Introduction to Cosmology

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LECTURE 4 - Standard cosmology I

In this lecture we will begin the study of the Hot Big Bang model within the framework of general relativity. Although we have been able to derive several dynamical solutions for an expanding universe in the context of newtonian cosmology, the geometrical properties of spacetime and expansion itself can only be fully understood within general relativity, as well as the properties of relativistic fluids such as radiation and the cosmological constant. The aim of this lecture is to analyze the geometrical properties of spacetimes satisfying the cosmological principle and the kinematics of particles moving in such spacetimes.

Friedmann-Robertson-Walker geometries

In general relativity, the geometry of spacetime is described by a (symmetric) metric tensor $g_{\mu\nu}$, with $\mu, \nu = 0, 1, 2, 3$. The metric tensor defines the way one measures spacetime distances in a given geometry, and yields the infinitesimal line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{1}$$

where we use Einstein's summation convention, i.e. that repeated upper and lower indices are summed. The simplest example is the flat Minkowski metric, where $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ in cartesian coordinates. Note that there is a freedom of choosing a 'signature' for the metric, and in our discussion we will use the $(-, +, +, +)$ signature corresponding to the signs of the time and spatial diagonal components of the metric.

We are thus interested in determining the line element corresponding to an expanding spacetime satisfying the cosmological principle, i.e. that is homogeneous and isotropic and describes the Universe on large scales. We are then looking for spacetimes with maximal symmetry, which are known as Friedmann-Robertson-Walker (FRW) or sometimes Friedmann-Lemaître-Robertson-Walker spacetimes. Besides the work of Friedmann and Lemaître that we have discussed previously, Howard Robertson (1903-1961) and Arthur Walker (1909-2001) proved rigorously that such spacetimes are the more general geometries consistent with the cosmological principle in 1935.

The maximal number of isometries, i.e. transformations that preserve the line element, in d dimensions is $d(d + 1)/2$. For a spacetime with $d = 3$ spatial dimensions, this yields 6 isometries, which can be divided into translations, rotations and boosts. This gives three possible cases for the spatial geometry:

- Spherical S^3: 6 rotations
- Euclidean E^3: 3 rotations + 3 translations
- Hyperbolic H^3: 3 boosts + 3 rotations

These correspond to the closed, flat and open geometries discussed in the context of newtonian cosmology. Let us analyze each case in more detail:

(a) Spherical spaces S^3

Consider the hypersurface in 4-dimensional Euclidean space E^4 with line element $ds^2 = dx^2 + dy^2 + dz^2 + dv^2$ defined by:

$$x^2 + y^2 + z^2 + v^2 = 1 \ . \tag{2}$$

This defines the surface of an S^3 sphere with unit radius and can be parametrized in terms of spherical coordinates:

$$v = \cos\chi \ , \qquad z = \sin\chi\cos\theta \ , \qquad y = \sin\chi\sin\theta\cos\phi \ , \qquad x = \sin\chi\sin\theta\sin\phi \ . \tag{3}$$

In these coordinates, the line element becomes:

$$ds^2 = d\chi^2 + \sin^2\chi d\Omega^2 \ , \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of S^2 and $0 \le \chi \le \pi$. We may also write this in terms of a radial coordinate $r = \sin\chi$, so that:

$$\begin{aligned} dr &= \cos\chi d\chi \\ dr^2 &= \cos^2\chi d\chi^2 = (1 - \sin^2\chi) d\chi^2 = (1 - r^2) d\chi^2 \ . \end{aligned} \tag{5}$$

This yields the following line element for a unit S^3:

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \ , \tag{6}$$

while one can extend this for a sphere of radius r_0 by rescaling the radial coordinate, giving:

$$ds^2 = \frac{dr^2}{1 - (r/r_0)^2} + r^2 d\Omega^2 \ , \tag{7}$$

(b) Hyperbolic spaces H^3

A 3-dimensional hyperboloid can be embedded in 4-dimensional Minkowski space $\mathcal{M}^4$ with line element $ds^2 = dx^2 + dy^2 + dz^2 - dv^2$ through the following condition:

$$x^2 + y^2 + z^2 - v^2 = -1 \ . \tag{8}$$

We may then use the parametrization:

$$v = \cosh\chi \ , \qquad z = \sinh\chi\cos\theta \ , \qquad y = \sinh\chi\sin\theta\cos\phi \ , \qquad x = \sinh\chi\sin\theta\sin\phi \ . \tag{9}$$

and show that the line element can be written in the form:

$$ds^2 = d\chi^2 + \sinh^2\chi d\Omega^2 \ , \tag{10}$$

so that the difference between the spherical and hyperbolic case lies in changing $\chi \to i\chi$ where now $0 \le \chi < +\infty$. Similarly to the spherical case, we can define a radial coordinate $r = \sinh\chi$ to get:

$$\begin{aligned} dr &= \cosh\chi d\chi \\ dr^2 &= \cosh^2\chi d\chi^2 = (1 + \sinh^2\chi) d\chi^2 = (1 + r^2) d\chi^2 \ . \end{aligned} \tag{11}$$

This then yields the line element:

$$ds^2 = \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \ , \tag{12}$$

whereas for finite radius r_0 we get:

$$ds^2 = \frac{dr^2}{1 + (r/r_0)^2} + r^2 d\Omega^2 \ . \tag{13}$$

(c) Euclidean space E^3

The flat geometry E^3 can be obtained from either the spherical or hyperbolic cases by taking the limit $r_0 \to +\infty$, yielding simply:

$$ds^2 = dr^2 + r^2 d\Omega^2 \ . \tag{14}$$

This means that we may write a generic spatial line element:

$$ds^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \ , \tag{15}$$

with $|k| = r_0^2$ and $k > 0$ for S^3, $k < 0$ for H^3 and $k = 0$ for E^3. As we will see when we derive the Friedmann equation from the Einstein field equations for these geometries, k corresponds to the integration constant introduced in the context of newtonian cosmology and that, as we anticipated, is associated with the geometry of the underlying spacetime.

Note that H^3 is an infinite space, as the area of a surface with constant $\chi = \chi_0$ is $4\pi \sinh^2 \chi_0$, which grows exponentially, whereas for S^3 we get $4\pi \sin^2 \chi_0$, which is always finite. In Euclidean space we obtain the familiar formula $4\pi\chi_0^2$, with χ coinciding with the radial coordinate r.

Furthermore, note that the assumption of homogeneity and isotropy only implies that the space must have a *local* geometry of the form S^3, E^3 or H^3. The global properties of the space could differ from these spaces, and for example in the flat case we could have a toroidal topology T^3 rather than E^3, which is obtained by identifying opposite sides of a fundamental volume element. Such non-trivial topologies may be relevant for theories with compact extra-dimensions, such as in the context of string theory and that may be of importance for the cosmology of the very early Universe. Only the local properties of spacetime will be relevant for our discussion, though, but one should keep in mind that there may be different topological possibilities!

To complete the line element for an isotropic and homogeneous spacetime we need to specify the temporal part of the metric and in general we may write:

$$ds^2 = g_{00}(\tilde{t})d\tilde{t}^2 + 2g_{0i}(\tilde{t})d\tilde{t}dx^i + a^2(\tilde{t})h_{ij}dx^i dx^j \ , \tag{16}$$

where h_{ij} corresponds to the generic spatial metric in Eq. (15). For isotropy we must require $g_{0i} = 0$, or otherwise the Universe could have a preferred direction, and we may always redefine the time coordinate to give $g_{00}(\tilde{t})d\tilde{t}^2 = -dt^2$. This gives the general form of the FRW metric:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right] \ , \tag{17}$$

where $a(t)$ is the familiar scale factor that determines how distances between objects at given coordinates (r, θ, ϕ) in a spatial slice varies in time. These are known as *comoving coordinates* since they are the coordinates measured by an observer moving along with the Hubble flow. Note that t is the time measured by such an observer and this is referred to as the *comoving frame.*

Cosmological horizons

Having derived the generic form of a metric for an expanding universe with maximal symmetry, we now wish to analyze how particles move in such FRW spacetimes. The simplest case is that of photons, or particles of light, which travel along null-rays at the speed of light $c = 1$ in natural units. This implies $ds^2 = 0$, and it is convenient to define the *conformal time* τ:

$$d\tau = \frac{dt}{a(t)} \ , \tag{18}$$

in terms of which the FRW metric can be written as:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right] \ , \tag{19}$$

the quantity between square brackets vanishing for the propagation of photons. For instance, consider a photon traveling in the radial direction with $d\Omega = 0$:

$$d\tau = -\frac{dr}{\sqrt{1-kr^2}} \ , \tag{20}$$

where we chose the solution such that r is decreasing with time. This implies that for a photon emitted at τ_e at comoving distance r_e and detected at τ_0 at the Earth, which we can put at the origin $r = 0$ without loss of generality:

$$\Delta\tau = \tau_0 - \tau_e = \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{k} r_e) \ , & S^3 \\ r_e \ , & E^3 \\ \frac{1}{\sqrt{-k}} \sinh^{-1}(\sqrt{-k} r_e) \ , & H^3 \end{cases} \ , \tag{21}$$

where the integral can be performed using the coordinate χ introduced earlier for the different cases. In particular, a photon in a closed universe can circle around S^3 in a conformal time $\Delta\tau = 2\pi/\sqrt{k}$.

Most importantly, since photons travel at the maximum possible speed, the maximum distance that a photon can travel since the origin of time, at $t = t_i$, determines the causal structure of the spacetime, i.e. which portion of the Universe is in causal contact at a given time t. This defines the particle horizon of the Universe:

$$d_H(t) = \int_0^{r_H} \sqrt{g_{rr}} dr = a(t) \int_{\tau_i}^{\tau} d\tau' = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')} \ . \tag{22}$$

If $d_H(t)$ is finite, our past light cone is limited by this particle horizon and this defines the part of the Universe that is visible, i.e. the light signals that have been able to reach us at a given time. For example, recalling our earlier discussion in newtonian cosmology, for a flat universe during matter or radiation domination the scale factor grows like $a(t) \propto t^p$, with $p = 2/3$ and $1/2$, respectively. Since the Universe begins with a Big Bang singularity at $t_i = 0$, this implies $d_H(t) \propto t$ and there is a finite particle horizon at any given time. For Λ-domination, with $a(t) \propto e^{Ht}$, there is no initial singularity and $t_i = -\infty$, being easy to check that no particle horizon exists.

The Universe may also have an *event horizon*, meaning a part of spacetime that will never be in causal contact with events at a given time t_0, i.e. we will never be able to send light signals into such parts of the universe. The present distance to this event horizon is then given by:

$$d_e(t_0) = \int_{t_0}^{+\infty} \frac{dt'}{a(t')} \ , \tag{23}$$

for $a_0 = 1$. It is clear that for a matter- or radiation-dominated flat universe d_e is infinite an there is no event horizon. For scenarios where the Universe recollapses in a finite time t_f, there will be an event horizon since the upper limit of the integral is t_f and the integral converges. In a Λ-dominated flat universe, although expansion is eternal, the integral in Eq. (23) converges and $d_e = H^{-1}$, so that there is an event horizon in this case and there are parts of the Universe that we will never see, which is what the present data suggests.

Particle kinematics in FRW

Let us turn now to the motion of arbitrary particles, which are not necessarily massless as the photon. The motion of a particle with mass m is governed by the action:

$$S = -m \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \ , \tag{24}$$

where λ is an affine parameter along the particle's worldline, i.e. the particle's path in spacetime, and $\dot{x}^\mu \equiv dx^\mu/d\lambda$. Note that this action corresponds to the integral of the infinitesimal line element along the worldline. The affine parameter λ is arbitrarily defined and the action must be invariant under reparametrizations $\lambda \to \tilde{\lambda}(\lambda)$, or equivalently:

$$x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda) \ . \tag{25}$$

The variation of the action with this reparametrization is given by:

$$\delta S = -m \int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} \left(-g_{\mu\nu,\alpha}\delta x^{\alpha}\dot{x}^{\mu}\dot{x}^{\nu} - 2g_{\alpha\nu}\delta\dot{x}^{\alpha}\dot{x}^{\nu}\right) = 0 \ , \tag{26}$$

where $g_{\mu\nu,\alpha} = dg_{\mu\nu}/dx^{\alpha}$. Integrating by parts, we obtain:

$$\begin{aligned}\delta S &= -m \int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} \left(-g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu} + 2g_{\alpha\nu,\sigma}\dot{x}^{\sigma}\dot{x}^{\nu} + 2g_{\alpha\nu}\ddot{x}^{\nu}\right)\delta x^{\alpha} = \\ &= -m \int \frac{d\lambda}{2\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}} \left(\left(-g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu}\right)\dot{x}^{\mu}\dot{x}^{\nu} + 2g_{\alpha\nu}\ddot{x}^{\nu}\right)\delta x^{\alpha} = 0 \ . \end{aligned} \tag{27}$$

Since this must hold for an arbitrary variation δx^{α}, we obtain:

$$\begin{aligned} \left(-g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu}\right)\dot{x}^{\mu}\dot{x}^{\nu} + 2g_{\alpha\nu}\ddot{x}^{\nu} = 0 \\ \ddot{x}^{\sigma} + \frac{1}{2}g^{\alpha\sigma}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right)\dot{x}^{\mu}\dot{x}^{\nu} = 0 \ , \end{aligned} \tag{28}$$

which is the *geodesic equation* typically written as:

$$\ddot{x}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \ , \tag{29}$$

where we define the Christoffel symbols or metric connection components:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}\left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\right) \ . \tag{30}$$

The worldline of a freely falling particle thus follows a geodesic curve, which minimizes the spacetime path followed by the particle and hence the action in Eq. (24). It is also common to write the geodesic equation in terms of the particle's 4-velocity $u^{\mu} = \dot{x}^{\mu}$:

$$\dot{u}^{\sigma} + \Gamma^{\sigma}_{\mu\nu}u^{\mu}u^{\nu} = 0 \ . \tag{31}$$

Note that the 4-velocity of a particle measured with respect to the comoving frame is called the peculiar velocity. The covariant derivative of a vector is defined by:

$$u^{\mu}{}_{;\nu} = u^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\nu}u^{\alpha} \ . \tag{32}$$

Similarly for a covector:

$$u_{\mu;\nu} = u_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu}u_{\alpha} \ . \tag{33}$$

This can be generalized for a generic tensor, and in particular the metric tensor is covariantly conserved:

$$g_{\mu\nu;\alpha} = g_{\mu\nu,\alpha} - \Gamma^{\sigma}_{\mu\alpha}g_{\sigma\nu} - \Gamma^{\sigma}_{\alpha\nu}g_{\mu\sigma} = 0 \ . \tag{34}$$

One can show that the geodesic equation corresponds to saying that a particle is freely falling or that the particle's 4-velocity is covariantly conserved long the worldline:

$$u^{\alpha}u^{\mu}{}_{;\alpha} = 0 \ . \tag{35}$$

The 4-velocity can be expressed in terms of the ordinary 3-velocity $v_i = dx^i/dt$ by the familiar relation $u^{\mu} = (u^0, u^i) = (\gamma, \gamma v^i)$, where the boost factor in curved space is $\gamma \equiv (1 - |\mathbf{v}|^2)^{-1/2}$ and $|\mathbf{v}|^2 = a^2 h_{ij}v^i v^j$.

Let us now take a closer look at the geodesic equation for the particular case of the FRW spacetimes. In particular, for the $\mu = 0$ component, we have:

$$\dot{u}^{0} + \Gamma^{0}_{\mu\nu}u^{\mu}u^{\nu} = 0 \ . \tag{36}$$

One can show that the only non-vanishing component of $\Gamma^0_{\mu\nu}$ is $\Gamma^0_{ij} = Ha^2 h_{ij}$, where H denotes the Hubble parameter, so that:

$$\frac{du^0}{d\lambda} + H|\mathbf{u}|^2 = 0 \ . \tag{37}$$

Now, let us use the fact that $(u^0)^2 - |\mathbf{u}|^2 = 1$ to show that $u^0 du^0 = |\mathbf{u}|d|\mathbf{u}|$, so that we find:

$$\frac{1}{u^0}\frac{d|\mathbf{u}|}{d\lambda} + H|\mathbf{u}| = 0 \ . \tag{38}$$

Finally, since $u^0 = dt/d\lambda$ this yields:

$$\frac{1}{|\mathbf{u}|}\frac{d|\mathbf{u}|}{dt} = -\frac{1}{a}\frac{da}{dt} \ , \tag{39}$$

which impies $|\mathbf{u}| \propto a^{-1}$. Recalling that the particle's 4-momentum is given by $p^\mu = mu^\mu$, this simply implies that the magnitude of the particle's 3-momentum redshifts with expansion. This is in agreement with the quantum mechanical picture where the particle's de Broglie wavelength, proportional to $|\mathbf{p}|$ is stretched by a factor $a(t)$ due to the Hubble expansion. It also implies that the peculiar velocity of a freely falling non-relativistic particle $v^i = dx^i/dt \sim u^i$ will be redshifted with expansion, and so any such particle will eventually come to rest in this frame. Note that this is not necessarily the case when we include perturbations about the homogeneous and isotropic spacetime, as we will see later on in the course.

Kinematical observables

To finalize our discussion of kinematics in FRW spacetimes, let us look at two important observables that allow us to measure the distance to luminous objects in the Universe:

(a) Luminosity distance

Recall from the first lecture that we defined the luminosity distance via:

$$d_L^2 = \frac{\mathcal{L}}{4\pi\mathcal{F}} \ , \tag{40}$$

where $\mathcal{L}$ is the absolute luminosity of an object, say a star, galaxy or galaxy cluster, and $\mathcal{F}$ is the energy flux measured at the Earth. Now, if the object is at a comoving distance r from the Earth, at $r = 0$ and is observed today $(a = a_0)$, we have:

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi(a_0 r^2)} \times \frac{1}{1+z} \times \frac{1}{1+z} \ , \tag{41}$$

where the first $(1+z)^{-1}$ factor arises from photon redshift and the second factor is due to the increase in the time of arrival of consecutive photons, corresponding to the increase in the distance they have to travel $a(t) \propto (1+z)^{-1}$. This then implies that the luminosity distance is given by:

$$d_L = a_0 r(1+z) \ . \tag{42}$$

We would like to express d_L solely in terms of the redshift factor, so let us begin by expanding the scale factor about the present time with $a_0 = 1$:

$$\begin{aligned} a(t) &= 1 + \dot{a}_0(t-t_0) + \frac{1}{2}\ddot{a}_0(t-t_0)^2 + \dots \\ &= 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)^2 + \dots \end{aligned} \tag{43}$$

where the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ is defined with a minus sign for historical reasons, since for a Universe filled with ordinary matter or radiation q_0 is positive. As $a(t)/a_0 = (1+z)^{-1}$, we have:

$$\begin{aligned} 1+z &= \left(1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)^2 + \dots\right)^{-1} \\ &= 1 - H_0(t-t_0) + \frac{1}{2}q_0 H_0^2(t-t_0)^2 + H_0^2(t-t_0)^2 + \dots\ , \end{aligned} \tag{44}$$

or equivalently:

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right) H_0^2(t_0-t)^2 + \dots\ . \tag{45}$$

We may invert this relation to yield:

$$t_0 - t = H_0^{-1}\left[z + \left(1 + \frac{q_0}{2}\right) z^2 + \dots\right]\ . \tag{46}$$

Now recall that for a photon at small comoving radius r, from Eq. (21), we have $r = \tau_0 - \tau$ to leading order in all three cases, yielding:

$$\begin{aligned} r + \dots &= \int_t^{t_0} \frac{dt}{a(t)} = \int_t^{t_0} \frac{1}{(1 + H_0(t-t_0) + \dots)^{-1}} = \\ &= (t_0 - t) + \frac{1}{2}H_0(t_0-t)^2 + \dots \\ &= H_0^{-1}\left(z - \frac{(1+q_0)}{2}z^2 + \dots\right)\ . \end{aligned} \tag{47}$$

We may now replace this into the expression for the luminosity distance in Eq. (42) to yield:

$$d_L = H_0^{-1}\left(z + \frac{1}{2}(1-q_0)z^2 + \dots\right)\ , \tag{48}$$

which is the expression that we anticipated in the first lecture, with the first term giving the linear relation between distance and redshift (velocity) that corresponds to the Hubble law.

(b) Angular diameter distance

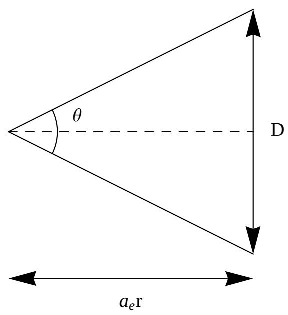

Figure 1: An object of physical size D at comoving distance r emits light at a time t_e and is observed with an angular diameter θ.

An object with physical size D at comoving distance r is viewed with an angular diameter θ given by:

$$\tan\left(\frac{\theta}{2}\right) = \frac{D/2}{a_e r}\ , \tag{49}$$

where a_e is the value of the scale factor at the time the object emits light. For $\theta \ll 1$, we have $\tan(\theta/2) \simeq \theta/2$ which implies:

$$\theta \simeq \frac{D}{a_e r} \ . \tag{50}$$

We thus define the angular diameter distance to an object by:

$$\begin{aligned} d_A &= \frac{D}{\theta} \\ &= a_e r \\ &= \frac{r}{1+z_e} \ . \end{aligned} \tag{51}$$

From our previous discussion this means that:

$$d_A = \frac{d_L}{(1+z)^2} \ . \tag{52}$$

Let us consider, for example, a matter-dominated flat universe, where $a(t) = (t/t_0)^{2/3}$ as we have derived previously. Hence,

$$\begin{aligned} r &= \int_{t_e}^{t_0} dt \left(\frac{t_0}{t}\right)^{\frac{2}{3}} \\ &= 3t_0 \left(1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right) \\ &= 3t_0 \left(1 - \sqrt{\frac{a_e}{a_0}}\right) \\ &= 3t_0 \left(1 - \frac{1}{\sqrt{1+z_e}}\right) \ . \end{aligned} \tag{53}$$

This gives:

$$\theta = \frac{D(1+z)}{3t_0} \left((1 - \frac{1}{\sqrt{1+z}}\right)^{-1} = \frac{D}{3t_0} \frac{(1+z)^{3/2}}{\sqrt{1+z}-1} \ , \tag{54}$$

which we plot in the figure below.

Note that θ diverges as $z \to 0$, as the object is infinitely close to the observer, and as $z \to \infty$, since the Big Bang singularity is very small, but nevertheless covers the whole sky. Also take into account that for an object which is tightly bound by gravity or other sufficiently strong forces, as is the case of the solar system, a galaxy or even a galaxy cluster, it decouples from the Hubble flow and its size D remains roughly constant during the evolution of the Universe.

Problem 4

(a) Relate the proper distance to an object $l_0 = \int_0^{l_0} \sqrt{g_{rr}} dr$ in a flat matter-dominated universe to the cosmological redshift and show that the luminosity distance is given by

$$d_L = \frac{l_0}{(1 - l_0/3t_0)^2} \ . \tag{55}$$

Discuss what happens as $l_0 \to 3t_0$.

(b) Compute all the non-vanishing Christoffel symbols for the general FRW metric in Eq. (17).

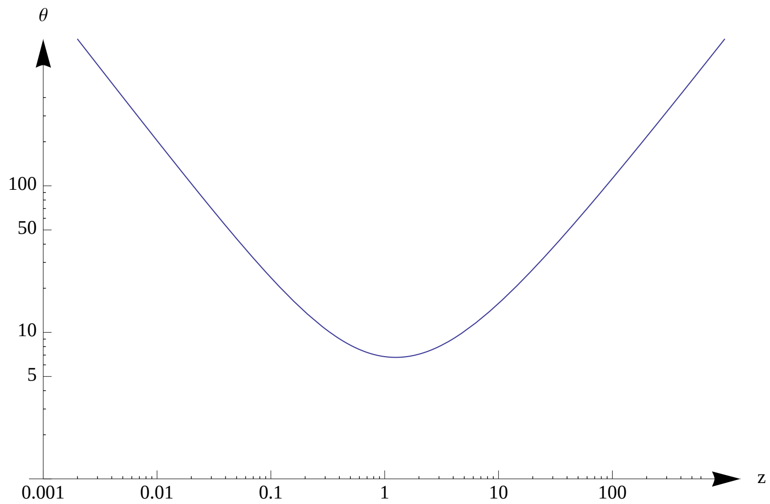

Figure 2: The angular diameter of an object with physical size $D = 3ct_0$ in a flat matter-dominated universe. The minimum angle occurs for $z_{min} = 5/4$.

References

[1] M. A. Strauss and J. A. Willick, Phys. Rept. **261**, 271 (1995) [astro-ph/9502079].