Introduction to Cosmology

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LECTURE 3 - Newtonian cosmology II

In this lecture we will continue to describe the properties of the Universe in the Hot Big Bang model derived from newtonian principles, in particular taking a first look at the expansion history of the Universe and the relevant nuclear and particle physics phenomena that take place as the Universe expands and cools down. We will also review the basic principles of perturbation theory that can be obtained within this framework and that we will develop later on in the course.

Age of the Universe

As we have seen in the previous lecture, the Friedmann equation describes the dynamics of an expanding Universe according to its matter and energy content. Recalling how the different relativistic and non-relativistic fluid components evolve with the scale factor, we may write the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\Omega_{\Lambda 0} + \frac{\Omega_{k0}}{a^2} + \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4}\right) , \tag{1}$$

where the subscript '0' denotes the present value of the relative abundace $\Omega_{i0} = \rho_i/\rho_c(t_0)$ of each component, normalizing $a_0 = 1$ and considering a general model with cosmological constant, curvature, matter and radiation. If we then measure the present abundances, we may solve this equation for the cosmic time t and obtain the age of the Universe t_0:

$$t_0 = H_0^{-1} f(\Omega_{\Lambda 0}, \Omega_{k0}, \Omega_{m0}, \Omega_{r0}) , \tag{2}$$

in natural units such that $\hbar = c = 1$. This gives a complicated expression in the general case, but it is easy to convince ourselves with a few simple examples with a single component that the Hubble time $H_0^{-1} \simeq 9.8h^{-1}$ Gyr sets the scale for the age of the Universe, as anticipated in the first lecture:

1. **Empty universe**

   A simple unphysical model with no matter, radiation nor cosmological constant but $\Omega_{k0} = 1$, yields:

   $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{a^2} \qquad \Rightarrow \qquad \dot{a} = H_0 , \tag{3}$$

   for an expanding universe. The solution satisfying $a_0 = 1$ is $a(t) = H_0 t$, which gives $H(t) = \dot{a}/a = 1/t$ and $t_0 = H_0^{-1}$. The scale factor then evolves from an initial singularity at $t = 0$ and the age of the Universe is exactly given by the Hubble time in this case.

2. **Matter-dominated flat universe**

   For $\Omega_{m0} = 1$ and no other components we have, as derived in this last lecture:

   $$a(t) = \left(\frac{t}{t_0}\right)^{2/3} . \tag{4}$$

Upon differentiation, this yields $H(t) = 2/3t$ and the age of the Universe is $t_0 = 2/3H_0^{-1}$. A similar result is obtained for a radiation-dominated universe replacing the factor 2/3 by 1/2.

3. **de Sitter Universe**

   If $\Omega_{\Lambda 0} = 1$ and no matter, radiation or curvature is present, we obtain a constant expansion rate $H(t) = H_0$, as derived in the last lecture:

$$a(t) = e^{H_0(t-t_0)} . \tag{5}$$

   In this de Sitter universe, there is no 'Big Bang' singularity, as the scale factor only vanishes in the limit $t \to -\infty$. The age of the universe is then formally infinite and, although this is not a fully realistic model, this is the key idea behind *inflation*, which we will study later on in the course.

Acceleration equation

Let us now differentiate the Friedmann equation:

$$2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2}{a^3}\dot{a}k . \tag{6}$$

Dividing by twice the Hubble parameter $H = \dot{a}/a$ and using the Friedmann equation we get:

$$\frac{\ddot{a}}{a} - \frac{8\pi G}{3}\rho + \frac{k}{a^2} = \frac{4\pi G}{3}\frac{\dot{\rho}}{H} + \frac{k}{a^2} , \tag{7}$$

so that the curvature terms cancel on both sides of the equation. Finally, recall that for a homogeneous and isotropic perfect fluid of energy density ρ and pressure p we have from the first law of thermodynamics $\dot{\rho} = -3H(\rho + p)$, so that we obtain:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) , \tag{8}$$

which is known as the acceleration equation or Raychaudhuri equation, after the Indian physicist Amal Kumar Raychaudhuri (1923-2005)[1].

From this equation, we can infer that expansion decelerates in a universe dominated by ordinary matter or radiation, so that the supernovae results described in the first lecture suggest the presence of an exotic dark energy component with *negative pressure* such that for $p < -\rho/3$ (or $w < -1/3$ for the equation of state parameter), we have $\ddot{a} > 0$. The cosmological constant with $w = -1$ is the simplest example of such a fluid, but even more exotic things may arise in the context of relativistic quantum field theory, as we will discuss in our last lecture.

Brief thermal history

So far we have discussed examples with single-component homogeneous and isotropic fluids dominating the energy density in the Universe, in which case we can derive simple analytic solutions for the evolution of the scale factor. The real (observed) Universe, however, seems to contain non-relativistic matter, radiation and a non-zero cosmological constant or a more general dark energy fluid. It is worth noting that these examples are nevertheless useful, since each of these components dominates at different epochs in the expansion history due to their different dilution rates. Radiation is diluted faster, as a^{-4}, while non-relativistic matter scales as a^{-3} and a cosmological constant has a constant energy density. This implies that the very early universe was dominated by relativistic particles, followed by heavy dark and baryonic matter and, finally, by dark energy, as illustrated in the following figure.

In the very early universe, for temperatures $T \gg 100$ GeV ($\approx 10^{15}$ K), all the Standard Model particles (photons, leptons, quarks, etc) behaved as relativistic particles or radiation, which dominated the energy density of the Universe. From equilibrium thermodynamics, the energy density in radiation is given by:

$$\rho_r = \frac{\pi^2}{30}g_* T^4 , \tag{9}$$

[1] Raychaudhuri derived a more general set of equations within the framework of general relativity that reduce to this equation in a homogeneous and isotropic cosmology.

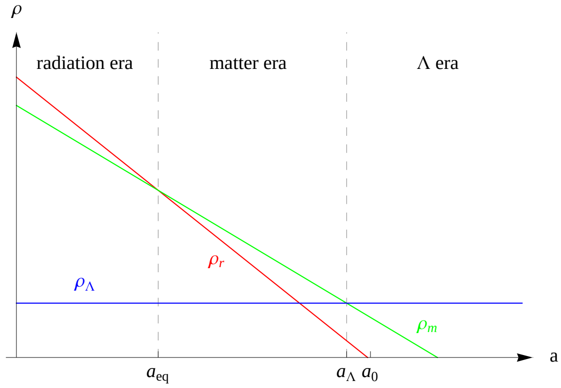

Figure 1: Evolution of the energy density for radiation, matter and cosmological constant, as a function of the scale factor $a(t)$.

in natural units setting $\hbar = c = k_B = 1$. The quantity g_* denotes the number of relativistic degrees of freedom, including all bosonic and fermionic particles with masses below the temperature of the radiation, the latter multiplied by a factor 7/8. For example, for a universe with photons, electrons, positrons and and three neutrinos we have:

$$g_* = \underbrace{2}_{photons} + \frac{7}{8}\left(\underbrace{3\times 2}_{\nu} + \underbrace{2+2}_{e^+e^-}\right) = 10.75 \ , \tag{10}$$

including two spin states for each particle. The full Standard Model set corresponds to $g_* = 106.75$. From Eq. (9), we find that $T \propto a^{-1}$, which is not surprising if we recall Wien's law for relation between the temperature of thermal blackbody radiation and the wavelength corresponding to the emission peak $\lambda_{max} \propto T^{-1}$. This implies that radiation cools down with expansion, as we had anticipated, so that the Standard Model particles become non-relativistic and stop contributing to the radiation energy density once their mass threshold $T \simeq m$ is crossed. Today, the only free relativistic particles in the Universe are photons and neutrinos, although as we will describe later in the course the latter are decoupled from the photon thermal bath and have a lower temperature. The present energy density of radiation at temperature $T = 2.73$ K is

$$\rho_{r0} = 7.8 \times 10^{-31} \ \mathrm{kg\,m^{-3}} \ , \tag{11}$$

which corresponds to $\Omega_{r0}h^2 \simeq 4\times 10^{-5}$. During the radiation era, we can then use the results derived in the previous lecture to relate cosmic time and temperature:

$$\begin{aligned} \rho_r(t) &= \frac{3}{32\pi G t^2} \qquad \Rightarrow \\ t &\simeq \frac{2.4}{\sqrt{g_*}}\left(\frac{1\ \mathrm{MeV}}{T}\right)^2 \ \mathrm{sec} \ . \end{aligned} \tag{12}$$

The radiation era terminates when $\rho_r(a_{eq}) = \rho_m(a_{eq})$, which is known as *matter-radiation equality*. Taking $\Omega_m h^2 \simeq 0.14$ [1], we find for the redshift z_{eq} of this epoch:

$$\begin{aligned} \rho_{r0}\left(\frac{a_{eq}}{a_0}\right)^4 &= \rho_{m0}\left(\frac{a_{eq}}{a_0}\right)^3 \qquad \Rightarrow \\ 1+z_{eq} &= \frac{a_0}{a_{eq}} = \frac{\rho_{m0}}{\rho_{r0}} = \frac{\Omega_{m0}}{\Omega_{r0}} \simeq 3500 \ , \end{aligned} \tag{13}$$

corresponding to a temperature $T \simeq 1$ eV ($\simeq 9500$ K). Soon after this non-relativistic matter, in particular cold dark matter with $\Omega_c h^2 \simeq 0.116$, takes over and becomes the dominant component until it is overcome by the cosmological constant (assuming this is the correct form of dark energy). One can show that $\rho_m = \rho_\Lambda$ at a redshift:

$$z_\Lambda = \left(\frac{1-\Omega_{m0}}{\Omega_{m0}}\right)^{\frac{1}{3}} - 1 \simeq 0.8 \ , \tag{14}$$

but that expansion begins accelerating earlier, at $z_{acc} \simeq 1.3$. Such a small value of the redshift for Λ-domination constitutes one of the most intriguing puzzles in modern cosmology - the *coincidence problem*: why has the cosmological constant only started to dominate recently in the cosmic history? We will come back to this problem at the end of the course.

CMB and nucleosynthesis

The synthesis of light elements begins during the radiation era, when the temperature falls below $T \simeq 1$ MeV. Above this threshold, neutrons and protons are essentially in thermal equilibrium with similar abundances. Equilibrium is maintained through weak interactions of the form:

$$n \longleftrightarrow p + e^- + \bar{\nu}_e \ , \tag{15}$$

as long as the interaction rate $\Gamma \gtrsim H$. Nuclear reactions are also in equilibrium at this stage, and in particular the fusion of a proton and a neutron into a deuterium bound state, $p + n \longleftrightarrow D$, is faster than expansion, so that Deuterium is produced essentially at the same rate that it is destroyed and keeps a small equilibrium abundance. Once these reactions fall out of equilibrium for $\Gamma \lesssim H$, as we will discuss in more detail later on, the inverse processes are suppressed and a constant abundance of elements is formed. Deuterium can then fuse into ^{4}He nuclei via:

$$\mathrm{D} + \mathrm{D} \longrightarrow {}^4\mathrm{He}. \tag{16}$$

However, this requires overcoming the Coulomb barrier felt by the two protons involved, which gives an exponential suppression of the reaction rate $\propto \exp(-2/T^{1/3}(\mathrm{MeV}))$, so that a constant abundance of ^{4}He can only be produced for $T \lesssim 0.1$ MeV. Once the temperature drops below this threshold, essentially all deuterons become bound into Helium nuclei and the average rate of nuclear reactions drops below the Hubble rate. This implies:

$$\Gamma_{nuc} = n_B \langle \sigma v \rangle \sim H. \tag{17}$$

To estimate the number of nucleons (baryons) present at this stage, we can use an average cross section for nuclear processes $\sigma \sim 0.01 m_p^{-2}$, where $m_p \simeq 1$ GeV denotes the proton mass, and an average thermal velocity $v \sim (T/m_p)^{1/2}$. As during the radiation era we have:

$$H = \frac{\pi}{\sqrt{90}} \sqrt{g_*} \frac{T^2}{M_P} \ , \tag{18}$$

where we have defined the reduced Planck mass $M_P = 1/\sqrt{8\pi G} \simeq 2.4 \times 10^{18}$ GeV, we then find:

$$n_B \sim 10^2 \frac{m_p^{5/2} T^{3/2}}{M_P} \ . \tag{19}$$

The number of photons is given by $n_\gamma = (2\zeta(3)/\pi^2)T^3$, so we obtain the following baryon-to-photon ratio at $T \simeq 0.1$ MeV:

$$\eta = \frac{n_B}{n_\gamma} \sim 10^2 \left(\frac{m_p}{M_P}\right) \left(\frac{m_p}{T}\right)^{3/2} \sim 2 \times 10^{-10} \ . \tag{20}$$

After this stage this ratio is conserved, since there are no known processes producing additional baryons or photons below this temperature. We may then use this result to estimate the present temperature of photons in the Universe. Noting that $\rho_B \simeq n_B m_p$, we find after some algebra and including the missing factors of $\hbar$, c and k_B:

$$T_0 = \left(\frac{\Omega_B h^2}{0.022}\right)^{\frac{1}{3}} \left(\frac{6 \times 10^{-10}}{\eta}\right)^{1/3} 2.73 \text{ K} \ , \tag{21}$$

so that our crude estimate yields a value remarkably close to the measured temperature of the CMB, which is one of the major successes of the Hot Big Bang model.

Cosmological perturbations

To complete our first glance at the Hot Big Bang model it is important to realize that, even though homogeneous and isotropic models can go a long way in describing the real Universe, it is from the structure of the Universe that we can truly learn about its evolution - fluctuations in the CMB, large scale structure, etc. Cosmological perturbation theory is an intricate subject that is well-defined within the framework of general relativity and that we will discuss in more detail later on the course, but we may already learn a great deal about the evolution of small perturbations in cosmology by considering the newtonian framework.

Let us then consider the evolution of small perturbations about a homogeneous and isotropic background using the Friedmann equation. We will consider only *adiabatic perturbations*, where all perturbations in the different fluids arise solely due to fluctuations of the background geometry, i.e. nothing in the equation of state of matter, radiation, etc, distinguishes one local region from another. We then expand the relevant geometric quantities about a given background value denoted by a subscript 'B':

$$\begin{aligned} a(t,\mathbf{x}) &= a_B(t)(1+\alpha(t,\mathbf{x}))\ , \\ k(\mathbf{x}) &= k_B + \delta k(\mathbf{x})\ . \end{aligned} \tag{22}$$

Substituting this in the Friedmann equation $\dot{a}^2 = (8\pi G/3)\rho a^2 - k$, we find to leading order and ignoring gradients, which holds for $\lambda \gg H^{-1}$:

$$[\dot{a}_B(1+\alpha) + a_B\dot{\alpha}]^2 = \frac{8\pi G}{3}\left[(\rho a^2)_B + a_B\alpha\frac{d}{da}(\rho a^2)_B\right] - (k_B + \delta k)\ . \tag{23}$$

From differentiating the background Friedmann equation, we can write the acceleration equation (8) in the form:

$$\ddot{a}_B = \frac{4\pi G}{3}\frac{d}{da}(\rho a^2)_B\ . \tag{24}$$

Replacing this result in the perturbed Friedmann equation, we find after some algebra and using the background Friedmann equation:

$$\dot{\alpha} + \left(\frac{\dot{a}_B}{a_B} - \frac{\ddot{a}_B}{\dot{a}_B}\right)\alpha = -\frac{1}{2}\frac{\delta k}{a_B\dot{a}_B}\ . \tag{25}$$

This is the differential equation defining the evolution of linear perturbations in a homogeneous and isotropic background. It is easy to check that it can be written as:

$$\frac{d}{dt}\left(\alpha\frac{a_B}{\dot{a}_B}\right) = -\frac{1}{2}\frac{\delta k}{\dot{a}_B^2}\ , \tag{26}$$

and hence has the general solution:

$$\alpha(t,\mathbf{x}) = C(\mathbf{x})\frac{\dot{a}_B}{a_B} - \frac{\delta k(\mathbf{x})}{2}\frac{\dot{a}_B}{a_B}\int\frac{dt}{\dot{a}_B^2}\ . \tag{27}$$

The homogeneous solution ($\delta k = 0$) is the *time-delay mode* proportional to the background Hubble parameter and which typically decays away for matter or radiation-domination, remaining constant in a Λ-dominated universe. The inhomogeneous term is the *curvature perturbation mode*, and it is easy to convince ourselves that this grows as $t^{2/3}$ for a matter-dominated background and t in the radiation era, while decaying in a Λ-dominated epoch. It is thus clear that linear perturbations may grow in an expanding Universe and form structure, while a cosmological constant inhibits structure formation.

For adiabatic perturbations, all species are perturbed in a fixed ratio, as from energy conservation we have:

$$\frac{\delta\rho_i}{\rho_i} \simeq -3(1+w_i)\alpha\ . \tag{28}$$

These are the simplest perturbations - adiabatic, linear with a growing mode and, as we will discuss later on, with a nearly gaussian distribution (random phase) and a scale-invariant spectrum.

Problem 3

Consider a flat universe ($k = 0$) filled with non-relativistic matter and a non-zero cosmological constant such that $\Omega_{m0} + \Omega_\Lambda = 1$.
(a) Integrate the Friedmann equation with respect to cosmic time t, with initial condition $a(t = 0) = 0$ to find:

$$a(t) = \left(\frac{\Omega_{m0}}{1-\Omega_{m0}}\right)^{1/3} \sinh^{2/3}\left(\frac{2}{3}H_0\sqrt{1-\Omega_{m0}}t\right) .$$

Verify that this solution has the correct limits as $\Omega_{m0} \to 0$ and $\Omega_{m0} \to 1$.
(b) Show that the age of the universe in this model is given by:

$$t_0 = \frac{2}{3}H_0^{-1}(1-\Omega_{m0})^{-1/2}\sinh^{-1}\left(\sqrt{\frac{1-\Omega_{m0}}{\Omega_{m0}}}\right)$$

and sketch this as a function of Ω_{m0}.
(c) Show that the energy density of the universe becomes Λ-dominated at a redshift:

$$1 + z_\Lambda = \left(\frac{1-\Omega_{m0}}{\Omega_{m0}}\right)^{1/3}$$

but begins accelerating earlier at $1 + z = 2^{1/3}(1 + z_\Lambda)$.

References

[1] G. Hinshaw, D. Larson, E. Komatsu, D. N. Spergel, C. L. Bennett, J. Dunkley, M. R. Nolta and M. Halpern *et al.*, arXiv:1212.5226 [astro-ph.CO].