Introduction to Cosmology

João G. Rosa
joao.rosa@ua.pt
http://gravitation.web.ua.pt/cosmo

LECTURE 11 - Inflation III

In the previous lectures we have discussed how the dynamics of a homogeneous scalar field may lead to a finite period of accelerated expansion in the early universe, thus solving the horizon and flatness problems, as well as diluting any unwanted relics such as monopoles. In this lecture, we will discuss how, as a bonus, inflation also provides an explanation for the origin of the cosmological structure, from the CMB anisotropies to the large scale structure that we observe today.

Quantum fluctuations of the inflaton field

Although so far we have focused on the evolution of a homogeneous scalar field obeying classical dynamics, we must now take into account that the inflaton is a quantum field. In quantum mechanics and its quantum field theory extension, Heisenberg's uncertainty principle prevents one from determining with absolute certainty the value of a field and its associated momentum. Hence, we expect a quantum field to exhibit small quantum fluctuations above its homogeneous value, even if macroscopic gradients in the field are quickly erased by expansion:

$$\phi = \bar{\phi} + \delta\phi \ , \tag{1}$$

where $\delta\phi \ll \bar{\phi}$ and the homogeneous part satisfies the classical equations of motion that we have analyzed in the previous lecture. To understand the dynamical evolution of these small quantum perturbations, let us expand the equation of motion $\Box\phi = -V'(\phi)$ about the classical value, yielding to linear order:

$$\Box\delta\phi = -V''(\bar{\phi})\delta\phi \ . \tag{2}$$

In the slow-roll regime, the second derivative of the potential for the classical field value can be neglected, corresponding to a light field $m^2 \ll H^2$ as we have seen earlier. This means that quantum fluctuations satisfy:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{1}{a^2}\nabla^2\delta\phi = 0 \ . \tag{3}$$

It will be convenient to express this in terms of conformal time, $d\tau = dt/a(t)$, and during inflation we have:

$$\tau = \int \frac{dt}{e^{Ht}} = -\frac{1}{aH} \ , \tag{4}$$

where we neglected the variation of the Hubble parameter in the slow-roll regime. For an arbitrary function $f(t) = f(\tau)$ we then have:

$$\begin{aligned} \dot{f} &= \frac{d\tau}{dt}\frac{df}{d\tau} = \frac{1}{a}f' \ , \\ \ddot{f} &= \frac{1}{a}\frac{d}{d\tau}\left(\frac{1}{a}f'\right) = \frac{1}{a^2}f'' - \frac{a'}{a^3}f' \ , \end{aligned} \tag{5}$$

where primes denote derivatives with respect to conformal time. We then have $H = a'/a^2$ and we find for the field fluctuations:

$$\delta\phi'' + 2\frac{a'}{a}\delta\phi' - \nabla^2\delta\phi = 0 \ . \tag{6}$$

We can then expand the field fluctuations in a basis of Fourier modes $\delta\phi_{\mathbf{k}} = A_k e^{i\mathbf{k}.\mathbf{x}}$ for some normalization factor to be determined, such that e.g. $\nabla^2\delta\phi_{\mathbf{k}} = -k^2\delta\phi_{\mathbf{k}}$. Each mode then satisfies a decoupled equation[1], given by:

$$\delta\phi_k'' + 2\frac{a'}{a}\delta\phi_k' + k^2\delta\phi_k = 0 \ . \tag{7}$$

Note that modes with the same wavenumber $k = |\mathbf{k}|$ satisfy the same equation independently of the direction in Fourier space. It is also useful to rescale the Fourier modes by the scale factor and define:

$$\chi_k(\tau) = a(t)\delta\phi_k(\tau) \ . \tag{8}$$

which after some algebra yield the following equation:

$$\chi_k'' - \frac{a''}{a}\chi_k + k^2\chi_k = 0 \ . \tag{9}$$

Finally, using that $a(\tau) = -1/H\tau$ during inflation and consequently $a''/a = 2/\tau^2$, we have the mode equation:

$$\chi_k'' + \left(k^2 - \frac{2}{\tau^2}\right)\chi_k = 0 \ , \tag{10}$$

which corresponds to a harmonic oscillator with a time-dependent frequency. Note that the physical wavelength of the modes is exponentially stretched by expansion during inflation:

$$\lambda_{phys} = \frac{2\pi}{k}a \ , \tag{11}$$

so that at some point it will become larger than the Hubble horizon, H^{-1}, which remains roughly constant during inflation:

$$\lambda_{phys} \gg H^{-1} \qquad \Leftrightarrow \qquad \frac{2\pi}{k}a \gg -a\tau \qquad \Leftrightarrow \qquad |k\tau| \ll 1 \ , \tag{12}$$

where we recall that τ takes negative values during inflation if we set $\tau = 0$ at the start of the radiation era following reheating. We then say that the mode is inside the horizon for $|k\tau| \gg 1$ ($k \gg aH$) and outside the horizon for $k\tau \ll 1$ ($k \ll aH$), with horizon-crossing happening for $|k\tau \simeq 1|$ ($k \sim aH$). This distinction is important as the evolution of the modes is distinct in both cases and is also related to the inflation solution to the horizon problem: modes were inside the horizon during inflation and hence in causal contact, but leave the horizon at some stage so that they become causally disconnected.

Deep inside the Hubble horizon, the second term in Eq. (10) is negligible and we obtain a simple harmonic equation:

$$\chi_k'' + k^2\chi_k = 0 \ , \tag{13}$$

with solutions $\chi_k = e^{\pm ik\tau}$. We can then make the following *ansatz* for the full solution:

$$\chi_k(k,\tau) = f_k(\tau)e^{\pm ik\tau} \ , \tag{14}$$

where $f_k(\tau) \to const.$ when the mode is deep inside the horizon. Substituting this into Eq. (10), we obtain:

$$f_k'' \pm 2ikf_k' - \frac{2}{\tau^2}f_k = 0 \ . \tag{15}$$

[1] Note that this is a consequence of working with a linearized equation.

It is not difficult to check that the solution is of the form $f_k(\tau) = A + B/\tau$, which upon replacing into Eq. (15) gives:

$$\frac{2B}{\tau^3} \pm 2ik\left(\frac{-B}{\tau^2}\right) - \frac{2}{\tau^2}\left(A + \frac{B}{\tau}\right) = 0 \qquad \Rightarrow \qquad B = \pm\frac{i}{k}A \ . \tag{16}$$

We thus obtain the following solutions:

$$\chi_k^{\pm}(\tau) = \left(1 \mp \frac{i}{k\tau}\right) e^{\mp ik\tau} \ , \tag{17}$$

and the inflaton fluctuations can be written as:

$$\delta\hat{\phi}(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^2} \frac{1}{\sqrt{2k}a} \left(\hat{a}_{\mathbf{k}}\chi_{\mathbf{k}}^+(\tau)e^{i\mathbf{k}.\mathbf{x}} + h.c.\right) \ , \tag{18}$$

with $(\chi_{\mathbf{k}}^+)^* = \chi_{\mathbf{k}}^-$. Note that we have introduced a hatted notation that indicates that the field perturbations become operators in the quantum theory, and the factors $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ represent creation and annihilation operators for the associated multi-particle states, similarly to the case of a simple harmonic oscillator, satisfying canonical commutation relations:

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \ . \tag{19}$$

This should be translated into commutation relations for the field and its conjugate momentum, defined as:

$$\delta\hat{\pi} = \frac{\delta S}{\delta\hat{\phi}'} = a^2\delta\hat{\phi}' \ , \tag{20}$$

and one can show that $[\delta\hat{\phi}(x), \delta\hat{\pi}(y)] = i\delta^3(\mathbf{x} - \mathbf{y})$ ($\hbar = 1$) implies the following Wronskian normalization condition for the modes:

$$\chi_{\mathbf{k}}^+(\chi_{\mathbf{k}}^{+'})^* - (\chi_{\mathbf{k}}^+)^*\chi_{\mathbf{k}}^{+'} = 2ik \ , \tag{21}$$

which yields the normalization in Eq. (18). The Bunch-Davies vacuum state is defined by:

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \ , \tag{22}$$

holding for all Fourier modes $\mathbf{k}$. Although the average fluctuation vanishes, the variance of the field fluctuations does not, being given by[2]:

$$\begin{aligned} \langle 0|\delta\hat{\phi}^2(0,\tau)|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2k}}\frac{1}{\sqrt{2k'}}H^2\tau^2\chi_{\mathbf{k}}^+\chi_{\mathbf{k}'}^-\langle 0|\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}'}^\dagger|0\rangle \ , \\ &= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\frac{H^2\tau^2}{2k}|\chi_{\mathbf{k}}^+|^2(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{H^2\tau^2}{2k}|\chi_{\mathbf{k}}^+|^2 \end{aligned} \tag{23}$$

Now, from the solution in Eq. (17), we see that when a mode exits the horizon, i.e. for $k\tau \ll 1$, its amplitude is given by $|\chi_{\mathbf{k}}| \simeq 1/k\tau$, which means that $|\delta\phi_{\mathbf{k}}^{\pm}| = |\chi_{\mathbf{k}}^{\pm}|/a \simeq H/k$, which approaches a constant value. Physically, this can be understood by noting that when a mode becomes larger than the horizon different wave crests become causally disconnected, and there should be no process that could change their relative amplitude. This a key feature of inflation, with more and more perturbations exiting the horizon during inflation and their amplitude freezing. Moreover, since the time of horizon exit depends only on the wavenumber k, all modes with the same wavelength are

[2] Translation invariance allows one to evaluate the variance at the origin without loss of generality.

frozen in phase, which will be a crucial feature of their evolution after inflation as we discuss below. On the other hand, the amplitude of subhorizon modes is suppressed relative to those that have already exited the horizon at any given stage, so that inflation not only stretches the quantum fluctuations to superhorizon classical values but also amplifies them to a constant value. Hence, we may take only superhorizon modes into account when computing the variance of the field fluctuations, yielding:

$$\langle 0|\delta\hat{\phi}^2(0,\tau)|0\rangle = \int \frac{dk}{k}\left(\frac{H}{2\pi}\right)^2 = \left(\frac{H}{2\pi}\right)^2 \log\left(\frac{k_{max}}{k_{min}}\right) . \tag{24}$$

This means that all logarithmic scales in momentum space contribute with the same factor $(H/2\pi)^2$ to the overall variance of the field. This means that inflation predicts a *scale-invariant* spectrum of inflaton perturbations, also known as the Harrison-Zeldovich spectrum. Note that the largest comoving scales (smallest wavenumber) are the first to exit the horizon during inflation, so that $k_{min} \sim |\tau_i|^{-1} = a_i H$, whereas the smallest scale is the last to leave the horizon at the end of inflation, $k_{max} \sim |\tau_e|^{-1} = a_e H$, taking a constant Hubble rate during slow-roll inflation. This gives:

$$\langle 0|\delta\hat{\phi}^2(0,\tau)|0\rangle = \left(\frac{H}{2\pi}\right)^2 \log\left(\frac{a_e}{a_i}\right) = \left(\frac{H}{2\pi}\right)^2 N_e \ , \tag{25}$$

so that the mean amplitude of the inflaton fluctuations grows linearly with the number of e-folds of inflation, as more and more scales exit the horizon and become frozen.

We define the power spectrum of inflaton perturbations as:

$$\langle 0|\delta\hat{\phi}^2(0,\tau)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\mathcal{P}_\phi(k) \ , \tag{26}$$

so that

$$\mathcal{P}_\phi(k) = \frac{H^2}{2k^3} \propto k^{-3} \ , \tag{27}$$

which characterizes the amplitude of the inflaton field fluctuations as a function of the comoving scale k.

Curvature and density perturbations from inflation

Since the inflaton field is the dominant source of energy density during inflation, these field perturbations will lead to perturbations in the energy-momentum tensor and consequently in the metric of the expanding spacetime. The small quantum fluctuations of the inflaton field have thus a backreaction on the underlying spacetime, making it slightly inhomogeneous. As inflation proceeds, these perturbations are stretched and amplified to macroscopic sizes, their amplitude freezing once they become larger than the Hubble horizon, so we expect the same to happen for the perturbations of the energy density and curvature of the universe. Hence, inflation provides the small seeds for the perturbations that will later grow to form the structure that we observe today in the universe and also the small perturbations in the temperature of the CMB.

The analysis of metric perturbations is more intricate than the analysis of fluctuations of a single scalar field, since the metric is a 2-tensor with 10 independent components. It is conventional to decompose metric perturbations in terms of scalar, vector and tensor perturbations in the following way:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1+2\Phi)dt^2 + 2aB_i dx^i dt + a^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j \ . \tag{28}$$

The quantities Φ and Ψ are scalar potentials, whereas B_i and E_{ij} are vector and tensor quantities, respectively. The vector B_i can be further decomposed into a divergence-free and a curl-free component:

$$B_i = \partial_i B - S_i \ , \qquad \partial_i S^i = 0 \ , \tag{29}$$

and similarly for the tensor part:

$$E_{ij} = 2\partial_i\partial_j E + 2\partial_{(i}F_{j)} + h_{ij} \ , \qquad \partial_i F^i = 0 \ , \qquad h^i_i = \partial^i h_{ij} = 0 \ , \tag{30}$$

where B and E are scalars, S_i and F_i are divergenceless vectors and h_{ij} a transverse and traceless tensor, which as we will see later on corresponds to gravitational waves. Vector perturbations are not generated by inflation, and in fact decay away (see e.g. [1]), whereas scalar and tensor perturbations do not. We will focus on the former in this lecture, and then discuss the generation of gravity waves in the next lecture.

The energy-momentum tensor for a perfect fluid can be decomposed in a similar way, and we have:

$$\begin{aligned} T^0_{\ 0} &= -(\rho + \delta\rho) \\ T^0_{\ i} &= (\rho + p) a v_i \\ T^i_{\ 0} &= -(\rho + p)(v^i - B^i)/a \\ T^i_{\ j} &= \delta^i_j (p + \delta p) + \Sigma^i_{\ j} \ , \end{aligned} \tag{31}$$

where v_i is the velocity perturbation and $\Sigma^i_{\ j}$ is the anisotropic stress tensor, while $\delta\rho$ and δp denote the perturbations of the energy density and pressure of the fluid. With this decomposition of the metric and energy-momentum tensor, we could determined the linearized form of Einstein's equations in flat FRW background dominated by the inflaton field. However, one must take into account that this decomposition depends on a particular choice of coordinates and we must be sure that the results obtained are independent of this choice. It is thus more convenient to find gauge invariant combinations of the metric and energy-momentum tensor perturbations. Tensor perturbations are actually gauge invariant but scalar perturbations are not, and it is useful to work with two gauge-invariant quantities [2, 3]:

1. *Curvature perturbation on uniform energy density surfaces*:

$$-\zeta = \Psi + \frac{H}{\dot{\rho}}\delta\rho \ . \tag{32}$$

   If we choose a time foliation of the spacetime such that the energy density is constant on constant time hypersurfaces, $\delta\rho|_t = 0$, this quantity corresponds to the spatial curvature of these hypersurfaces, since one can show that $R^{(3)} = 4\nabla^2\Psi/a^2$ (see e.g. [4]).

2. *Comoving curvature perturbation*:

$$\mathcal{R} = \Psi - \frac{H}{\rho + p}\delta q \ , \tag{33}$$

   where δq is the scalar part of the 3-momentum density, $T^0_{\ i} = \partial_i(\delta q)$.

One can show that these quantities are equal and remain constant on superhorizon scales [4]. Moreover, during slow-roll inflation we have $T^0_{\ i} = -\partial_i(\delta\phi)\dot{\bar{\phi}}$, so that $\delta q = -\dot{\bar{\phi}}\delta\phi$, $\delta\rho \simeq V'(\bar{\phi})\delta\phi \simeq -3H\dot{\bar{\phi}}\delta\phi$ and $\dot{\rho} = -3H(\rho + p) = -3H\dot{\bar{\phi}}^2$, which imply:

$$\mathcal{R} \simeq -\zeta \simeq \Psi + \frac{H}{\dot{\bar{\phi}}}\delta\phi \ , \tag{34}$$

so that both quantities are equivalent during slow-roll inflation. Gauge invariance of these quantities can be easily verified in this form. Suppose we start with a time slicing with spatially flat constant time hypersurfaces, $\Psi|_t = 0$ and $\delta\phi|_t \neq 0$, and make a time coordinate transformation $t \to \tilde{t} = t + \delta t$. Since the inflaton is a scalar quantity which remains invariant under coordinate transformations, we have:

$$\begin{aligned} \tilde{\phi}(\tilde{t}) &= \phi(t) \\ \bar{\phi}(t) + \dot{\bar{\phi}}\delta t + \delta\tilde{\phi}(t) &= \bar{\phi}(t) + \delta\phi(t) \\ \delta\tilde{\phi}(t) &= \delta\phi(t) - \dot{\bar{\phi}}\delta t \ . \end{aligned} \tag{35}$$

We may then choose $\delta t = \delta\phi/\dot{\bar{\phi}}$ such that in the new time slicing $\delta\tilde{\phi} = 0$. Noticing that $a(\tilde{t}) = a(t) + \dot{a}\delta t = a(t)(1 + H\delta t)$, invariance of the spatial line element under coordinate transformations implies:

$$a^2(t)d\mathbf{x}^2 = a^2(\tilde{t})(1 - 2\tilde{\Psi})d\mathbf{x}^2 = a^2(t)(1 + 2H\delta t - 2\tilde{\Psi})d\mathbf{x}^2 \qquad \Rightarrow \qquad \tilde{\Psi} = H\frac{\delta\phi}{\dot{\bar{\phi}}}, \tag{36}$$

such that $\tilde{\mathcal{R}} = \mathcal{R}$ in both coordinate systems. From this it is also clear that inflaton perturbations generate to curvature perturbations. Our previous computation of the spectrum of inflaton perturbations neglected scalar metric perturbations, which holds in the spatially flat gauge where $\Psi = 0$. Using gauge invariance we may then infer the following power spectrum for the comoving curvature perturbation:

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\phi}}}\right)^2 \mathcal{P}_\phi(k) \ . \tag{37}$$

It is also common to express this in terms of the dimensionless power spectrum:

$$\Delta^2_{\mathcal{R}} \equiv \frac{k^3}{2\pi^2}\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\bar{\phi}}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \ , \tag{38}$$

where the normalization is chosen such that the variance is expressed as $\langle \mathcal{R}_k^2 \rangle = \int \Delta^2_{\mathcal{R}}(k) d\ln k$ and the scale-invariance of the spectrum after horizon crossing is manifest.

A heuristic way of understanding the generation of curvature and hence density perturbations during inflation is to note that different patches of the universe, where the inflaton takes slightly different values, will follow the same slow-roll evolution but the end of inflation will occur at slightly different times, with $\delta t = \delta\phi/\dot{\bar{\phi}}$. Different regions will thus have expanded by a slightly different amount and there is a consequent spread in the energy density proportional to the fluctuations in the scale factor at the end of inflation:

$$\frac{\delta\rho}{\rho} \sim \frac{\delta a}{a} = H\delta t = \frac{H^2}{2\pi\dot{\bar{\phi}}} \ , \tag{39}$$

taking $\delta\phi \simeq H/2\pi$ for the average contribution to the overall fluctuation amplitude of each logarithmic momentum scale as we obtained earlier.

Contact with observations

As we have discussed above, the amplitude of the inflaton fluctuations, and hence that of curvature and density perturbations, freezes once each Fourier mode becomes larger than the Hubble horizon, $k \ll aH$. Once inflation ends and the universe enters the standard cosmological evolution, the quantity aH decreases in time, i.e. the cosmic horizon grows, as for example in the radiation era $aH \propto t^{-1/2}$. This means that all scales will eventually re-enter the horizon during the cosmological evolution and their amplitude will "unfreeze" once again, as illustrated in Figure 1.

Since inflation produces curvature perturbations in the underlying spacetime, this will induce perturbations on all fluids propagating in the universe - photons, baryons, neutrinos, cold dark matter... The relative size of each type of perturbation is fixed by the corresponding equation of state, since $\delta\rho_i/\rho_i = -3(1+w_i)\delta a/a$ as we have seen earlier in this course, so that inflation generates *adiabatic* perturbations. The quantum nature of the process means that fluctuations also begin and remain mostly *gaussian* throughout their evolution. Finally, as we will discuss later on, the combined effect of gravity and pressure will make density perturbations oscillate once they re-enter the horizon after inflation. Since comoving modes with the same wavelength will re-enter the horizon at the same time, when $k = aH$, their oscillating phases will be aligned and they will oscillate in phase. In particular, this will produce a *coherent phase structure* in the perturbations of the photon fluid, such that at the time of recombination and last scattering different modes will be in different phases of their oscillation, producing a series of peaks and throughs that we observe today in the spectrum of CMB temperature anisotropies and which is one of the most remarkable features of the horizon-crossing/re-entry dynamics intrinsic to the inflationary paradigm [5].

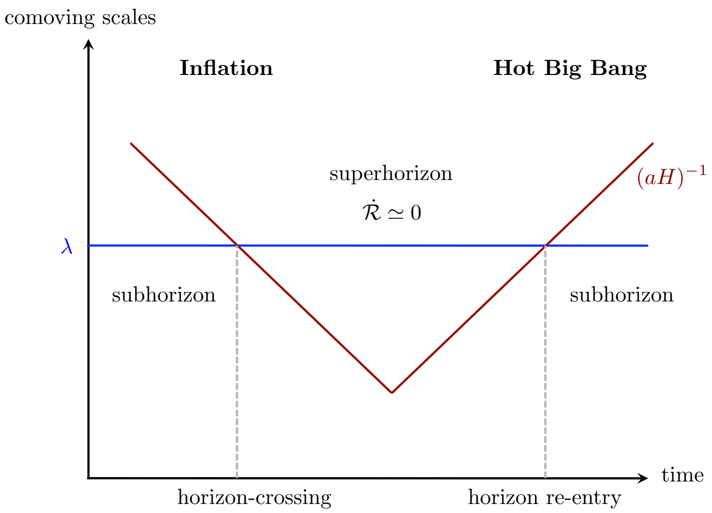

Figure 1: Diagram illustrating the evolution of a density/curvature perturbation during and after inflation, starting from a small quantum fluctuation of the inflaton field which is smaller than the Hubble horizon, $k \gg aH$, and which gets stretched and amplified, until it freezes out after horizon crossing, $k = aH$. The curvature perturbation remains approximately constant while it is superhorizon and, after inflation ends and the standard cosmological evolution begins, the Hubble horizon grows and eventually the now macroscopic density perturbation re-enters the horizon.

Although a detailed analysis of the spectrum of CMB temperature fluctuations is outside the scope of this course, it is clear from the arguments above that inflation provides the primordial seeds for CMB perturbations and large-scale structure. The spectrum of perturbations that we observe today with satellites such as WMAP and Planck is the result of the evolution of this primordial spectrum through the radiation and matter eras, which follows from the linearized Einstein equations as we will discuss in the next lectures. The initial amplitude and spectral properties of the fluctuations are nevertheless crucial features that can be extracted from the data. In particular, the amplitude of the comoving curvature power spectrum was first measured by the COBE satellite, the most recent measurement by Planck [6] yielding:

$$\Delta^2_{\mathcal{R}} = 2.2 \times 10^{-10} \ . \tag{40}$$

From Eq. (38) and using the slow-roll equations one can show that:

$$\Delta^2_{\mathcal{R}} \simeq \frac{1}{24\pi^2} \frac{V(\phi)}{M_P^4} \frac{1}{\epsilon_\phi} \bigg|_{k=aH} \ . \tag{41}$$

This shows that the amplitude of density perturbations can be used to constrain the parameters of the inflaton potential, in particular establishing a relation between its absolute scale, V and slope, given by the slow-roll parameter ϵ_ϕ. An important fact to observe is that all quantities are evaluated when the relevant Fourier modes visible in the CMB cross the horizon during inflation, since after this the spectrum will remain frozen until horizon re-entry after inflation. This corresponds to between 50 and 60 e-folds before the end of inflation, in order to address the horizon and flatness problems as we have discussed previously, so that for practical purposes horizon-crossing defines the beginning of inflation. Eq. (41) also shows explicitly that in the limit of an exact cosmological constant, $\epsilon_\phi \to 0$, the curvature perturbation is ill-defined, which just reflects the fact that inflation never ends in the exact de Sitter limit, so that curvature and density perturbations are meaningless in this case.

Another crucial feature of the primordial spectrum is that exact scale invariance is only obtained in the pure de Sitter limit, as since the inflaton is slowly rolling, the field value and hence the associated potential is slightly different for modes crossing the horizon at different times. Deviations from scale invariance are characterized by the scalar spectral index, defined as:

$$n_s - 1 = \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} \ , \tag{42}$$

so that the full power spectrum takes the form:

$$\Delta^2_{\mathcal{R}} = A_s \left(\frac{k}{k_\star}\right)^{n_s-1} \ , \tag{43}$$

where $k_\star$ is a chosen pivot scale that can be measured in the CMB and the amplitude A_s corresponds to evaluating Eq. (41) at this scale and takes the observed value in Eq. (40). Note that the power of k is only defined as $n_s - 1$ for historical reasons. Since the amplitude of the spectrum is evaluated at horizon-crossing, $k = aH$, we have:

$$\frac{d}{d\ln k} = k\frac{d}{dk} = aH\frac{d}{d(aH)} \simeq a\frac{d}{da} = \frac{d}{d\ln a} = \frac{1}{H}\frac{d}{dt} = \frac{\dot\phi}{H}\frac{d}{d\phi} \ , \tag{44}$$

where for simplicity we have dropped the barred notation, keeping in mind that ϕ denotes the background value. The slow-roll equation $3H\dot\phi \simeq V'(\phi)$ and the Friedmann equation $H^2 \simeq V(\phi)/3M_P^2$ then yield:

$$\frac{\dot\phi}{H} \simeq -\frac{V'(\phi)}{3H^2} \simeq -M_P^2\frac{V'(\phi)}{V(\phi)} \ . \tag{45}$$

Thus, we obtain for the scalar spectral index in the slow-roll approximation:

$$\begin{aligned} n_s - 1 &\simeq -M_P^2\frac{V'(\phi)}{V(\phi)}\frac{1}{\Delta^2_{\mathcal{R}}}\frac{d\Delta^2_{\mathcal{R}}}{d\phi} \\ &\simeq -M_P^2\frac{V'(\phi)}{V(\phi)}\left[\frac{V'(\phi)}{V(\phi)} - \frac{\epsilon'_\phi}{\epsilon_\phi}\right] \\ &\simeq -M_P^2\frac{V'(\phi)}{V(\phi)}\left[\frac{V'(\phi)}{V(\phi)} - \frac{2}{M_P^2}\frac{V(\phi)}{V'(\phi)}(\eta_\phi - 2\epsilon_\phi)\right] \\ &\simeq 2\eta_\phi - 6\epsilon_\phi \ . \end{aligned} \tag{46}$$

We hence see explicitly that in the pure de Sitter case the spectrum would be scale-invariant, $n_s = 1$, but that in slow-roll inflation one expects to find small deviations given by the size of the slow-roll parameters at horizon-crossing. The amplitude and "tilt" of the primordial spectrum may thus give important information about the form of the inflationary potential. In particular, the Planck mission has recently made a measurement of the scalar spectral index with the best accuracy so far, giving [6]:

$$n_s \simeq 0.9603 \pm 0.0073 \ . \tag{47}$$

This has firmly established that the spectrum is slightly *red-tilted*, i.e. $n_s < 1$ and there is more power on larger scales (smaller comoving wavenumber k), whereas if n_s exceeded unity we would say the spectrum is *blue-tilted*.

Let us consider our working example of a quadratic potential, $V(\phi) = m^2\phi^2/2$. Recalling that in this case $\epsilon_\phi = \eta_\phi = 2(M_P/\phi)^2$ and the number of e-folds of inflation after horizon-crossing at $\phi = \phi_\star$ is $N_e \simeq \phi_\star^2/4M_P^2$, we have:

$$n_s - 1 \simeq -8\frac{M_P^2}{\phi_\star^2} = -\frac{2}{N_e} \ , \tag{48}$$

which for $N_e = 50$ yields $n_s \simeq 0.96$, in very good agreement with the Planck results, while for $N_e = 60$ we obtain a slightly larger value $n_s \simeq 0.967$. Also, from Eq. (41), we obtain for the amplitude of the primordial spectrum with a quadratic model:

$$A_s \simeq \frac{1}{96\pi^2}\frac{m^2}{M_P^2}\frac{\phi_\star^4}{M_P^4} \simeq \frac{N_e^2}{6\pi^2}\frac{m^2}{M_P^2} . \quad (49)$$

Taking $N_e = 50$ and the observed value for the amplitude, we get for the inflaton mass $m \simeq 2.3 \times 10^{-6} M_P \simeq 5 \times 10^{12}$ GeV. This means that the inflaton corresponds to a very heavy scalar boson compared to known particle species (recall that $m_p \simeq 1$ GeV), but it is still light compared to the Hubble parameter during inflation, in this case given by, at horizon-crossing:

$$H_\star \simeq \sqrt{\frac{V(\phi_\star)}{3M_P^2}} \simeq \frac{m}{\sqrt{6}}\frac{\phi_\star}{M_P} \simeq 2\sqrt{\frac{N_e}{6}}m \simeq 6m , \quad (50)$$

taking $N_e = 50$ as above, implying that the field is in the overdamped regime as required for slow-roll inflation. We also obtain for the scale of inflation:

$$V_\star^{1/4} \simeq 3^{1/4}\sqrt{H_\star M_P} \sim 10^{16}\ \text{GeV}, \quad (51)$$

which is thus close to the GUT scale, a feature common to several inflationary models and that points towards a possible connection between inflationary physics and grand unification.

The quadratic is a simple model of inflation with only two free parameters, m and $\phi_\star$, that can be completely determined by observations, but for generic potentials we typically have some undetermined parameters. It is also possible to consider higher-order deviations from scale invariance, in particular the *running of the spectral index*:

$$n_s' = \frac{dn_s}{d\ln k} \simeq -16\epsilon_\phi\eta_\phi + 24\epsilon_\phi^2 + 2\xi_\phi^2 , \quad (52)$$

which we leave as an exercise to check, and where we have defined a new slow-roll parameter

$$\xi_\phi^2 = M_P^4 \frac{V'(\phi)V'''(\phi)}{(V(\phi))^2} , \quad (53)$$

involving the third derivative of the scalar potential. Being quadratic in the slow-roll parameters, we expect the running to be generically small, and in particular smaller than deviations from scale-invariance. For the quadratic potential, it is easy to show that $n_s' \simeq 2/N_e^2 \simeq 10^{-3}$. The Planck collaboration has found no statistically significant evidence for such a running, being compatible with zero at 95% C.L. [6]. In the next lecture, we will see how the observation of primordial tensor perturbations or gravity waves, which so far have not been detected, may further help constraining inflationary models.

Problem 11

Consider a time coordinate transformation $t \to \tilde{t} = t + \alpha$, where $\alpha = \alpha(x^\mu)$.

(a) By taking into account the invariance of the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, show that the metric scalar perturbation functions defined in Eq. (28) transform as follows:

$$\begin{aligned} \Phi &\to \Phi - \dot{\alpha} \\ B &\to B + a^{-1}\alpha \\ E &\to E \\ \Psi &\to \Psi + H\alpha \ . \end{aligned} \tag{54}$$

(b) Show that for a generic scalar function $f(x) = \bar{f}(t) + \delta f(x)$, perturbations about a time-dependent background value transform as:

$$\delta f \to \delta f - \dot{\bar{f}}\alpha \ . \tag{55}$$

(c) Use the results above to show that the curvature perturbation on uniform energy density surfaces ζ is invariant under time coordinate transformations.

References

[1] R. Durrer, Fund. Cosmic Phys. **15**, 209 (1994) [astro-ph/9311041].

[2] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[3] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[4] D. Baumann, arXiv:0907.5424 [hep-th].

[5] S. Dodelson, AIP Conf. Proc. **689**, 184 (2003) [hep-ph/0309057].

[6] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1303.5082 [astro-ph.CO].