Introduction to Cosmology

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LECTURE 5 - Standard cosmology II

In this lecture we will continue our study of the standard cosmological model in the context of general relativity. We will review the basic concepts and equations underlying Einstein's gravitational field equations and apply them to the case of expanding FRW spacetimes. We will recover most of the results we have obtained in the context of newtonian cosmology, but many of the geometrical and physical meaning of the equations governing the evolution of an expanding universe that we previously postulated will become clear in this framework.

Review of basic concepts in general relativity

Einstein developed his theory of general relativity in the beginning of the 20th century and it prevails today as the standard description of gravitational interactions in the universe, having already passed many stringent tests. In general relativity, spacetime is a dynamical geometrical entity, described in terms of a pseudo-riemannian manifold[1], which locally resembles flat Minkowski spacetime, $\mathcal{M}^4$. Although the geometrical structure of spacetime is physically meaningful, the coordinates used to describe it are not. A change of coordinates $x^\mu \to x'^\mu$ changes vectors, co-vectors and tensors defined on the spacetime, and in particular:

$$\begin{aligned} dx'^\mu &= \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \ , \\ \frac{\partial}{\partial x'^\mu} &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \ . \end{aligned} \tag{1}$$

The metric used to compute spacetime distances also changes under local coordinate transformations (or diffeomorphisms), but the line element itself, being a measurable observable, is preserved:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g'_{\mu\nu} dx'^\mu dx'^\nu \ . \tag{2}$$

This is the local gauge invariance or diffeomorphism invariance that is the basis of general relativity. Local flatness implies that, at a given spacetime point p, we can choose coordinates such that:

$$g_{\mu\nu}(p) = \eta_{\mu\nu} \ , \qquad g_{\mu\nu,\alpha}(p) = 0 \ . \tag{3}$$

Recall that commas denote ordinary derivatives with respect to spacetime coordinates, $v^\mu{}_{,\alpha} = \partial v^\mu / \partial x^\alpha$ for an arbitrary vector vield v. To convince ourselves that this is indeed the case, let us notice that under a change of coordinates:

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \ , \\ x'^\mu &= x^\mu_p + \left.\frac{\partial x'^\mu}{\partial x^\nu}\right|_p (x^\nu - x^\nu_p) + \ldots \end{aligned} \tag{4}$$

[1]For a pseudo-riemannian manifold, the metric tensor need not be positive-definite.

Now, Eqs. (3) correspond to 10 and 40 equations, respectively, recalling that the metric is symmetric in its two indices. To solve them we can choose $\partial x'^{\alpha}/\partial x^{\mu}$ and $\partial^2 x'^{\alpha}/\partial x^{\mu}\partial x^{\nu}$ at point p, which corresponds to 16 and 40 numbers, respectively. Hence, choosing a locally flat coordinate system implies choosing 56 adjustable parameters satisfying 50 equations, which always has a solution with 6 remaining degrees of freedom, corresponding to the 3 rotations and 3 boosts that are the symmetries of Minkowski space.

In practice we use different coordinate systems that are more appropriate to the physical system that we are trying to describe. Some useful coordinate systems are:

(a) Local inertial or geodesic coordinates

As we have seen in the last lecture, freely falling particle follow geodesics in a given spacetime described by a worldline $x^{\mu}(\lambda)$ for some affine parameter λ. An appropriate choice of coordinate system $\tilde{x}^{\mu}$ for a freely falling observer has the following properties:

1. The observer is at $\tilde{\mathbf{x}} = \mathbf{0}$
2. The observer's 4-velocity is, hence, $\tilde{u}^{\mu} = \tilde{u}^0 \delta^{\mu}_0$
3. The geodesic equation then reads:

$$\begin{aligned}\frac{d}{d\lambda}(g_{\mu\nu}u^{\nu}) &= \frac{1}{2}g_{\alpha\beta,\mu}u^{\alpha}u^{\beta} \\ \frac{d}{d\lambda}(\tilde{g}_{\mu 0}\tilde{u}^0) &= \frac{1}{2}\tilde{g}_{00,\mu}\tilde{u}^0\tilde{u}^0 \ ,\end{aligned} \tag{5}$$

   and $\tilde{g}_{00}\tilde{u}^0\tilde{u}^0 = -1$ choosing the affine parameter to be the observer's proper time $ds^2/d\lambda^2 = \tilde{g}_{\mu\nu}\tilde{u}^{\mu}\tilde{u}^{\nu} = -1$.
4. We may require that

$$\begin{aligned}&\tilde{g}_{\mu\nu} = \eta_{\mu\nu} \\ &\tilde{g}_{\mu\nu,i} = 0 \ ,\end{aligned} \tag{6}$$

   yielding a total of $10 + 30 = 40$ equations. For this we can choose a total of 37 parameters, namely $\partial\tilde{t}/\partial t$ (1), $\partial\tilde{x}^{\mu}/\partial x^i$ (12) and $\partial^2\tilde{x}^{\mu}/(\partial x^i\partial x^j)$ (24). Note that the 4-velocity of the observer fixes $\partial\tilde{x}^i/\partial t$. From the geodesic equation we obtain automatically $\tilde{u}^0 = 1$ and $\tilde{g}_{00,i} = 0$, leaving only 37 equations to solve and no unfixed degrees of freedom in the end.

(b) Synchronous coordinates

Let us choose a spacelike surface Σ, $t = const.$, and coordinates x^i on this surface. The surface is intersected by the worldline of a particle at each point. Hence, we can choose the time coordinate at each point in spacetime to be the time measured by a standard clock carried by the particle whose worldline passes through that point. We also choose its spatial coordinates to be the coordinates x^i of the intersection point of the worldline with Σ. Finally, we choose the origin of time also at the intersection point. Thus:

1. For each clock, $\delta x^i = 0$, and so:

$$d\tau^2 = -g_{00}dt^2 \ , \tag{7}$$

   but we chose $t = \tau$, so that $g_{00} = -1$.
2. On the surface Σ, $g_{0i} = 0$, since $\delta t = \delta x_0 = g_{0\mu}\delta x^{\mu} = g_{0i}\delta x^i = 0$ for any δx^i. Using the geodesic equation along each worldline:

$$\frac{d}{d\lambda}(g_{\mu 0}u^0) = \frac{1}{2}g_{00,\mu}u^0u^0 \ , \tag{8}$$

   which implies for $\mu = i$ that $dg_{0i}/d\lambda = 0$, i.e. that $g_{0i} = 0$ everywhere.

Hence, in synchronous coordinates, $g_{0\mu} = -\delta^0{}_\mu$. Note that this is only valid in a neighbourhood of Σ where the worldlines of different particles do not intersect each other and the coordinates are well-defined. There is also still a residual freedom to choose the initial surface Σ and the spatial coordinates on it, which is known as a *gauge freedom*.

Einstein-Hilbert action and gravitational field equations

In general relativity, the geometry of spacetime determined by the metric tensor $g_{\mu\nu}$ and the metric connection which, as we have seen previously, defines the covariant derivative on the spacetime. Recall that the metric connection is written in terms of the Christoffel symbols:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}\left(g_{\mu\alpha,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \ . \tag{9}$$

The metric connection allows one to define the Riemann curvature tensor:

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \ . \tag{10}$$

The Riemann tensor encodes the curvature properties of the spacetime, in particular determining how geodesics deviate from each other in the spacetime. It can be defined for an arbitrary connection, but for a torsionless symmetric connection as is the case of the metric connection, the Riemann curvature tensor yields the commutator of covariant derivatives:

$$A_{\mu;\rho\sigma} - A_{\mu;\sigma\rho} = A_\nu R^\nu{}_{\mu\rho\sigma} \ . \tag{11}$$

A contraction of the first and third indices of the Riemann tensor yields the Ricci tensor:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho} \ . \tag{12}$$

The Ricci tensor is symmetric in its indices as a consequence of the various symmetries of the Riemann tensor for the metric connection, and upon contraction with the metric one obtains the Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu} \ . \tag{13}$$

The Ricci scalar is gauge invariant, i.e. invariant under general coordinate transformations, and is the simplest such geometrical quantity of second order in derivatives of the metric. It is thus the ideal candidate from which to construct the action for the gravitational field, which is known as the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) \ . \tag{14}$$

Note that g denotes the determinant of the metric, which is a tensor density, and Λ is a constant that can be added to the Lagrangian in the general case - Einstein's famous cosmological constant that we have seen in the previous lectures. To obtain the field equations for the metric, one can use the following set of identities:

$$\begin{aligned}
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \ , \\
\delta g^{\mu\nu} &= -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} \ , \\
\delta R_{\mu\nu} &= \left(\delta\Gamma^\alpha_{\mu\nu}\right)_{;\alpha} - \left(\delta\Gamma^\alpha_{\alpha\mu}\right)_{;\nu} \ , \\
\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}V^\mu\right) &= V^\mu_{;\mu} \ ,
\end{aligned} \tag{15}$$

where the last equation is valid for arbitrary vector fields V^μ. Varying the action (14) with respect to the metric using these identities, we obtain Einstein's equations in vacuum:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} \ . \tag{16}$$

The quantity $G_{\mu\nu}$ is a symmetric 2-tensor known as the Einstein tensor, and which is covariantly conserved, $G^{\mu\nu}_{\ ;\nu} = 0$. If we include an action S_m for matter fields, we may define the energy-momentum tensor:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}} \ , \tag{17}$$

which is also covariantly conserved, $T^{\mu\nu}_{\ ;\nu} = 0$, reflecting energy and momentum conservation in curved spacetime. Varying the total action $S = S_{EH} + S_m$ with respect to the metric, we obtain the general form of Einstein's equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{18}$$

Recall that the metric is also covariantly conserved for the Christoffel connection, so that the addition of the cosmological constant is consistent with covariant conservation of both sides of the equations. These equations are tensor identities, holding in arbitrary coordinate systems, and provide a set of 10 equations yielding the dynamics of spacetime. Actually, the equations involving $G_{0\nu}$ are simply constraint equations and there are only 6 dynamical equations for the 10 metric components. There is, thus, a 4-fold ambiguity related of course to the choice of coordinates x^μ throughout spacetime.

Einstein's equations thus relate the geometry of spacetime and its dynamical evolution to the local energy and momentum of matter fields, and we shall now apply this to the case of FRW spacetimes.

Energy-momentum tensor for perfect fluids

Before computing the curvature tensors in FRW spacetimes, let us first focus on the form of the right hand side of the Einstein equations. As we have seen previously, the evolution of the universe is well described by homogeneous and isotropic perfect fluids to a first approximation. Perfect fluids have no heat conduction nor viscosity and are completely described by their energy density ρ and pressure p.

Let us think about the simplest case of dust - a collection of particles at rest with respect to each other, or a perfect fluid with zero pressure. Since all particles have the same velocity in any inertial frame, we can imagine a 4-velocity field $u^\mu(x)$ defined throughout spacetime. This gives a number-flux 4-vector $n^\mu = nu^\mu$, where n is the number density of particles in their rest frame. This implies n^0 gives the number density of particles in other frames, while n^i gives the flux of particles in the direction x^i. If particles have the same mass m, in the rest frame the energy density $\rho = nm$. Since n and m are the zero components of 4-vectors n^μ and p^μ, respectively, we see that ρ is the $\mu = \nu = 0$ component of a 2-tensor:

$$T^{\mu\nu}_{dust} = p^\mu n^\nu = nmu^\mu u^\nu = \rho u^\mu u^\nu \ , \tag{19}$$

which is thus the energy-momentum tensor for dust. For more general perfect fluids with pressure, the form is not much more complicated. The term "perfect" can be thought of as meaning "isotropic in the rest frame", implying that $T^{\mu\nu}$ is diagonal in this frame, there being no net flux of momentum in any orthogonal direction. Isotropy also implies that all 3 components T^{ii} are equal, and in fact give the pressure p of the fluid. Comparing with what we learn from dust, in the rest frame of a perfect fluid we thus have

$$T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p) \ . \tag{20}$$

For a general frame, the generalization is not difficult:

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \ . \tag{21}$$

A more concrete derivation can be done starting from the action for a collection of free particles of mass m interacting only through gravity:

$$S = m \sum_i \int d^4x \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, \delta^3(\mathbf{x} - \mathbf{x}_i(t)) \ , \tag{22}$$

where dots denote derivatives with respect to time t. From the definition of the energy-momentum tensor in Eq. (17), we have:

$$
\begin{aligned}
T^{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = -\frac{2}{\sqrt{-g}} m \sum_i \delta^3(\mathbf{x}-\mathbf{x}_i(t))\frac{1}{2}\frac{1}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}(-\dot{x}^\mu \dot{x}^\nu) = \\
&= \frac{m}{\sqrt{-g}}\sum_i \delta^3(\mathbf{x}-\mathbf{x}_i(t))\frac{dx^\mu}{d\tau}\frac{dx^\nu}{dt} \ , \qquad (23)
\end{aligned}
$$

where $d\tau = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}dt$ is the proper time of an observer moving with the fluid. In an inertial coordinate system, $g_{\mu\nu} = \eta_{\mu\nu}$ and $u^\mu = dx^\mu/d\tau$, yielding:

$$
\begin{aligned}
T^{00} &= m\sum_i u_i^0 \delta^3(\mathbf{x}-\mathbf{x}_i(t)) \ , \\
T^{0j} &= m\sum_i u_i^0 v_i^j \delta^3(\mathbf{x}-\mathbf{x}_i(t)) \ , \\
T^{jk} &= m\sum_i u_i^j v_i^k \delta^3(\mathbf{x}-\mathbf{x}_i(t)) = m\sum_i u_i^0 v_i^j v_i^k \delta^3(\mathbf{x}-\mathbf{x}_i(t)) \ , \qquad (24)
\end{aligned}
$$

where we used that $u_i^j = dx_i^j/d\tau = (dt/d\tau)(dx^j/dt) = u_i^0 v_i^j$. We can identify the number density of particles with:

$$
n = \sum_i \delta^3(\mathbf{x}-\mathbf{x}_i(t)) \qquad (25)
$$

and recalling that the pressure is the average value of the momentum transfer $|p^j|^2/E$ in a given direction, we get for an isotropic fluid:

$$
T^{jj} = nm\langle u^0 (v^j)^2\rangle = n\langle \frac{(p^j)^2}{E}\rangle = p \ . \qquad (26)
$$

Since the average velocity is zero in an inertial frame, $T^{jk} = p\delta^{jk}$. In the non-relativistic case, $u^0 = \gamma \simeq 1$ and so:

$$
p = \frac{1}{3}nm\langle v^2\rangle \ll 1 \ , \qquad (27)
$$

where we used isotropy in three spatial dimensions. Also, $T^{00} = n\gamma m = \rho$, so that we recover the result above in the rest frame of the fluid. In the relativistic case, $\langle v^2\rangle = c^2 = 1$, and we get the equation of state for radiation:

$$
p = \frac{1}{3}nm = \frac{1}{3}\rho \ , \qquad (28)
$$

as we had anticipated in our study of newtonian cosmology.

As we can see from Eq. (21), for a fluid with equation of state $p = -\rho$, we have $T^{\mu\nu} = pg^{\mu\nu}$. This means that the cosmological constant term in the Einstein equations (16) can be taken to the right hand side of the equations to yield an effective energy-momentum tensor:

$$
T_\Lambda^{\mu\nu} = -\frac{\Lambda}{8\pi G}g^{\mu\nu} = p_\Lambda g^{\mu\nu}, \qquad (29)
$$

so that $\rho_\Lambda = -p_\Lambda = \Lambda/8\pi G$. A positive cosmological constant thus has the same effect as a fluid with a negative pressure that can lead to accelerated expansion as we have seen before. This also implies that the cosmological constant corresponds to a contribution to the energy-momentum tensor in the absence of matter fields, i.e. to the energy density and pressure of vacuum. This is extremely important in the context of quantum field theory, where there is a non-vanishing contribution to the zero-point energy of vacuum fields, analogously to the zero-point energy of a quantum harmonic oscillator, $E_0 = \hbar\omega/2$. In fact, we can separate two contributions to the cosmological constant - a bare geometrical term on the left hand side of the equations and a vacuum term on the right hand side. If these have the same sign they may cancel each other and lead to the small cosmological constant that we infer from observations today. Of course as we have discussed the fine-tuning required is enormous, yielding the famous cosmological constant problem to which we will come back at the end of the course.

Einstein's equations in FRW spacetimes

To obtain the Einstein tensor in FRW spacetimes, we need to compute the metric connection components and the associated curvature tensors. Recall the form of the FRW metric with curvature k:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \ , \qquad \gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \ . \tag{30}$$

From this we can show, for example, that:

$$\begin{aligned}\Gamma^0_{\mu\nu} &= \frac{1}{2}g^{0\beta}\left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right) = \\ &= \frac{1}{2}g^{00}\left(g_{\mu 0,\nu} + g_{0\nu,\mu} - g_{\mu\nu,0}\right) = \\ &= \frac{1}{2}g_{\mu\nu,0}\end{aligned} \tag{31}$$

which implies that the only non-vanishing components of this form are:

$$\Gamma^0_{ij} = \frac{1}{2}\frac{\partial}{\partial t}(a^2\gamma_{ij}) = a\dot{a}\gamma_{ij} \ . \tag{32}$$

Similarly, one can show that the remaining non-vanishing components are:

$$\begin{aligned}&\Gamma^r_{rt} = \frac{\dot{a}}{a} \ , \qquad \Gamma^r_{rr} = \tfrac{kr}{1-kr^2} \ , \qquad \Gamma^r_{\theta\theta} = -r(1-kr^2) \ , \qquad \Gamma^r_{\phi\phi} = -\sin^2\theta r(1-kr^2) \ , \\ &\Gamma^\theta_{\theta t} = \frac{\dot{a}}{a} \ , \qquad \Gamma^\theta_{\theta r} = \tfrac{1}{r} \ , \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ , \\ &\Gamma^\phi_{\phi t} = \frac{\dot{a}}{a} \ , \qquad \Gamma^\phi_{\phi r} = \tfrac{1}{r} \ , \qquad \Gamma^\phi_{\phi\theta} = \cot\theta \ .\end{aligned} \tag{33}$$

From this we can compute the components of the Ricci tensor. For example:

$$\begin{aligned}R_{00} &= \Gamma^\rho_{00,\rho} - \Gamma^\rho_{\rho 0,0} + \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{00} - \Gamma^\rho_{\sigma 0}\Gamma^\sigma_{0\rho} = \\ &= -3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 = \\ &= -3\left(\frac{\ddot{a}a - \dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2}\right) = \\ &= -3\frac{\ddot{a}}{a} \ .\end{aligned} \tag{34}$$

Similarly, one can show that $R_{0i} = 0$ for all spatial indices and that:

$$R_{ij} = a^2\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right)\gamma_{ij}. \tag{35}$$

It is thus straightforward to compute the Ricci scalar:

$$\begin{aligned}R &= g^{00}R_{00} + g^{ij}R_{ij} = \\ &= 3\frac{\ddot{a}}{a} + \frac{\gamma^{ij}}{a^2}a^2\gamma_{ij}\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right) = \\ &= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \ ,\end{aligned} \tag{36}$$

where we used $\gamma^{ij}\gamma_{ij} = \mathrm{Tr}(\mathbb{I}) = 3$.

Finally, the Einstein tensor components are given by:

$$
\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} \ , \\
G_{ij} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) a^2\gamma_{ij} \ .
\end{aligned} \tag{37}
$$

From Eq. (21) for the energy-momentum tensor in the rest frame of the fluid, we have $T_{00} = T^{00} = \rho$ and $T_{ij} = pg_{ij} = pa^2\gamma_{ij}$. We may now write the full Einstein equations (18) collecting all of the above results. It is easy to see that the $\mu = \nu = 0$ component yields the general form of the Friedmann equation:

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} - \frac{\Lambda}{3} \ , \tag{38}
$$

where the cosmological constant can be written as an effective contribution to the energy density as discussed before. The $\mu = i, \nu = j$ components yield:

$$
-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = 8\pi G p \ , \tag{39}
$$

and using the Friedmann equation we obtain the Raychaudhuri or acceleration equation:

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \ , \tag{40}
$$

where we can explicitly see that the cosmological constant contributes to a positive acceleration.

Hence, we have recovered the basic equations determining the evolution of the scale factor in a homogeneous and isotropic expanding universe using the Einstein equations for the FRW metric. The results are surprisingly identical to what we had obtained using simple newtonian cosmology, except that quantities such as the curvature k, which was a simple integration constant in the newtonian case, acquires a simple geometric meaning within the framework of general relativity.

Finally, the covariant conservation of the energy-momentum tensor, $T^{\mu\nu}{}_{;\nu} = 0$, corresponds to energy and momentum conservation in an expanding universe. In particular, the $\mu = 0$ component yields:

$$
\begin{aligned}
T^{0\nu}{}_{;\nu} &= T^{0\nu}{}_{,\nu} + \Gamma^0_{\alpha\nu}T^{\alpha\nu} + \Gamma^\nu_{\alpha\nu}T^{0\alpha} = \\
&= T^{00}{}_{,0} + \Gamma^0_{ij}T^{ij} + \Gamma^\nu_{0\nu}T^{00} = \\
&= \dot{\rho} + (a\dot{a}\gamma_{ij})\left(p\frac{\gamma^{ij}}{a^2}\right) + 3\frac{\dot{a}}{a}\rho = \\
&= \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \ ,
\end{aligned} \tag{41}
$$

which is the equation we had obtained using the first law of thermodynamics with $dS = 0$, and that applies for generic (isentropic) perfect fluids.

Problem 5

Deduce the Einstein field equations in vacuum by varying the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R-2\Lambda)\ . \tag{42}$$

with respect to the inverse metric $g^{\mu\nu}$, following the steps below:
(a) Show that the variation of the action involves three different contributions:

$$\delta S_{EH} = \delta S_1 + \delta S_2 + \delta S_3\ , \tag{43}$$

where

$$\begin{aligned}
\delta S_1 &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\ , \\
\delta S_2 &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}\ , \\
\delta S_3 &= \frac{1}{16\pi G}\int d^4x(R-2\Lambda)\delta(\sqrt{-g})\ .
\end{aligned} \tag{44}$$

(b) Verify that

$$\delta R_{\mu\nu} = \left(\delta\Gamma^{\alpha}_{\mu\nu}\right)_{;\alpha} - \left(\delta\Gamma^{\alpha}_{\alpha\mu}\right)_{;\nu}\ , \tag{45}$$

and use Stokes Theorem to show that $\delta S_1 = 0$.
(c) Use the property $\mathrm{Tr}(\log M) = \log(\det M)$ valid for an arbitrary matrix M to show that:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\ . \tag{46}$$

(d) Using the results above, find that

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right]\delta g^{\mu\nu} = 0 \tag{47}$$

thus yielding the Einstein field equations in vacuum for arbitrary variations of the metric tensor.

References

[1] S. M. Carroll, *Lecture notes on general relativity*, arXiv:gr-qc/9712019.