Introduction to Cosmology

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LECTURE 2 - Newtonian cosmology I

As a first approach to the Hot Big Bang model, in this lecture we will consider a cosmological model based on newtonian dynamics. Although general relativity will become imperative in order to describe the geometrical structure of an expanding spacetime filled with relativistic fluids, we will see that the basic features of the standard cosmological model can be described using Newton's laws for non-relativistic matter and some added ingredients, which will be justified later on.

As we have seen in the previous lecture, there is a lot of observational evidence for an expanding universe that is isotropic and homogeneous on large scales, and this *cosmological principle* will form the basis of all our discussion.

Friedmann equation

The simplest model of the Universe one can think of is that of a universe filled with dust, i.e. non-relativistic pressureless matter, $p = 0$. We can think of this dust as a collection of point particles, and on the cosmological scales of approximately $10 - 1000$ Mpc these point particles are a sufficiently good approximation for galaxies or even galaxy clusters. The cosmological principle states that there is no preferred place or direction in the Universe on large scales, so we can pick any coordinate system with respect to which we can measure the positions and velocities of these test particles.

Let us then consider a point particle of mass m at a radius $a(t)$ from an arbitrarily defined origin, a distance we shall refer to as the *scale factor*. Using Newton's 2nd law $F = m\ddot{a}$, this yields:

$$m\ddot{a} = -\frac{GMm}{a^2} \ , \tag{1}$$

since the particle feels the gravitational attraction of all the matter contained inside a sphere of radius a, with total mass M. This is of course independent of the mass of the test particle, and we can multiply each side of Eq. (1) by the particle's velocity $\dot{a}$ and integrate once, giving:

$$\begin{aligned} \ddot{a}\dot{a} &= -\frac{GM\dot{a}}{a^2} \Leftrightarrow \\ \frac{1}{2}\dot{a}^2 &= \frac{GM}{a} - \frac{k}{2} \end{aligned} \tag{2}$$

where we have written the integration constant as $-k/2$ for later convenience. Now, if dust is homogeneously and isotropically distributed with density ρ, the mass contained inside the radius a is given by:

$$M = \frac{4}{3}\pi\rho a^3 \ , \tag{3}$$

which is a constant, since no matter is being created or destroyed. This implies that $\rho \propto a^{-3}$ for non-relativistic matter. Using this, we may write Eq. (2) in the form:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \tag{4}$$

where the Hubble parameter $H = \dot{a}/a$ gives the rate of expansion as we discussed in the previous lecture. This is known as the *Friedmann equation* after the Russian physicist Alexander Friedmann (1888-1925) and is the fundamental equation in standard cosmology.

Although we have derived this equation using newtonian dynamics for non-relativistic matter, which holds for small velocities $v/c \ll 1$ and weak gravitational fields $\Phi/c^2 \ll 1$, this turns out to be the most general form of the Friedmann equation, with $\rho = \sum \rho_i$ denoting the total energy density of the Universe, filled with different relativistic and non-relativistic fluids of densities ρ_i.

Open, closed and flat universes

The constant of integration k that we have introduced earlier actually acquires an important meaning within general relativity, as we will justify later on, representing the *spatial curvature of the Universe*. There are three possible cases:

- **closed universe**: for $k > 0$, the Universe has the geometry of a 3-sphere, S^3, with radius of curvature given by:
$$\frac{k}{a^2} = \frac{1}{R^2} \ , \tag{5}$$
and associated volume $V = 2\pi^2 R^3$.
- **flat universe**: for $k = 0$, the Universe is flat, as can be seen by taking the limit $R \to \infty$, obeying the laws of 3-dimensional Euclidean geometry, E^3.
- **open universe**: for $k < 0$, the Universe has the geometry of a 3-dimensional hyperboloid, H^3, which is unbounded.

These three possible cases are illustrated in the figure below and we will take a closer look at their properties when we discuss Friedmann-Robertson-Walker spacetimes in more detail.

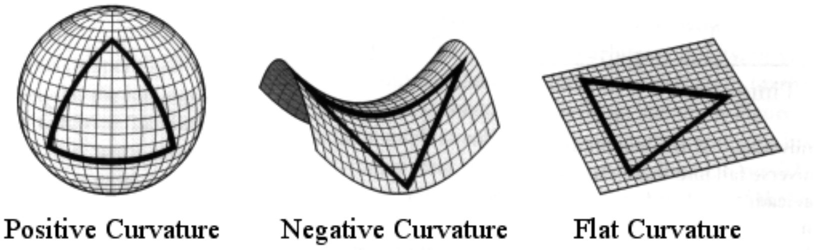

Figure 1: The three possible cases for the geometry of the Universe depending on the sign of the curvature constant k. A simple consequence of the geometry is that, while in flat space the sum of the angles in a triangle is 180°, for positive (negative) curvature it is larger (smaller) than 180°.

To better understand the evolution of the Universe in these three possible cases, let us rewrite the Friedmann equation in the form:
$$\frac{1}{2}\dot{a}^2 + V_{eff}(a) = -\frac{k}{2} \ , \tag{6}$$
which is simply the equation for a test particle of total energy $-k/2$ in an effective potential $V_{eff}(a) = -(4\pi G)/3\rho a^2$. For a dust-filled universe, $\rho = \rho_m$, the effective potential falls with the inverse distance $V_{eff}(a) \propto -1/a$, and this is just the familiar Kepler problem (with vanishing angular momentum). We know, for example, that there are *no static* solutions in this problem and that, taking into account that the kinetic energy is always positive:

- For $k < 0$, the energy is positive and trajectories are unbounded, which implies eternal expansion;

- For $k > 0$, the energy is negative and the trajectories are bounded, i.e. the universe will expand up to a maximum size and then contract.

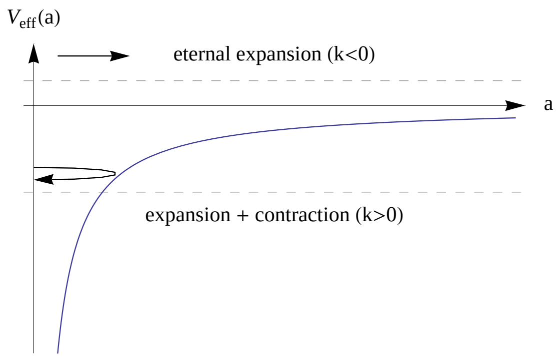

Figure 2: Analysis of the dust-filled universe using the effective potential approach.

The special case $k = 0$ of a flat dust-filled Universe is marginal, separating an eternally expanding Universe from one that will eventually recollapse into a 'big crunch' singularity. Figure 3 illustrates the evolution of the scale factor for the three different cases in a matter-dominated universe.

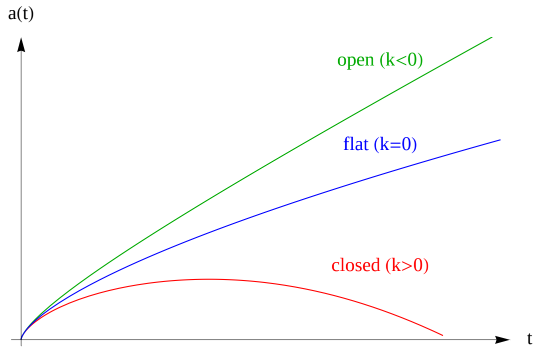

Figure 3: Evolution of the scale factor with time in a matter-dominated universe for the three possible cases.

Einstein's static universe

The fact that a universe filled with simple pressureless matter either expands or contracts constituted, as we discussed in the previous lecture, a big problem for the scientific community before Hubble's discovery in 1929. Einstein, in particular, was quite troubled by this conclusion being an immediate consequence of his gravitational field equations. However, general relativity was actually defined only up to a constant term Λ in the gravitational action, which we now know as the cosmological constant and in fact corresponds to vacuum energy. Contrary to non-relativistic matter, the total energy of vacuum in a given volume does not remain constant with expansion, since as spacetime expands more vacuum is being created. In practice, the cosmological constant corresponds to a fluid with constant energy density, $\rho_\Lambda = const$, as will become clear when we discuss the Hot Big Bang model within the framework of general relativity.

For now, let us discuss the consequences of adding vacuum energy into the game. Adding a constant ρ_Λ to the dust-filled universe above, it is easy to show that the effective potential becomes:

$$V_{eff}(a) = -\frac{4\pi G}{3}(\rho_m + \rho_\Lambda)a^2 = -\frac{GM}{a} - \frac{4\pi G}{3}\rho_\Lambda a^2 \ , \tag{7}$$

which is illustrated in Figure 4.

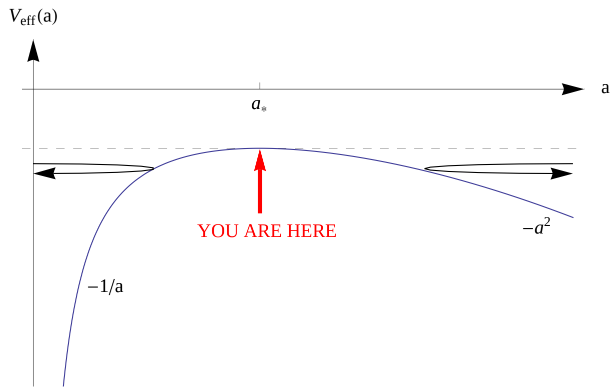

Figure 4: Effective potential for a universe with non-relativistic matter and a cosmological constant.

It is clear from this figure that there are two types of bouncing solutions - (i) at small values of a the universe can expand up to a maximum size and (ii) at large values of a the universe can contract down to a minimum size. Most importantly, the effective potential has a local maximum:

$$\frac{\partial V_{eff}}{\partial a}(a_*) = 0 \qquad \Rightarrow \qquad a_* = \left(\frac{3M}{8\pi\rho_\Lambda}\right)^{\frac{1}{3}} \ , \tag{8}$$

where we can find a static universe, which was Einstein's primary reason for considering a cosmological constant term in the gravitational field equations. One can show that at this point we have $\rho_m = 2\rho_\Lambda$, and a static solution $\dot{a} = 0$ implies $V_{eff}(a_*) = -k/2$, which gives a closed universe with positive curvature:

$$\begin{aligned} k &= -2V_{eff}(a_*) = \\ &= 8\pi G\rho_\Lambda a_*^2 \end{aligned} \tag{9}$$

The curvature radius can be computed using Eq. (5), yielding:

$$R = \frac{a_*}{k^{1/2}} = (8\pi G\rho_\Lambda)^{-\frac{1}{2}} \ , \tag{10}$$

which is independent of the amount of non-relativistic matter and is thus the curvature radius for a closed Universe dominated by a cosmological constant. The problem with this solution is obviously that it is not stable, and any small deviation will lead to a bouncing solution of the type described above. This was Einstein's 'biggest' blunder but, as we will see, an important solution in modern cosmology.

Evolution of the energy density

As mentioned above, the Friedmann equation in the general form Eq. (4) admits not only non-relativistic dust but also other types of matter and energy, and it is important to investigate how different fluids evolve in an expanding universe. This can be derived simply using the First Law of Thermodynamics:

$$dE = -pdV + TdS \ , \tag{11}$$

where the variation of the energy in a given fluid volume corresponds to the work and heat transferred in or out of it. If the rate of interactions between the particles in the fluid is small, we may neglect the heat transfer and consider that the expansion of the Universe is *adiabatic* or *isentropic*, $TdS = 0$. For a fluid of energy density ρ and pressure p enclosed in a volume $V = a^3$, we thus have:

$$\frac{d}{dt}(\rho a^3) = -p\frac{d}{dt}(a^3) \ , \tag{12}$$

which can be written as a conservation equation:

$$\dot{\rho} = -3H(\rho + p) \ . \tag{13}$$

It is convenient to introduce the equation of state parameter w defined as:

$$p = w\rho \ , \tag{14}$$

so that we may write Eq. (13) as:

$$\frac{\dot{\rho}}{\rho} = -3H(1 + w) \ . \tag{15}$$

The most relevant fluids for cosmology are:

- **dust/non-relativistic matter**:

  This describes a fluid of cold and heavy particles ($T \ll m$), in particular cold dark matter (CDM) and baryons, for which the pressure is negligible, with $w = 0$, so that $\rho_m \propto a^{-3}$ as we have seen previously;

- **radiation**:

  This describes hot relativistic particles ($T \gg m$), such as photons or in fact all Standard Model particles in the very early Universe at high temperatures. Radiation has an equation of state $w = 1/3$, so that $\rho_r \propto a^{-4}$. One can understand this behaviour recalling that expansion redshifts the wavelength of radiation, so that the total energy $E \propto \lambda^{-1} \propto a^{-1}$ is not conserved, as opposed to the total dust mass in Eq. (3).

- **cosmological constant:** Vacuum energy corresponds to a fluid with negative pressure $p_\Lambda = -\rho_\Lambda$ or $w = -1$. This pressure counteracts the attractive gravitational effect of matter to produce the static solution found earlier and may in fact lead to accelerated expansion as we will discuss in the next lecture.

Critical density

A universe with a flat geometry plays a prominent role in cosmology, not only as it leads to a critical evolution as described above but also since there is substancial observational evidence for our Universe having a very small curvature, as discussed in Lecture 1. It is then convenient to define the *critical density* corresponding to a flat universe:

$$\rho_c = \frac{3H^2}{8\pi G} \ , \tag{16}$$

where we have simply used the Friedmann equation for $k = 0$. Like the Hubble parameter, this critical value varies during the cosmological evolution and today its measured value is $1.88 \times 10^{-29} h^2 \ \mathrm{gcm}^{-3}$, which is extremely small! It is then common to normalize the energy density of each fluid component to the critical density at a given time, defining the relative abundance:

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \ , \tag{17}$$

with the index i denoting matter, radiation, cosmological constant, etc. The Friedmann equation is then often written in the form:

$$\Omega_T = \sum_i \Omega_i = 1 + \frac{k}{aH^2} \ , \tag{18}$$

so that a closed (open) universe has $\Omega_T > 1$ ($\Omega_T < 1$) and the flat universe yields the critical value $\Omega_T = 1$. Note that sometimes one defines $\Omega_k = -k/(a^2H^2)$ as the relative contribution of curvature to the energy density, giving $\tilde{\Omega}_T = \Omega_T + \Omega_k = 1$, although this is not particularly useful. It is common to denote as Ω_0 the present energy density normalized to the critical value (excluding curvature), and observational data (namely the CMB) yields $\Omega_0 = 1.002 \pm 0.011$ [1], which is thus extremely close to a flat universe.

Simple solutions in a flat universe

Let us then consider a few simple examples of solutions in a universe with Euclidean geometry. If the universe is dominated by a fluid with equation of state parameter w, we have:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} , \tag{19}$$

which follows from the conservation equation (13), with a_0 denoting the present value of the scale factor, which without loss of generality we can set to unity. The Friedmann equation can then be written as:

$$\dot{a}^2 = H_0^2 a^{-1-3w} , \tag{20}$$

yielding

$$a^{\frac{1}{2}(1+3w)} da = H_0 dt . \tag{21}$$

It is straightfoward to integrate this to find:

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} , \qquad w \neq -1 , \tag{22}$$

where we chose the integration constant such that $a(t_0) = a_0 = 1$. This yields, for example, $a(t) = (t/t_0)^{2/3}$ for non-relativistic matter and $a(t) = (t/t_0)^{1/2}$ for radiation. The case of a cosmological constant $w = -1$ is special and it is easy to see that:

$$a(t) = e^{H_0(t-t_0)} , \tag{23}$$

corresponding to exponentially fast, rather than power-law, expansion. In this case the Hubble parameter is actually a constant $H(t) = H_0$, while in general we have:

$$H(t) = \frac{2}{3(1+w)t} , \qquad w \neq -1 , \tag{24}$$

giving $H = 2/3t, 1/2t$ for a matter- or radiation-dominated universe, respectively. This allows one to relate the measured value of the Hubble parameter H_0 to the present cosmic time, i.e. the age of the univese, t_0, which we will discuss more generally in the next lecture. Finally, one should note that for $w \neq 1$, we have:

$$\rho(t) = \frac{3}{8\pi G} H^2 = \frac{3}{8\pi G} \left(\frac{2}{3(1+w)}\right)^2 \frac{1}{t^2} , \tag{25}$$

which is diluted as t^{-2} independently of the fluid that dominates the energy density of the Universe.

Problem 2

Consider a universe filled with non-relativistic matter, $w = 0$, and positive curvature, $k > 0$. Normalize the present day scale factor $a_0 = 1$ to find $k = H_0^2(\Omega_0 - 1)$.

(a) Write the Friedmann equation in terms of the conformal time defined by $d\tau/dt = 1/a(t)$.

(b) Solve the equation to show that:

$$a(\tau) = \frac{\Omega_0}{2(\Omega_0 - 1)}\left(1 - \cos(\sqrt{k}\tau)\right)$$

(c) Integrate to obtain:

$$t(\tau) = H_0^{-1}\frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}\left(\sqrt{k}\tau - \sin(\sqrt{k}\tau)\right)$$

and from this find the time t_{BC} it takes for the Universe to collapse to a 'big crunch' singularity.

(d) How do the results change for $k < 0$?

References

[1] J. Beringer *et al.* [Particle Data Group Collaboration], *Review of Particle Physics (RPP)*, Phys. Rev. D **86**, 010001 (2012).