Introduction to Cosmology

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LECTURE 9 - Inflation I

In the previous lectures we have discussed the main properties of the dynamical and thermodynamical evolution of our universe within the frameworks of newtonian and Einstein's gravity. For a universe filled with radiation, baryonic and cold dark matter and a cosmological constant, this forms the basis of the Hot Big Bang model - an expanding universe which is isotropic and homogeneous on large scales. Although this model is extremely successful in describing the observable universe, in particular predicting the abundances of the light nuclear elements and the blackbody spectrum of the Cosmic Microwave Background, it is not completely free of problems. In this lecture, we will thus address the main shortcomings of the standard cosmological model and discuss how a period of inflation in the early universe may solve them.

Shortcomings of the Hot Big Bang Model

The Hot Big Bang model is extremely successful in describing the observable universe. A simple model of a homogeneous and isotropic Friedmann-Robertson-Walker spacetime expanding according to the relative abundances of both relativistic and non-relativistic matter, as well as a small but non-negligible vacuum energy, is enough to explain most of what we see when we point our telescopes to the night sky. However, when extrapolating the present conditions back in time, we are faced with some serious theoretical challenges in setting the particular initial conditions that dynamically yield the universe that we observe today.

(a) Horizon problem

Measurements of the cosmic background of microwave radiation have shown to a remarkable level of precision that all points in the sky are at the same average temperature of about 2.73 K, with small deviations of the order of 10^{-5}. Since photons have freely streamed throughout the universe since the time of last scattering, we may infer that all points in the last scattering surface must have been at the same temperature as well. This remarkable homogeneity and isotropy could be understood if causal processes such as Compton scattering could have smoothed out any fluctuations in the primordial plasma. However, within standard cosmology we know that a particle horizon exists at early times, when the universe was dominated by either non-relativistic matter or radiation. Recall that the causal horizon at conformal time τ is $l = c\tau$, as this corresponds to the maximum distance that light can travel since the Big Bang at $\tau = 0$. This corresponds to a proper distance:

$$d_H = a(t)l = a(t)\int_0^t \frac{dt'}{a(t')} \ . \tag{1}$$

As we have seen previously, for a radiation dominated universe the scale factor grows as $a(t) \propto t^{1/2}$, so that the particle horizon grows faster, with $d_H = 2t$. More and more scales come into causal contact as the universe evolves, meaning that the largest scales that we see today are only now coming into causal contact! To quantify this, we can observe that points that were in causally connected at the time of last scattering were separated by a comoving distance τ_{LSS} in natural units. As illustrated in the spacetime diagram in Figure 1, this corresponds to an angle in the present night sky given by:

$$\tan(\theta/2) = \frac{\tau_{LSS}/2}{\tau_0 - \tau_{LSS}} \simeq \frac{\tau_{LSS}/2}{\tau_0} = \frac{1}{2}\sqrt{\frac{a_{LSS}}{a_0}} = \frac{1}{2\sqrt{1+z_{LSS}}} \ , \tag{2}$$

which for $z_{LSS} \simeq 1100$ as we have seen earlier yields $\theta \sim 2^\circ$. This means that we would expect to find significant fluctuations in the CMB temperature on angular scales larger than 2° and is equivalent to saying that the present Hubble volume corresponded to about 10^5 causally disconnected regions at the time of recombination [1]. Hence, unless we require the presently observed smoothness to be present already at the Big Bang, there is no physical process capable of transferring energy between these different regions by the time of last scattering to bring them to a common average temperature.

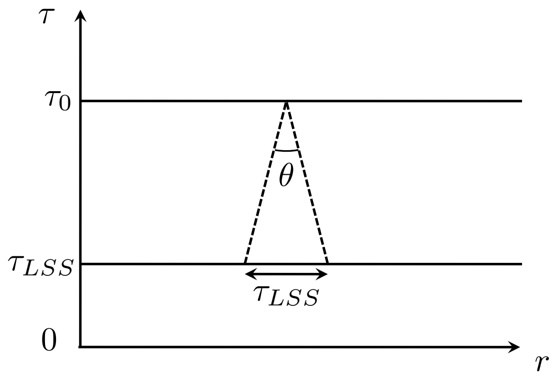

Figure 1: Spacetime diagram illustrating the angular size of the comoving horizon at the time of last scattering in the present sky. The horizontal axis corresponds to the comoving distance r, while the vertical axis gives the conformal time τ starting at the Big Bang at $\tau = 0$ in standard cosmology.

(b) Flatness problem

Recall that the Friedmann equation for an expanding FRW universe is given in natural units by:

$$H^2 = \frac{\rho}{3M_P^2} - \frac{k}{a^2} \ . \tag{3}$$

It is not difficult to express this in terms of the total abundance $\Omega = \rho/\rho_c$, giving:

$$\Omega^{-1} - 1 = -\frac{3M_P^2 k}{\rho a^2} \ , \tag{4}$$

which yields $\Omega = 1$ for a flat universe with $k = 0$. In a matter-dominated universe, $\rho \propto a^{-3}$ and so

$$\Omega^{-1} - 1 \propto a \ , \tag{5}$$

while for a radiation-dominated universe, $\rho \propto a^{-4}$ and so

$$\Omega^{-1} - 1 \propto a^2 \ . \tag{6}$$

In both cases we see that any small deviation from unity at early times will grow as the universe expands. Since we do not observe any significant amount of curvature in the present universe today, this implies that the initial value of Ω must be incredibly fine-tuned. For example, at the Planck scale we must have:

$$|\Omega(T = M_P) - 1| \lesssim 10^{-60} , \tag{7}$$

and even at the time nucleosynthesis begins, at $t = 1$ sec, deviations from an exactly flat universe must already be smaller than 1 part in 10^{16}. It seems that within standard cosmology the universe could not be as flat as presently observed unless very special initial conditions are assumed.

(c) Small scale inhomogeneity

Although the universe is homogeneous and isotropic on large scales, there is a wide variety of structures in the universe as we have discussed previously, ranging from large clusters and superclusters to voids and super-voids. As we will discuss in more detail later in the course, this structure should arise from the non-linear gravitational growth of density fluctuations that were small and linear at the time of last scattering, as evidenced by the small fluctuations in the CMB temperature. Fluctuations arise at different wavelengths and can be analyzed in terms of a Fourier decomposition. Since all physical wavelengths $\lambda(t) \propto a(t)$ are stretched by expansion, the presently observed large scale structure corresponded to fluctuations on much smaller scales at the time of last scattering, but the existence of a particle horizon in standard matter- or radiation-dominated cosmology implies that they must have been outside the causally allowed region at some point in the past. Similarly to our previous discussion of the large-scale smoothness, there is also no known mechanism to generate fluctuations on super-horizon scales, so they were either produced when they came into causal contact or some modification of the standard cosmological model is required. However, the statistical properties of the CMB anisotropies suggest an underlying coherent structure for the primordial seeds of the large scale structure that would not be present if fluctuations were randomly produced when they came within the particle horizon. So what produced these primordial seeds?

(d) Unwanted relics

In standard cosmology, the temperature of the universe, defined as the temperature of the photon fluid as we have seen earlier, can reach arbitrarily large values, well above the energy scale at which the laws of particle physics have so far been tested at around 1 TeV and up to the Planck scale, $M_P \sim 10^{18}$ GeV, where the description of gravitational interactions in terms of general relativity is expected to break down due to quantum effects. The Standard Model of particle physics gives a very good description of fundamental interactions between elementary particles at the quantum level in terms of three apparently distinct forces - the strong, weak and electromagnetic forces, which are described in terms of the exchange of gauge bosons between particles of different charges according to the underlying gauge symmetries of the theory. The mass of these gauge bosons and the associated coupling constants (as the magnitude of the electric charge in the case of electromagnetic interactions) determine the range and the strength of these interactions. For example, we know that weak interactions governing radioactive decay are much weaker and have a much shorter range than electromagnetic interactions because the weak gauge bosons, $W^\pm$ and Z^0, are heavy, while the photon is massless. In reality, the weak gauge bosons are only heavy due to their interactions with the Higgs field, associated with the recently discovered Higgs boson, which has a non-zero value at low energies, whereas at temperatures above a few hundred GeV they are massless such that both interactions have a unique description in terms of what is called the electroweak theory.

Another important property of the Standard Model is that coupling constants vary with energy as a result of their quantum nature - more and more quantum processes become relevant at higher energies and their effects amount to a renormalization of couplings and other parameters of the theory. In particular, our present understanding of the three fundamental forces suggests that the associated couplings become comparable at high energies/temperatures around 10^{16} GeV, possibly signaling the existence of a more fundamental theory with a higher degree of symmetry. Although the properties of such a theory remain unknown, several scenarios of Grand Unified Theories (GUT) have been proposed in the literature as we discussed in the context of baryogenesis. Although we will not explore in depth the properties of such theories in this course, generically one expects that they are spontaneously broken into the

Standard Model interactions by a process analogous to the electroweak Higgs mechanism and that is associated to one or more phase transitions in the early universe. Phase transitions may lead to the formation of several different topological defects, magnetic monopoles being the most notable example. These correspond to stable superheavy particles produced during such phase transitions and interact very weakly, such that they decouple and freeze-out early in the history of the universe, yielding a too large relic density that typically will dominated the universe, $\Omega_{mon} \gg 1$. Although GUT models will probably not be directly testable in the near future, this poses a challenging problem for the Hot Big Bang model since there is no mechanism to avoid the overproduction of GUT monopoles nor any process capable of diluting their abundance. In other extensions of the Standard Model similar problems arise, such as the overproduction of gravitinos in supersymmetric theories or scalar moduli fields in theories with additional compact dimensions.

Basic picture of inflation

The shortcomings of the Hot Big Bang model suggest looking for a modification of the standard cosmological evolution at early times, in order to set the apparently fine tuned initial conditions that yield the required flatness and smoothness of the observed universe, provide the seeds for the cosmic structure and avoid large the overproduction of unwanted relics.

Let us then take a closer look at the horizon and flatness problems. First, notice that the comoving horizon distance can be rewritten as an integral over the scale factor:

$$l = \int_{a_i}^{a} \frac{da'}{a'^2 H(a')} \ , \tag{8}$$

where we used that $dt = da/\dot{a}$ and $H = \dot{a}/a$, and taken an arbitrary value for the initial scale factor, a_i. In the radiation era, $H \propto a^{-2}$, so that the integral converges when we take $a_i \to 0$, yielding a finite particle horizon. If, on the other hand, H decreases more slowly than a^{-1}, the integral would diverge when taking the limit of an initial singularity, so that there would be no finite horizon and all scales could be have been in causal contact. For a power law expansion, $a(t) \propto t^p$,

$$H = \frac{p}{t} \propto a^{-1/p} \ , \tag{9}$$

so that we get a finite horizon for $p > 1$. Since $\ddot{a} \propto p(p-1)t^{p-2}$, we see that the condition $p > 1$ corresponds to accelerated expansion, so that if the universe was accelerating at early times one could make the particle horizon arbitrarily large.

Similarly, from Eq. (4), we see that deviations from flatness would not grow with time if the energy density of the dominant fluid decreased more slowly than a^{-2}. Recalling that $\rho \propto a^{-3(1+w)}$, this implies $w < -1/3$ for the equation of state parameter, which is also the condition for accelerated expansion.

Hence, the solution to the horizon and flatness problems seems to lie in a period of accelerated expansion in the early universe, preceding the radiation era, and which from the rapid expansion it implies is known as *inflation*. The simplest example that we have seen already in this course is a cosmological constant-dominated universe, where the scale factor grows exponentially and $a \to 0$ only for $t \to -\infty$, so that there is no particle horizon and any deviations from a flat geometry are exponentially diluted away. An exact cosmological constant would not, however, redshift away, so that the conventional cosmology dominated by matter and radiation could not follow a period of Λ-dominated expansion. As we will see below, this can be achieved by considering a transient cosmological constant, i.e. a fluid that behaves like Λ only for a finite period of time and then decays away.

Before we discuss the mechanism behind inflation in more detail, let us try to quantify the amount of accelerated expansion required. To solve the horizon problem, we need to ensure that all scales visible in the present night sky were in causal contact before the end of inflation and the beginning of the standard cosmological evolution. From Eq. (2) and extrapolating the spacetime diagram in Figure 1, this means that the size of the comoving horizon at the end of inflation must exceed the comoving distance that light can travel from that moment until today. If we take the origin of conformal time at the end of inflation, so that accelerated expansion begins at some negative time

$-\tau_i$, this requires:

$$\tau_i \geq \tau_0 \ . \tag{10}$$

During inflation, the Hubble parameter H_{inf} remains approximately constant for a fluid that behaves like Λ, so that:

$$\tau_i = \int_{a_i}^{a_f} \frac{da}{a^2 H} \simeq -\frac{1}{H_{inf}}\left(\frac{1}{a_f} - \frac{1}{a_i}\right) \simeq \frac{1}{H_{inf} a_i} \ , \tag{11}$$

where $a_f \gg a_i$ are the values of the scale factor at the beginning and end of inflation. After inflation one should recover the standard cosmology, and one can show that most of the expansion occurs in the radiation dominated era, so that we may for simplicity we may neglect the later matter and dark energy dominated epochs[1]. For radiation domination, we have $H \propto a^{-2}$ as seen above, so that $a^2 H = a_f^2 H_{inf} = const.$ and we have

$$\tau_0 = \int_{a_f}^{a_0} \frac{da}{a^2 H} \simeq \frac{1}{a_f^2 H_{inf}}(a_0 - a_f) \simeq \frac{1}{a_f^2 H_{inf}} \ , \tag{12}$$

where we have taken $a_0 = 1 \gg a_f$. Hence, the condition for solving the horizon problem in Eq. (10) corresponds to requiring $a_f^2 > a_i$ or equivalently:

$$\frac{a_f}{a_i} > \frac{a_0}{a_f} = \frac{T_R}{T_0}, \tag{13}$$

where we have used that in the radiation era we have $a \propto T^{-1}$ and defined the temperature at the end of inflation as the *reheating* temperature, T_R, as we will explain later in more detail. It is conventional to express the ratio on the left hand side in terms of the number of e-folds of accelerated expansion:

$$\frac{a_f}{a_i} = e^{N_e} \ , \tag{14}$$

as the scale factor grows exponentially during inflation. As $T_0 \sim 10^{-4}$ eV, if we require the reheating temperature to be around the GUT scale, $M_{GUT} \sim 10^{16}$ GeV, we require a minimum of $N_e > 66$ e-folds of inflation. It is easy to see that for lower reheating temperatures the required amount of accelerated expansion is smaller, and typical reference values are around 50-60 e-folds, which quantifies the duration of the inflationary period.

It is not difficult to check that solving the flatness problem requires the same number of e-folds. During inflation, for an approximate cosmological constant, $\Omega^{-1} - 1 \propto a^{-2}$ from Eq. (4), while for radiation-domination deviations from $\Omega = 1$ grow like a^2 as seen previously. Hence, if inflation is to compensate for the growth of deviations during the radiation epoch, we require the same amount of expansion during and after inflation, i.e. $a_f/a_i \geq a_0/a_f$, which is exactly what we require to solve the horizon problem. Note that although the same amount of expansion is required, this expansion occurs very fast during inflation, with the scale factor growing is an quasi-exponential fashion, whereas $a \propto t^{1/2}$ during the radiation era. So, in a tiny fraction of a second we must obtain the same amount of expansion as during the next few billions of years! Additionally, any particle species (matter or radiation) present before inflation takes place will quickly redshift away, so that inflation may also solve the problem of overproducing unwanted relics, provided however that they are not produced after inflation ends.

Inflation and scalar fields

The presently observed cosmological constant has only recently come to dominate the energy balance in the universe, as we have discussed in previous lectures. Hence, it could not have dominated the early universe before radiation domination and we need a new physical mechanism to provide accelerated expansion in the early universe. This is necessarily a fluid with negative pressure and it must have non-trivial dynamics (as opposed to a cosmological

[1] Note that matter-radiation equality occurs at a redshift $z_{eq} \simeq 3500$, so that the universe has expanded only for 8 e-folds in the matter and dark energy epochs, which is much less than the 50-60 e-folds in the radiation era as we compute below.

constant), since it must decay after it has produced the required amount of accelerated expansion and give rise to the standard cosmological evolution.

The simplest inflationary paradigm was developed by Guth, Albrecht, Steinhardt, Linde, and many others in the 1980's [2, 3, 4], and is based on the dynamics of a scalar field known as the *inflaton*. Modern particle physics models are based on quantum field theory - all matter and gauge particles are described in terms of fields, defined in the whole spacetime, and which provide the relativistic generalization of the Schrödinger wavefunction. Excitations of these fields are quantized and corresponds to single and multi-particle states, and the properties of such fields reflect the properties of the associated particles, such as their mass, spin and charges. Scalar fields are the simplest fields, giving a description of relativistic bosons of zero spin. Classical values for a scalar field $\phi(x)$ represent the collective behaviour of Bose-Einstein condensates of spinless particles, and as such are invariant under boosts and rotations, thus preserving Lorentz invariance.

The action for a scalar field is defined in terms of its kinetic and gradient energy and a potential that describes the self-interactions between the scalar bosons:

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \ . \tag{15}$$

To obtain the thermodynamic properties of scalar fields, we may compute the associated energy-momentum tensor by varying the action with respect to the metric tensor:

$$\begin{aligned} \delta S_\phi &= \int d^4x\left[\delta(\sqrt{-g})\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) + \sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\delta g^{\mu\nu}\right)\right] = \\ &= \int d^4x\left[\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right) + \sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi\right)\left(-g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\right)\right] = \\ &= \int d^4x\frac{\sqrt{-g}}{2}\left[\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right)g^{\mu\nu} + \partial^\mu\phi\partial^\nu\phi\right]\delta g_{\mu\nu} \ . \end{aligned} \tag{16}$$

This thus gives:

$$T_\phi^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}} = -\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right)g^{\mu\nu} + \partial^\mu\phi\partial^\nu\phi \ . \tag{17}$$

For an FRW geometry this yields:

$$\begin{aligned} T_{00} &= = \frac{1}{2}\dot\phi^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi) \ , \\ T_{ij} &= \partial_i\phi\partial_j\phi - a^2\gamma_{ij}\left(-\frac{1}{2}\dot\phi^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi)\right) \ , \end{aligned} \tag{18}$$

where gradients are taken with respect to comoving coordinates and $(\nabla\phi)^2 = \gamma^{ij}\partial_i\phi\partial_j\phi$. Analogously with a perfect fluid, from this we can obtain the energy density $\rho_\phi = T_{00}$ and the pressure $p_\phi = g^{ij}T_{ij}/3$ of the scalar field:

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot\phi^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi) \ , \\ p_\phi &= \frac{1}{2}\dot\phi^2 - \frac{1}{6}\frac{(\nabla\phi)^2}{a^2} - V(\phi) \ . \end{aligned} \tag{19}$$

From this we see that a scalar field may have different equations of state depending on which component of the energy density dominates. If kinetic terms are dominant, $p_\phi = \rho_\phi$ ($w = 1$); if gradient energy is the dominant component, we have $p_\phi = -\rho_\phi/3$ ($w = -1/3$); finally, if the potential energy dominates, we have $p_\phi = -\rho_\phi$ ($w = -1$). It is thus the latter case that may mimic the behaviour of a cosmological constant and yield accelerated expansion - if there is a patch of the universe where the field is homogeneous and rolling slowly down its potential, it will lead to a period of accelerated expansion that quickly turns this small patch into a macroscopically smooth and flat region!

The smoothness requirement is easy to fulfill, as from Eq. (19) any gradient energy present originally will be exponentially diluted and quickly become irrelevant. All we need is a smooth patch comparable to the horizon size during inflation, which as we will see is typically minuscule on cosmological scales. On the other hand, negligible kinetic energies require the field value to be moving slowly, which in turn implies a very flat potential. The advantage of this picture is that a period of accelerated expansion will last only as long as the kinetic energy remains subdominant. If at some point the potential steepens and the field starts moving quickly (typically towards the minimum of the potential), inflation will end and, as we will discuss, the potential energy originally stored in the inflaton field may be transferred into ordinary matter and radiation. This is why scalar field models give a simple picture of the inflationary universe - accelerated expansion lasts only for a short period, where the universe expands by a very large factor in order to solve the above mentioned problems, but the Λ-like behaviour is only transient and the standard cosmological evolution begins at the end of inflation.

The original proposal of Guth actually considered a case where the scalar field is trapped in a false minimum of its potential, so that the kinetic energy vanishes completely and it behaves exactly as a cosmological constant. For inflation to end the field had to undergo a quantum tunneling transition into the true minimum of the potential, which typically has a very small probability. On one hand this meant that a sufficiently long period of inflation could be easily obtained but that the transition into a radiation-dominated era was very improbable. In the "new inflation" picture developed by Linde the scalar field slowly rolls down a shallow region of the potential until the latter becomes too steep and reheating may occur. We will analyze the necessary conditions for slow-roll inflation and subsequent reheating in the next lecture, but we illustrate its main dynamical features in Figure 2 for a typical inflationary potential.

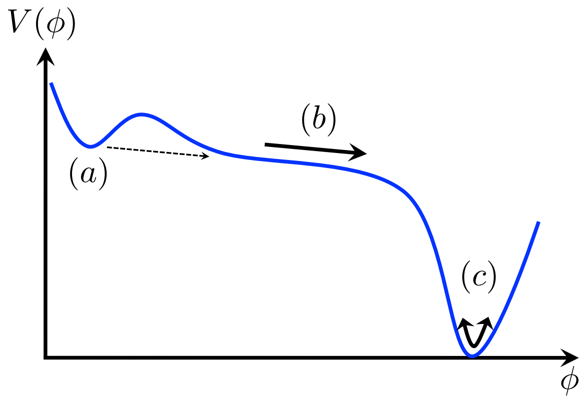

Figure 2: Typical potential for a scalar field leading to inflation. In the "old inflation" picture, the field is trapped in a false minimum as in (a), which is metastable and may decay through quantum tunneling. Although this yields an effective cosmological constant, reheating is very ineffective in this case. On the other hand, the field may tunnel into a shallow region of the potential where it slow rolls and behaves approximately like a cosmological constant, as in (b). This lasts until the field comes close to the absolute minimum of the potential, where it acquires a large velocity so that inflation ends. While it oscillates about its minimum, as in (c), its energy can be transferred into relativistic particles that come to dominate and start the conventional cosmological evolution.

Problem 9

(a) Consider a one-dimensional harmonic oscillator of mass m and natural frequency ω, such that its potential energy is $V(x) = \omega^2 x^2/2$. Use Newton's law to show that its equation of motion is:

$$\ddot{x} + \omega^2 x^2 = 0 \ , \tag{20}$$

with general solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

(b) If there are friction forces acting on this harmonic oscillator, the equation of motion includes an additional damping term:

$$\ddot{x} + 2\omega\zeta\dot{x} + \omega^2 x^2 = 0 \ , \tag{21}$$

where ζ is the damping ratio. Using an exponential *ansatz* $x(t) = Ae^{\alpha t}$, find the values of the coefficient α as a function of the damping ratio and frequency of the harmonic oscillator. Verify, in particular, that you obtain the standard oscillating solutions obtained in (a) for weak damping and that for strong damping you obtain slowly decaying solutions.

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