.

Determine the weight diagram for the irreducible representations in the following tensor products:

$$\mathbf{3}\times\mathbf{3}\ , \qquad \mathbf{3}\times\bar{\mathbf{3}} \tag{245}$$

d) Assuming that the singlet meson is described by the state:

$$|\eta'\rangle = \frac{1}{\sqrt{3}}\left(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle\right) \tag{246}$$

and that there is only one other flavourless meson that includes the strange quark in its composition, determine the states in the irreducible representations obtained in the previous question.

9. Consider the group $SU(5)$ and its Lie algebra.

a) Determine the Young tableaux associated with the following irreducible representations:

$$\mathbf{1}\ ,\mathbf{5}\ ,\bar{\mathbf{5}}\ ,\mathbf{10}\ ,\mathbf{15}$$

b) Using Young tableaux, decompose the following tensor products into irreducible representations of the group:

$$\mathbf{5}\times\mathbf{5}\ , \qquad \mathbf{5}\times\bar{\mathbf{5}}\ , \qquad \bar{\mathbf{5}}\times\mathbf{10}$$

c) The group $SU(3)\times SU(2)$ can be embedded as a sub-group of $SU(5)$ such that, for $U_3 \in$ SU(3) and $U_2 \in$ SU(2):

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} \in \text{ SU}(5)\ .$$

Determine the decomposition of the irreducible representations $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{24}$ of $SU(5)$ in terms of irreducible representations of $SU(3)\times SU(2)$.

10. Consider the group $SU(5)$ and its Lie algebra.

a) Using Young tableaux, decompose the following tensor products into irreducible representations:

$$\mathbf{5}\times\bar{\mathbf{5}}$$
$$\mathbf{5}\times\mathbf{5}$$
$$\bar{\mathbf{5}}\times\mathbf{10}$$

b) Identify the Young tableau corresponding to the adjoint representation from amongst the Young tableaux obtained in the previous question.

c) The group $SU(3) \times SU(2)$ can be embedded in $SU(5)$ as:

$$U_5 = \begin{pmatrix} U_3 & 0 \\ 0 & U_2 \end{pmatrix} ,$$

where $U_n \in SU(n)$. Determine the branching of the irreducible representations $\mathbf{5}$, $\bar{\mathbf{5}}$, $\mathbf{10}$ and $\mathbf{15}$ of $SU(5)$ in irreducible representations of $SU(3) \times SU(2)$ of the form $(R_{SU(3)}, R'_{SU(2)})$.

c) Is the Lie algebra of $SU(3) \times SU(2)$ simple? Justify your answer.