Introduction to Cosmology

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LECTURE 12 - Cosmological perturbation theory I

Throughout this course, we have mainly focused on the evolution of a homogeneous and isotropic FRW universe filled with perfect fluids that give the dominant contribution to the energy density at different stages in the cosmological evolution - the scalar inflaton, radiation (photons, neutrinos, etc), non-relativistic matter (baryons and cold dark matter) and dark energy. However, as we have seen in the last lecture in the context of inflation, perturbations about this homogeneous and isotropic background may give crucial information about the physical mechanisms driving the cosmological evolution.

In the previous lecture, we have studied the evolution of fluctuations of the scalar inflaton field during the quasi-de Sitter period of accelerated expansion, showing that they are stretched and amplified, becoming constant once the wavelength of each Fourier component becomes larger than the Hubble horizon. As the inflaton is assumed to be the dominant source of energy density at this stage, this then yields a scale-invariant spectrum of curvature perturbations in the underlying spacetime that provides adiabatic and gaussian seeds for the formation of structure in the universe. In the next two lectures, we will describe how these small seeds evolve in the standard Hot Big Bang cosmological evolution for the different fluid components. We will show, in particular, that these small fluctuations grow in the radiation and matter eras, leading first to small anisotropies in the CMB temperature spatial distribution and later on, when non-linear dynamics becomes relevant, to gravitational collapse and the formation of structure on large scales, such as galaxies and galaxy clusters. We will also show that, besides scalar curvature perturbations, a stochastic background of gravity waves is also produced during inflation and describe how its properties may help constrain the physics behind accelerated expansion.

Basic setup: metric and energy-momentum tensor perturbations

In the previous lecture we have already introduced the general form of metric and energy-momentum tensor perturbations expressed in terms of scalar, vector and tensor quantities. We have seen that the form of the perturbations is in general dependent on the choice of coordinates, so that it is better to express all results in terms of gauge-invariant quantities. However, to study the evolution of perturbations it becomes more convenient to work with a particular gauge choice. A particularly useful coordinate system that we have described earlier in the course is the synchronous gauge, where ${g^0}_\mu = \delta^0_\mu$, so that we only have to consider perturbations about the spatial part of a flat FRW metric:

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i x^j\right] \ , \tag{1}$$

where $|h_{ij}| \ll 1$. This corresponds to setting $\Phi = B = 0$ for the scalar metric perturbations in the form that we have described in the previous lecture. Another popular gauge choice is the longitudinal or conformal newtonian gauge, where the scalar metric perturbations are given by:

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + (1-2\Psi)\delta_{ij}dx^i x^j\right] \ , \tag{2}$$

corresponding to alternatively setting $E = B = 0$, apart from vector and tensor perturbations. Here we will focus on perturbations in the synchronous gauge and in problem 12 we discuss how the results change in the conformal newtonian gauge.

As in the general case, it is useful to decompose the synchronous gauge perturbations h_{ij} into components that behave differently under rotations, yielding scalars, vectors and tensors. To better understand how this can be achieved, recall that a vector can be decomposed into a curl-free (gradient) and a divergence-free component:

$$\mathbf{V} = \nabla\theta + \mathbf{V}_v \ , \qquad \nabla \cdot \mathbf{V}_v = 0 \ , \tag{3}$$

where θ is a scalar and $\mathbf{V}_v$ is a (transverse) vector. In Fourier space, we can write this as:

$$\mathbf{V} = i\mathbf{k}\theta + \mathbf{V}_v \ , \qquad \mathbf{k} \cdot \mathbf{V}_v = 0 \ , \tag{4}$$

where a subscript $\mathbf{k}$ is implicit in all quantities. For a symmetric 2-tensor, we can do a similar decomposition according to how each component transforms under spatial rotations:

$$h_{ij} = \frac{1}{3} h\delta_{ij} + \left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right) h_s + \left(\hat{k}_i h^v_j + \hat{k}_j h^v_i\right) + h^t_{ij} \ , \tag{5}$$

where $\hat{k}_i h^v_i = \hat{k}_i h^t_{ij} = h^t_{ii} = 0$ and $\hat{k}_i = k_i/|\mathbf{k}|$. These components are know as the:

- *scalar trace*: $h = h_{ii}$
- *anisotropic scalar*: $h_s = (3\hat{k}_i\hat{k}_j h_{ij} - h)/2$
- *(transverse) vector*: $h^v_i = \hat{k}_j h_{ij} - \hat{k}_j h_{jl}\hat{k}_l\hat{k}_i$
- *(transverse-traceless) tensor*: h^t_{ij}, which gives the remaining degrees of freedom.

Taking into account the constraints, we see that the spatial metric perturbations correspond to 2 scalar, 2 vector and 2 tensor degrees of freedom.

To express the perturbed energy-momentum tensor, recall that for a perfect fluid we have:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \ . \tag{6}$$

We may write the perturbed energy density as:

$$\rho(\mathbf{x}, \tau) = \bar{\rho}(\tau)\left(1 + \delta(\mathbf{x}, \tau)\right) \ , \tag{7}$$

and where the background value can be expressed in terms of the abundance:

$$\bar{\rho} = \Omega\rho_c = \Omega\frac{3H^2}{8\pi G} = \frac{3\Omega}{8\pi G a^2}\left(\frac{a'}{a}\right)^2 \ . \tag{8}$$

Similarly, the perturbed pressure is given by:

$$p = \bar{p} + \delta p = w\bar{\rho} + \bar{\rho}\frac{\delta\rho}{\bar{\rho}}\frac{\delta p}{\delta\rho} = w\bar{\rho}(1 + \delta) \ , \tag{9}$$

for an equation of state $\bar{p} = w\bar{\rho}$ with corresponding sound speed:

$$c_s^2 = \frac{\delta p}{\delta\rho} = w \ . \tag{10}$$

The perturbed 4-velocity can be written as:

$$u^\mu = \bar{u}^\mu + \delta u^\mu = \frac{1}{a}\left(1, \mathbf{v}\right) \ , \tag{11}$$

which follows from $g_{\mu\nu}u^\mu u^\nu = -1$ to leading order in the synchronous gauge perturbations. Hence, we have for the perturbed energy-momentum tensor, to leading order:

$$\begin{aligned} T^{00} &= \frac{1}{a^2}\bar{\rho}(1+\delta) \ , \\ T^{0i} &= \frac{1}{a^2}(1+w)\bar{\rho}v^i \ , \\ T^{ij} &= \frac{1}{a^2}w\bar{\rho}\left[(1+\delta)\delta_{ij} - h_{ij}\right] \ . \end{aligned} \tag{12}$$

Notice that $\mu = 0$ corresponds to the conformal time coordinate, which gives the different normalization factors $1/a^2$ with respect to the results with cosmic time t given in the previous lecture.

Having written down the metric and energy-momentum tensor perturbations, we can compute the associated curvature tensors to linear order and obtain the linearized form of the Einstein field equations. Although we will not go through the derivation in detail, we will give the main equations that we will explore below. The equation for $R_{00} = 8\pi G(T_{00} - g_{00}T/2)$, where $T = T^\mu_{\ \mu}$ is the trace of the energy momentum tensor, yields the scalar trace equation:

$$h'' + \frac{a'}{a}h' + 3\left(\frac{a'}{a}\right)^2 \sum_\alpha (1+3w_\alpha)\Omega_\alpha\delta_\alpha = 0 \ , \tag{13}$$

where we sum over all fluid components, with the subscript α denoting photons, cold dark matter, baryons, neutrinos, dark energy, etc. Similarly, the equation for $R_{ij} = 8\pi G(T_{ij} - g_{ij}T/2)$ yields the equation for the anisotropic scalar:

$$h_s'' + 2\frac{a'}{a}h_s' + \frac{1}{3}k^2 h^- = 0 \ , \tag{14}$$

where $h^- = h - h_s$. Also, the equation for R_{0i} gives the momentum constraint equation:

$$\frac{1}{3}(h' - h_s') + 3\left(\frac{a'}{a}\right)^2 \sum_\alpha (1+w_\alpha)\Omega_\alpha\theta_\alpha = 0 \ , \tag{15}$$

where we assumed that all fluid 3-velocities have only a scalar component $\mathbf{v}_\alpha = \nabla\theta_\alpha$. Eqs. (13-15) form a set of differential equations for the scalar metric components, which are thus completely determined by the density perturbations of the different fluids, as we have already seen for scalar curvature perturbations during inflation.

The remainder of Einstein's equations give the vector equation:

$$h_i^{v}{}'' + 2\frac{a'}{a}h_i^{v}{}' = 0 \ , \tag{16}$$

and the tensor equation

$$h_{ij}^t{}'' + 2\frac{a'}{a}h_{ij}^t{}' + k^2 h_{ij}^t = 0 \ , \tag{17}$$

as well as a constraint equation for the vector perturbations:

$$ikh_i^{v}{}' = 2\left(\frac{a'}{a}\right)^2 \sum_\alpha (1+w_\alpha)\Omega_\alpha v_{\alpha i}^v \ , \tag{18}$$

such that the right-hand side vanishes if there is no vector component for the perturbed 3-velocities.

These equations could be modified by additional terms on the right hand side corresponding to sources other than the cosmological perfect fluids, such as topological deffects, but we will not consider this case in our discussion. One important thing to notice is that scalar, vector and tensor components satisfy decoupled equations, which justifies the usefulness of this decomposition. Also, we see that the scalar and vector components are completely determined by perturbations in the energy-momentum tensor, so do not really represent gravitational degrees of freedom. The pure gravitational components correspond to the transverse-traceless tensor part, which gives the two polarizations associated with gravity waves and we will explore in more detail below. As the vector components of the metric perturbations are not sourced by ordinary perfect fluids without vorticity, we can consistently set them to zero and ignore them in the ordinary cosmological evolution, although one should keep in mind that they could be sourced in other non-trivial models.

The dynamical evolution of cosmological perturbations is completed by using conservation of the energy-momentum tensor, $T^{\mu\nu}{}_{;\nu}=0$. To lowest order, the $\mu=0$ component yields the familiar continuity equation for the background energy density and pressure $\bar{\rho}'_\alpha+3(a'/a)(1+w_\alpha)\bar{\rho}_\alpha=0$, while to leading order in the perturbations we obtain the perturbed continuity equation:

$$\delta'_\alpha-(1+w_\alpha)k^2\theta_\alpha+\frac{1}{2}(1+w_\alpha)h'=0\ , \tag{19}$$

where we again ignore vorticity for all fluids. The $\mu=i$ component yields to leading order in the perturbations the perturbed Euler equation:

$$\theta'_\alpha+(1-3w_\alpha)\frac{a'}{a}\theta_\alpha+\frac{w_\alpha}{1+w_\alpha}\delta_\alpha=0\ . \tag{20}$$

In the remainder of this and in the next lecture we will explore the solutions of these equations in different cases.

Primordial gravitational waves

As clear from Eq. (17), transverse-traceless modes satisfy the same equation as a massless scalar field in an expanding universe. This is related to the fact that they represent gravity waves, the gravitational analogous of electromagnetic waves, with two independent polarizations. Notice, in particular, that in a non-expanding universe they satisfy a simple wave equation $(\partial_t^2-\nabla^2)h^t_{ij}=0$. Just like electromagnetic waves are associated with photons in the quantum theory, so are gravity waves associated with a quantum particle known as the graviton. Although gravity waves have not yet been detected, there is significant indirect evidence for their existence, since they are emitted by dynamical gravitating systems such as the binary pulsar studied by Hulse and Taylor (see e.g. [1] and references therein).

Since tensor perturbations satisfy the same equations as inflaton fluctuations, which are effectively massless during slow-roll inflation, they will exhibit the same behaviour, starting from small quantum fluctuations that are stretched and amplified, yielding a frozen nearly scale-invariant spectrum once they become superhorizon. Moreover, since they interact very weakly with ordinary matter or radiation, as evident from the absence of matter perturbations on the right-hand side of Eq. (17), these perturbations will remain basically unchanged until today, providing a direct window into the physics of inflation. Since h^t_{ij} is a gauge invariant quantity, since there are no tensor-type coordinate transformations, we can immediately infer the power spectrum for tensor perturbations from the results derived in the previous lecture for inflaton fluctuations:

$$\langle {h^t_{ij}}^2\rangle=2\times\frac{M_P^2}{4}\langle\delta\phi^2\rangle \qquad\Rightarrow\qquad \Delta_t^2(k)=\frac{M_P^2}{2}\Delta_\phi^2(k)=\frac{2}{\pi^2}\frac{H^2}{M_P^2}\ , \tag{21}$$

where we note that the factor 2 accounts for both graviton polarizations and we introduce a normalization $M_P/2$ in h^t_{ij} to account for the relative normalization of the second-order Einstein-Hilbert action for tensor perturbations,

$$S^t_{(2)}=\frac{M_P^2}{8}\int d\tau d^3x a^2\left[({h^t_{ij}}')^2-(\partial_l h^t_{ij})^2\right]\ , \tag{22}$$

relative to the action for inflaton fluctuations. Since the tensor power spectrum only depends on the Hubble parameter H, evaluated at horizon-crossing, observing the background of gravity waves generated during inflation could yield a direct measurement of the scale of inflation. This spectrum is stochastic (gaussian), due to the quantum nature of the generation process, and since it modifies the underlying spacetime geometry it will also change the spectrum of CMB temperature anisotropies. It is common to express this in terms of the *scalar-to-tensor ratio*, defined as:

$$r = \frac{\Delta_t^2}{\Delta_s^2} \equiv \frac{\Delta_t^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_\phi \ , \tag{23}$$

where we used the result for the scalar curvature perturbations derived in the previous lecture $\Delta_{\mathcal{R}}^2 = (V/M_P^4)/(24\pi^2\epsilon_\phi)$. This ratio thus gives us directly the slope of the inflationary potential at horizon-crossing for the relevant CMB scales. This also shows that the contribution of tensor perturbations is suppressed with respect to scalar perturbations in the slow-roll regime, which explains why its effects on the CMB have not been detected so far. Since we do know the amplitude of scalar perturbations, $\Delta_{\mathcal{R}}^2 \simeq 2 \times 10^{-9}$, and Δ_t^2 is proportional to the scale of inflation using the Friedmann equation, we obtain:

$$V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\ \mathrm{GeV} \ , \tag{24}$$

so that for large values of the tensor-to-scalar ration $r \gtrsim 0.01$ inflation occurs close to the GUT scale, as we had seen for the quadratic potential. For this example, we have $r = 16\epsilon_\phi = 32(M_P/\phi)^2 = 8/N_e \simeq 0.13 - 0.16$ for $N_e = 50 - 60$. Although no tensor perturbations have been observed so far, the Planck satellite placed an upper bound $r < 0.12$ at 95% C.L. [2], which means the simple quadratic model is already in some tension with the observational data. Although we will not discuss this in detail, it is worth mentioning that the Planck satellite will also measure the polarization of the CMB photons, which also reflects the spacetime structure at the time of recombination. Tensor modes induce a particular type of photon polarization known as B-mode polarization, due to the divergence-free nature of the polarization vectors analogously to a magnetic field, whereas scalar curvature perturbations only induce E-mode or curl-free polarization. Thus, observing the statistical distribution of the CMB polarization may give us a unique insight into the physics of inflation, and new results are expected to become available next year.

An interesting observation comes from noticing that:

$$r = \frac{\frac{8}{M_P^2}\left(\frac{H}{2\pi}\right)^2}{\left(\frac{H}{\dot\phi}\right)^2\left(\frac{H}{2\pi}\right)^2} = \frac{8}{M_P^2}\left(\frac{d\phi}{dN_e}\right)^2 \ , \tag{25}$$

where we used that $d/dN_e = H^{-1}d/dt$. Since r does not vary much during the $N_e = 50 - 60$ e-folds of slow-roll inflation, this yields approximately:

$$\frac{\Delta\phi}{M_P} \simeq 2\left(\frac{r}{0.01}\right)^{1/2} \ , \tag{26}$$

which is known as the Lyth bound [3], showing that a large tensor-to-scalar ratio implies a superplanckian variation of the inflaton field, and hence can only occur for large field models. This illustrates the power of measuring the tensor-to-scalar ratio in constraining inflationary models.

Most importantly, like the spectrum of scalar curvature perturbations, the spectrum of tensor modes exhibits deviations from scale-invariance, and we may write:

$$\Delta_t^2 = A_t \left(\frac{k}{k_\star}\right)^{n_t} \ , \tag{27}$$

where the *tensor spectral index* is given by:

$$\begin{aligned} n_t &= \frac{d\ln\Delta_t^2}{d\ln k} \\ &= \frac{\dot\phi}{H}\frac{1}{\Delta_t^2}\frac{d\Delta_t^2}{d\phi} \\ &= -M_P^2\frac{V'(\phi}{V(\phi))}\frac{1}{V(\phi)}\frac{dV(\phi)}{d\phi} \\ &= -2\epsilon_\phi \ , \end{aligned} \tag{28}$$

where we used that $\Delta_t^2 \propto H^2 \propto V(\phi)$. This is a remarkable feature of slow-roll inflation, since we may establish a simple consistency relation between the tensor-to-scalar ratio and the tensor spectral index:

$$r = -8n_t \ , \tag{29}$$

where $n_t < 0$ since $\epsilon_\phi > 0$. This means that if both these observables can be accurately measured in the near future, we will be able to establish whether the inflationary paradigm is the correct mechanism for generating perturbations in the early universe and explaining the problems of the Hot Big Bang model. Actually, some variations of the basic paradigm of a single slowly-rolling scalar field involving multiple scalar fields or other non-trivial components/dynamics yield different consistency relations, so observations may actually pinpoint if the single slowly-rolling scalar field description is the correct one.

Finally, it is also worth noting that any massless or nearly massless field will exhibit the same type of quantum fluctuations during the inflationary stage and it may, for example, be possible that other light fields are present in the early universe, even though giving a subdominant contribution to the energy density during inflation. However, if these fields can later decay into ordinary matter and/or radiation, their perturbations will also become imprinted in some of the cosmological fluids. Since no global curvature perturbations are generated during inflation by this field, these are called *isocurvature perturbations* and, as opposed to adiabatic fluctuations, they may yield perturbation spectra that are different for distinct fluid components. Scenarios where the fluctuations generated by the inflaton field are not the only source of density perturbations in the universe are generically known as *curvaton* models [4].

Radiation and cold dark matter perturbations

Having derived the fundamental equations describing the linearized evolution of perturbations in an FRW universe, we must now look at the particular fluids that give the dominant components of the energy density in the universe after inflation. As we have seen before, these correspond to either relativistic particles, or radiation, or non-relativistic particles, which are mainly cold dark matter (CDM). Baryons also behave as non-relativistic matter at low-temperatures, but since they give a subdominant contribution to the matter density in the universe today we will consider only a simplified analysis with only radiation and pressureless cold dark matter.

For CDM, $\bar{p}_c \simeq 0$ and since they are non-relativistic we may neglect the 3-velocity component. From the perturbed continuity equation (19) we thus obtain:

$$\delta_c' = -\frac{1}{2}h' \ , \tag{30}$$

which for appropriate initial conditions gives $\delta_c = -h/2$. Since the perturbed volume element is given by:

$$\sqrt{\det(g_{ij})} = \sqrt{a^6(1+h)} \simeq a^3\left(1+\frac{h}{2}\right) = 0 \ , \tag{31}$$

we conclude that density changes in CDM are uniquely due to volume changes. Substituting this result into the scalar trace equation (13), and using $w_r = 1/3$ for radiation, we get:

$$\delta_c'' + \frac{a'}{a}\delta_c' - \frac{3}{2}\left(\frac{a'}{a}\right)^2(\Omega_c\delta_c + 2\Omega_r\delta_r) = 0 \ , \tag{32}$$

which gives the general evolution of CDM perturbations in a universe with both CDM and radiation describing the background evolution.

For radiation perturbations, the perturbed continuity equation gives:

$$\delta_r' - \frac{4}{3}k^2\theta_r + \frac{2}{3}h' = 0 \ , \tag{33}$$

since for relativistic particles the velocities approach the speed of light $c = 1$ and cannot be neglected. The perturbed Euler equation (20) yields:

$$\theta_r' + \frac{1}{4}\delta_r = 0 \ . \tag{34}$$

Differentiating Eq. (33) with respect to conformal time and using Eq. (34), we obtain the equation determining the evolution of radiation perturbations:

$$\delta_r'' + \frac{1}{3}k^2\delta_r - \frac{4}{3}\delta_c'' = 0 \ . \tag{35}$$

The second term corresponds to the effect of the perturbed velocity of the radiation fluid and it is related to its pressure, while the last term corresponds to a volume change effect as for CDM. This immediately tells us that because of the non-zero pressure term, which is absent for CDM, radiation perturbations will resist gravitational collapse, in particular for subhorizon scales (large k) where this term is more important. As we will see in the next lecture, this will lead to acoustic oscillations that are imprinted in the CMB anisotropy spectrum. Moreover, this shows that if dark matter was made of a hot relativistic component such as neutrinos it would also resist gravitational collapse and may not lead to the formation of structure in the universe - which is why the cold dark matter hypothesis is preferred. On superhorizon scales, on the other hand, the pressure term is negligible and $\delta_r'' = (4/3)\delta_c''$, which for appropriate initial conditions yields:

$$\delta_r = \frac{4}{3}\delta_c \ . \tag{36}$$

This shows that perturbations are *adiabatic* on superhorizon scales, as occurs if the initial conditions for the evolution are set by inflation, which produces a frozen spectrum on superhorizon scales. This is a consequence of the fact that on superhorizon scales all perturbations are generated by changes in the scale factor or spatial volume with $\delta a/a = h/6$ from Eq. (31), and the perturbed continuity equation yields, as we had seen previously:

$$\delta_\alpha = -3(1+w_\alpha)\frac{\delta a}{a} \qquad \Rightarrow \qquad \delta_r = -4\frac{\delta a}{a} \ , \qquad \delta_c = -3\frac{\delta a}{a}. \tag{37}$$

The adiabaticity of the perturbations on superhorizon scales is also related to the fact that the number of photons per CDM particle is fixed, i.e. no relative entropy is generated:

$$\delta\left(\frac{n_r}{n_c}\right) = 0 \qquad \Rightarrow \qquad \frac{\delta n_r}{\bar{n}_r} = \frac{\delta n_c}{\bar{n}_c} \ . \tag{38}$$

Since, as we have seen several times before, $\rho_r \propto T^4$ and $n_r \propto T^3$, while $\rho_c = m_c n_c$ for non-relativistic matter, this gives:

$$\frac{\delta\rho_r}{\bar{\rho}_r} = 4\frac{\delta T}{\bar{T}} = \frac{4}{3}\frac{\delta n_r}{\bar{n}_r} = \frac{4}{3}\frac{\delta n_c}{\bar{n}_c} = \frac{4}{3}\frac{\delta\rho_c}{\bar{\rho}_c} \ , \tag{39}$$

in agreement with Eq. (36). If, on the other hand, we have isocurvature perturbations as we mentioned earlier, with $h = 0$, the total density perturbation must vanish and hence $\delta_r = -\delta_c$, so that the ratio of photons to CDM particles must change, which is why these are also called *entropy perturbations*. We will nevertheless assume that inflation produces a dominant adiabatic component, as in the simplest models that we have studied and as supported by the observational data.

In the next lecture we will study the solutions of these equations in the three different post-inflationary cosmological eras - radiation, matter and dark energy, in order to determine the form of the spectrum of matter perturbations that can be compared to the present distribution of galaxies and galaxy clusters, as well as the fluctuations in the radiation temperature at the time of recombination that yield the observed CMB anisotropies.

Problem 12

As discussed in Lecture 11, scalar metric perturbations can be parametrized, in Fourier space, as:

$$ds^2 = -(1+2\Phi)dt^2 + 2iak_iBdx^idt + a^2[(1-2\Psi)\delta_{ij} - 2k_ik_jE]dx^idx^j \ . \tag{40}$$

(a) Comparing with the decomposition of the spatial metric components in Eq. (5) for the synchronous gauge, show that $\Phi = B = 0$ and

$$\Psi = -\frac{1}{6}(h-h_s) \ , \qquad E = -\frac{h_s}{2k^2} \ . \tag{41}$$

(b) Consider a scalar gauge transformation of both the time and spatial coordinates of the form:

$$t \to \tilde{t} = t + \alpha \qquad x^i \to \tilde{x}^i = x^i + \partial_i\beta \ , \tag{42}$$

where α and β are scalar functions of both t and x^i. Show that in the new coordinates the scalar metric perturbations transform as:

$$\begin{aligned} \Phi &\to \Phi - \dot{\alpha} \\ B &\to B + a^{-1}\alpha - a\dot{\beta} \\ E &\to E - \beta \\ \Psi &\to \Psi + H\alpha \ , \end{aligned} \tag{43}$$

which generalizes the result obtained in problem 11.

(c) In the conformal newtonian gauge, one sets $\tilde{E} = \tilde{B} = 0$. Deduce the form of the gauge transformation parameters that bring the synchronous gauge metric into this form and conclude that the non-vanishing perturbations are given by:

$$\begin{aligned} \tilde{\Phi} &= \frac{1}{2k^2}\left(h_s'' + \frac{a'}{a}h_s'\right) \ , \\ \tilde{\Psi} &= -\frac{1}{6}(h-h_s) - \frac{a'}{a}\frac{h_s'}{k^2} \ , \end{aligned} \tag{44}$$

where primes denote derivatives with respect to conformal time. Use Eq. (14) to show that $\tilde{\Phi} = \tilde{\Psi}$.

(d) Use the transformation for a scalar quantity to show that density perturbations in the synchronous and newtonian gauges are related by:

$$\tilde{\delta}_\alpha = \delta_\alpha - \frac{3}{2}\frac{a'}{a}(1+w_\alpha)\frac{h_s'}{k^2} \ , \tag{45}$$

so that the difference between the two gauge choices is negligible on superhorizon scales.

References

[1] J. M. Weisberg and J. H. Taylor, ASP Conf. Ser. **328**, 25 (2005) [astro-ph/0407149].

[2] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1303.5082 [astro-ph.CO].

[3] D. H. Lyth, Phys. Rev. Lett. **78**, 1861 (1997) [hep-ph/9606387].

[4] D. H. Lyth and D. Wands, Phys. Lett. B **524**, 5 (2002) [hep-ph/0110002].