Introduction to Cosmology

João G. Rosa
joao.rosa@ua.pt
http://gravitation.web.ua.pt/cosmo

LECTURE 10 - Inflation II

In the previous lecture we began to analyze how a period of accelerated expansion in the early universe, prior to radiation domination, could solve the shortcomings of the conventional Big Bang model, namely the horizon and flatness problems. This period is known as inflation and, as we began to discuss, can be implemented in models with non-standard fluids that mimic the effect of a cosmological constant at early times but may decay into radiation and ordinary particles after a sufficiently long period of quasi-exponential expansion that dilutes any initial curvature effects and increases the size of the causal horizon at the time of last scattering. Today we will describe in more detail how the dynamics of a scalar field provides a simple mechanism to implement this construction and discuss particular field theory models where this can be realized.

Dynamics of slow-roll inflation

Recall that the action for a real scalar field in a curved spacetime is given by:

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \ , \tag{1}$$

from which we determined the corresponding energy-momentum tensor, with an energy density $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and pressure $p_\phi = \dot{\phi}^2/2 - V(\phi)$ for a homogeneous scalar field. This implies that such a scalar field can mimic the effect of a cosmological constant if its potential energy $V(\phi)$ gives the dominant contribution to the energy density and pressure, thus yielding an effective Λ if it overcomes all other fluid components at early times. Physically, it is easy to understand that this will be the case if the field value is varying slowly. As any physical system, the value of the field, which describes the collective motion of the associated Bose-Einstein condensate, will adjust itself in order to minimize its energy, in particular decreasing its potential energy. If the function $V(\phi)$ varies considerably with the value of ϕ, the field will move quickly down the potential curve; if, on the other hand, the potential energy remains approximately constant for a range of field values, the inflaton will roll slowly towards the minimum of the potential, which is the case we are interested in.

To better quantify this physical picture, we need to derive the equation of motion that describes the dynamics of the inflaton field in an expanding FRW spacetime. As we have seen in several cases before, this can be obtained by minimizing the variation of the action in Eq. (1) for small field variations. We then have:

$$\begin{aligned}
\delta S_\phi &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu(\delta\phi)\times 2 - V'(\phi)\delta\phi\right] \\
&= \int d^4x\left[\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right)(\delta\phi) - \sqrt{-g}V'(\phi)\delta\phi\right] \\
&= \int d^4x\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) - V'(\phi)\right]\delta\phi \ ,
\end{aligned} \tag{2}$$

where we have integrated the first term by parts and set to zero variations of the field at infinity. Hence, for arbitrarily small variations of the field, we require $\delta S_\phi = 0$, so that:

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) - V'(\phi) = 0\ , \tag{3}$$

where the second order differential operator acting on the field is known as the d'Alembertian and usually denoted as $\Box\phi$. Since inflation will dilute any curvature effects, we may consider for simplicity a flat FRW universe with metric $g_{\mu\nu} = diag(-1,\ a^2(t),\ a^2(t),\ a^2(t))$ in cartesian coordinates, such that $\sqrt{-g} = a^3(t)$ and so:

$$\begin{aligned} \tfrac{1}{a^3}\partial_t\left(-a^3\dot{\phi}\right) + \tfrac{1}{a^3} \times a^3 \times a^{-2}\nabla^2\phi - V'(\phi) = 0 \\ \ddot{\phi} + 3\tfrac{\dot{a}}{a}\dot{\phi} - \tfrac{1}{a^2}\nabla^2\phi + V'(\phi) = 0\ . \end{aligned} \tag{4}$$

As we discussed in the previous lecture and can see explicitly in this equation, spatial variations will become less important as the universe expands, so that the field becomes essentially homogeneous. Hence, neglecting gradients, the equation of motion for the inflaton field can be written as:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\ . \tag{5}$$

One can also easily show that this equation of motion follows trivially from conservation of the energy-momentum tensor, which for an expanding FRW universe reduces to $\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0$ as we have obtained before:

$$\begin{aligned} \partial_t\left(\tfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) + 3H\dot{\phi}^2 = 0 \\ \dot{\phi}\left(\ddot{\phi} + 3H\dot{\phi} + V'(\phi)\right) = 0\ , \end{aligned} \tag{6}$$

which for $\dot{\phi} \neq 0$ reduces to Eq. (5). We hence see that the time variation of the homogeneous inflaton field is determined by a competition between two effects $-$ the slope of the potential $V'(\phi)$ and Hubble expansion. The expansion rate is determined by the Friedmann equation, which assuming that the energy density of the inflaton is dominant fluid component in the early universe is given by:

$$H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\ , \tag{7}$$

where one should recall that the reduced Planck mass is given by $M_P = 1/\sqrt{8\pi G}$ in natural units.

From Eq. (5) one concludes that the variation of the inflaton field is driven by two effects - Hubble expansion and the slope of the potential. The former acts as an effective friction term proportional to the field velocity, $3H\dot{\phi}$, that damps the variation of the field and, from Eq. (7), is sourced by the inflaton itself. On the other hand, the slope of the potential always tends to drive the field towards the minimum of the potential (assuming it is bounded), increasing the magnitude of the acceleration $|\ddot{\phi}|$.

To better understand the effect of these two terms, let us consider a particular case of a quadratic inflaton potential, $V(\phi) = m^2\phi^2/2$, for which the equation of motion corresponds to a damped harmonic oscillator:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0\ . \tag{8}$$

The parameter m denotes the mass of the scalar field, or equivalently the mass of the particles that make up the associated condensate. Note, in particular, that the generic relativistic equation describing the propagation of massive waves is given by $(\partial_t^2 - \nabla^2 + m^2)\phi = 0$ in Minkowski (non-expanding) spacetime. This yields the well-known relativistic formula $E^2 = |\mathbf{p}|^2 + m^2$ with the standard operator identifications in quantum mechanics $E \to i\partial_t$ and $\mathbf{p} \to -i\nabla$. Following the procedure described in Problem 9, it is easy to see that Eq. (8) has two types of solutions depending on the value of the mass m:

- $m^2 \gg H^2$: The field is *underdamped* in this regime, as the Hubble friction term can be neglected and the field has an oscillating profile with frequency m, $\phi(t) = A\cos(mt) + B\sin(mt)$;

- $m^2 \ll H^2$: The field is *overdamped* in this regime and the mass term is negligible, so that the field varies slowly, $\ddot{\phi} \simeq \dot{\phi} \simeq 0$.

For inflation to occur, we are interested in the latter case where the field mimics a cosmological constant, since the kinetic energy will be negligible compared to the potential energy and $p_\phi \simeq -\rho_\phi$. More complicated potentials are simply non-linear generalizations of the damped harmonic oscillator case, and in the generic case we require the following two conditions to be satisfied:

1. The kinetic energy of the inflaton is much smaller than its potential energy:

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \ , \tag{9}$$

   ensuring the equation of state is close to a cosmological constant, $w_\phi \simeq -1$.

2. The acceleration of the field is small, or otherwise the field velocity will grow quickly and the first condition cannot be satisfied during a sufficiently long period:

$$\ddot{\phi} \ll 3H\dot{\phi} \ . \tag{10}$$

These two conditions are known as the *slow-roll* conditions, ensuring the motion of the field is overdamped for sufficiently long to allow for quasi-exponential expansion. If these conditions are satisfied, the inflaton and Friedmann equations that the following form:

$$\begin{aligned} 3H\dot{\phi} &\simeq -V'(\phi) \\ H^2 &\simeq \frac{V(\phi)}{3M_P^2} \ . \end{aligned} \tag{11}$$

Hence, the potential energy sets the expansion rate and the size of the causal horizon during inflation, while its slope sets the velocity of the field and hence the duration of the slow-roll period. The slow-roll conditions are conventionally expressed in terms of two slow-roll parameters, defined as:

$$\begin{aligned} \epsilon_\phi &= \frac{1}{2}M_P^2\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ , \\ \eta_\phi &= M_P^2\left(\frac{V''(\phi)}{V(\phi)}\right) \ . \end{aligned} \tag{12}$$

These parameters measure the relative slope and curvature of the potential function $V(\phi)$ in units of the Planck mass. Using Eqs. (11), one can show that:

$$\frac{\dot{\phi}^2/2}{V(\phi)} \simeq \frac{1}{2V(\phi)}\left(\frac{V'(\phi)}{3H}\right)^2 \simeq \frac{M_P^2}{6}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{\epsilon_\phi}{3} \ll 1 \tag{13}$$

Similarly, the relative variation of the Hubble parameter in a Hubble time H^{-1} can be obtained by differentiating the Friedmann equation and using the slow-roll equation for the field:

$$2H\dot{H} \simeq \frac{V'(\phi)\dot{\phi}}{3M_P^2} \simeq -\frac{[V'(\phi)]^2}{9HM_P^2} \ , \tag{14}$$

and dividing by $2H^4 \simeq 2(V(\phi)/3M_P^2)^2$ we obtain:

$$\frac{\dot{H}}{H^2} \simeq -\frac{M_P^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = -\epsilon_\phi \ll 1 \ . \tag{15}$$

Hence, requiring $\epsilon_\phi \ll 1$ ensures that equation of state approaches that of a cosmological constant, where the Hubble parameter remains roughly constant during inflation, as required. On the other hand, the field acceleration can be obtained from differentiating the slow-roll equation:

$$\ddot{\phi} \simeq \partial_t \left(-\frac{V'(\phi)}{3H} \right) \simeq -\frac{V''(\phi)\dot{\phi}}{3H} + \frac{V'(\phi)}{3}\frac{\dot{H}}{H^2} , \tag{16}$$

where the second term is proportional to $\epsilon_\phi \ll 1$ and may be neglected. For consistency, we then require:

$$\left| \frac{\ddot{\phi}}{3H\dot{\phi}} \right| \simeq \left| \frac{V''(\phi)}{9H^2} \right| = \frac{1}{3} M_P^2 \left| \frac{V''(\phi)}{V(\phi)} \right| = \frac{|\eta_\phi|}{3} \ll 1 . \tag{17}$$

Thus, up to numerical factors the slow-roll conditions can be expressed as conditions on the slope and curvature of the potential, measured in units of the Hubble scale:

$$\epsilon_\phi \ll 1 , \qquad |\eta_\phi| \ll 1. \tag{18}$$

If we go back to our example of a quadratic potential function, we see that the second slow-roll condition corresponds to:

$$\eta_\phi = M_P^2 \frac{m^2}{V(\phi)} \simeq \frac{1}{3}\frac{m^2}{H^2} \ll 1 , \tag{19}$$

which is exactly the condition for an overdamped harmonic oscillator found earlier, where the field varies slowly in time. In general, the second derivative of the potential corresponds to the effective mass of the field, so we say that inflation requires a light scalar field (as compared to the Hubble scale) dominating the energy density in the early universe.

Note that other types of fields could also be overdamped and dominate the energy balance in the early universe. For example, particles like the photon are described by vector fields $A^\mu(x)$, whereas fermions such as the electron are described by spinor fields $\psi(x)$. The scalar field case is special since, having no intrinsic spin, it remains unchanged under Lorenz transformations, i.e. it does not pick up a special reference frame in the universe. A constant value for a vector field would, on the other hand, yield a preferred direction that could break the isotropy of the universe, which is one of the basic postulates of the Hot Big Bang model.

Inflation begins when the potential energy of the inflaton overcomes any pre-existing components, and we will last as long as the slow-roll conditions are satisfied and the field acts as an effective cosmological constant. Afterwards, the inflaton equation of state will change and the field decays into ordinary particles as we discuss below. Suppose that inflation begins at time t_i when the field has the value ϕ_i in a Hubble-sized patch of the universe, and ends at time t_e where the field value is such that $\epsilon_\phi(\phi_e) \sim 1$ or $|\eta_\phi(\phi_e)| \sim 1$. The total number of e-folds of accelerated expansion is then given by:

$$\begin{aligned} N_e &= \log\left(\frac{a_e}{a_i}\right) = \int_{a_i}^{a_e} \frac{da}{a} \\ &= \int_{t_i}^{t_e} \frac{1}{a}\frac{da}{dt} dt \\ &= \int_{t_i}^{t_e} H(t) dt \\ &= \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi \\ &\simeq -\int_{\phi_i}^{\phi_e} \frac{3H^2}{V'(\phi)} d\phi \\ &\simeq -\frac{1}{M_P^2}\int_{\phi_i}^{\phi_e} \frac{V(\phi)}{V'(\phi)} d\phi , \end{aligned} \tag{20}$$

where we have successively changed the integration variable from the scale factor $a(t)$ to cosmic time t and field value $\phi(t)$, and used the slow-roll and Friedmann equations, Eq. (11). It is thus explicit that the flatter the inflaton potential, i.e. the smaller its slope in the relevant range of field values, the longer inflation will last. Let us consider our working example with a quadratic potential, for which:

$$\epsilon_\phi = \eta_\phi = 2\left(\frac{M_P}{\phi}\right)^2 , \tag{21}$$

so that inflation requires super-planckian values of the field, $\phi \gg M_P$. This implies that $\phi_e = \sqrt{2}M_P \ll \phi_i$, and we have from Eq. (20):

$$N_e \simeq -\frac{1}{2M_P^2}\int_{\phi_i}^{\phi_e} \phi d\phi = \frac{\phi_i^2 - \phi_e^2}{4M_P^2} \approx \frac{\phi_i^2}{4M_P^2} , \tag{22}$$

so that inflation will last longer if it begins at larger field values. In particular, $N_e \simeq 50-60$ requires $\phi_i/M_P \simeq 14-15$.

Inflationary model building

As we have seen so far, the main ingredient for a successful model of inflation solving the problems of the standard cosmological model is a scalar field with a sufficiently flat potential. Scalar fields have a long standing history in particle physics, describing for example neutral pions, which are bound states of quarks and antiquarks. Elementary scalar fields have been postulated to exist for several decades, but only recently have we found experimental evidence for the long-sought Higgs boson - the scalar field responsible for spontaneously breaking the electroweak symmetry and giving mass to elementary particles. Although it is not yet clear that this discovery corresponds to an elementary (as opposed to composite) particle, it gives a new motivation to take the slowly-rolling scalar field inflationary paradigm seriously.

Extensions of the Standard Model (SM) of particle physics typically include several additional scalar fields. For example, supersymmetry postulates the existence of scalar partners for all the quarks and leptons, while in theories with compact extra-dimensions the size and shape of these invisible dimensions are effectively described by the dynamics of scalar fields. Several models of inflation have been developed since the 1980's, either motivated in extensions of the SM or simply looking for potential functions that satisfy the slow-roll requirements. While an exhaustive review of inflationary models is outside the scope of thus course, there are three large classes of models that group the many models found in the literature with similar features:

1. **Large field models**

   In these models, inflation occurs for large field values, typically super-planckian. The quadratic potential is an example of such a model, as we have seen previously, and is part of a larger class of models with monomial potentials:

   $$V(\phi) = \lambda\phi^n , \tag{23}$$

   for which we have

   $$\epsilon_\phi = \frac{n^2}{2}\left(\frac{M_P}{\phi}\right)^2 , \qquad \eta_\phi = n(n-1)\left(\frac{M_P}{\phi}\right)^2 , \tag{24}$$

   yielding a number of e-folds:

   $$N_e \simeq \frac{1}{2n}\frac{\phi_i^2}{M_P^2} , \tag{25}$$

   such that we need $\phi_i \gg M_P$ in order to obtain 50-60 e-folds of inflation. Monomial potentials are the canonical example of what is known as chaotic inflation, originally proposed by Andrei Linde [1], who argued that, according to Heisenberg's uncertainty principle, the universe should emerge from the quantum gravity era, for energies above the Planck scale, with stochastic initial conditions, such that energy densities $\langle V\rangle \sim \langle\dot{\phi}^2\rangle \sim M_P^4$. This corresponds to $\phi > M_P$ for small λ in the case of monomial potentials, and in some patches of the post-planckian universe the potential energy would be the dominant component and inflate.

2. **Small field models**

   In this class of models, inflation occurs near the origin of field space, $\phi \ll M_P$. A particular example are hilltop models:

$$V(\phi) = V_0\left(1 - \frac{\gamma}{n}\left(\frac{\phi}{M_P}\right)^n\right) , \tag{26}$$

   with $n \geq 2$. These typically correspond to an expansion of a more general potential about the origin, as is the case of the Higgs potential $V(\phi) = \lambda(\phi^2 - v^2)^2$, where the origin corresponds to an unstable maximum of the potential, so that the field will evolve towards the true minimum at $\phi = v$. Close to the origin the constant term V_0 is the leading contribution to the potential, which may thus have the required flatness. In particular, we have:

$$\epsilon_\phi = \frac{\gamma^2}{2}\left(\frac{\phi}{M_P}\right)^{2n-2} , \qquad \eta_\phi = -\gamma(n-1)\left(\frac{\phi}{M_P}\right)^{n-2} , \tag{27}$$

   with slow-roll occuring for $\phi \ll M_P$ and/or small values of γ. As the field evolves towards non-zero values we have $\phi_e \gg \phi_i$ and the number of e-folds is given by:

$$N_e \simeq \frac{1}{\gamma(n-2)}\left(\frac{\phi_i}{M_P}\right)^{2-n} . \tag{28}$$

3. **Hybrid models**

   Hybrid models consider two or more coupled scalars with a potential of the form [2]:

$$V(\phi,\chi) = f(\phi) + \frac{g^2}{2}\phi^2\chi^2 + \frac{\lambda^2}{4}(\chi^2 - v^2)^2 , \tag{29}$$

   where ϕ acts as the inflaton field and χ is known as the waterfall field. The function $f(\phi)$ is arbitrary but typically includes contributions from quantum effects. The main idea is that the χ field has a Higgs-like potential whose minimum value depends on the inflaton field, ϕ. In particular, the origin $\chi = 0$ is either a minimum or a maximum of the potential. To see this, let us evaluate the mass of the waterfall field at this point, i.e. the curvature of the potential:

$$m_\chi^2(\chi = 0) = \frac{\partial^2 V(\phi,\chi)}{\partial\chi^2}(\phi, 0) = g^2\phi^2 - \lambda^2 v^2 , \tag{30}$$

   which means that for $g\phi > \lambda v$ the field is heavy and the field χ sits at a (meta)stable minimum at the origin $\chi = 0$. This yields for the inflaton a potential of the form $V(\phi, 0) = f(\phi) + \lambda^2 v^4/4$, which is constant up to corrections included in $f(\phi)$, which are model-dependent but can be made small. Hence, the inflaton potential is naturally flat for large field values. If $f(\phi)$ makes the inflaton decrease, the system will eventually reach a stage where the waterfall field becomes massless and the origin $\chi = 0$ becomes an unstable maximum. This triggers a phase transition that ends inflation with the waterfall field moving towards the true minimum at $\chi = v$. The number of e-folds of inflation depends on the function $f(\phi)$ so we will not compute here.

We illustrate the form of the inflaton potential in these three cases in Figure 1.

As mentioned earlier, the list of possible inflationary potentials is not exhausted by these three types of model, but the main dynamical features of slow-roll inflation are indeed captured in these examples.

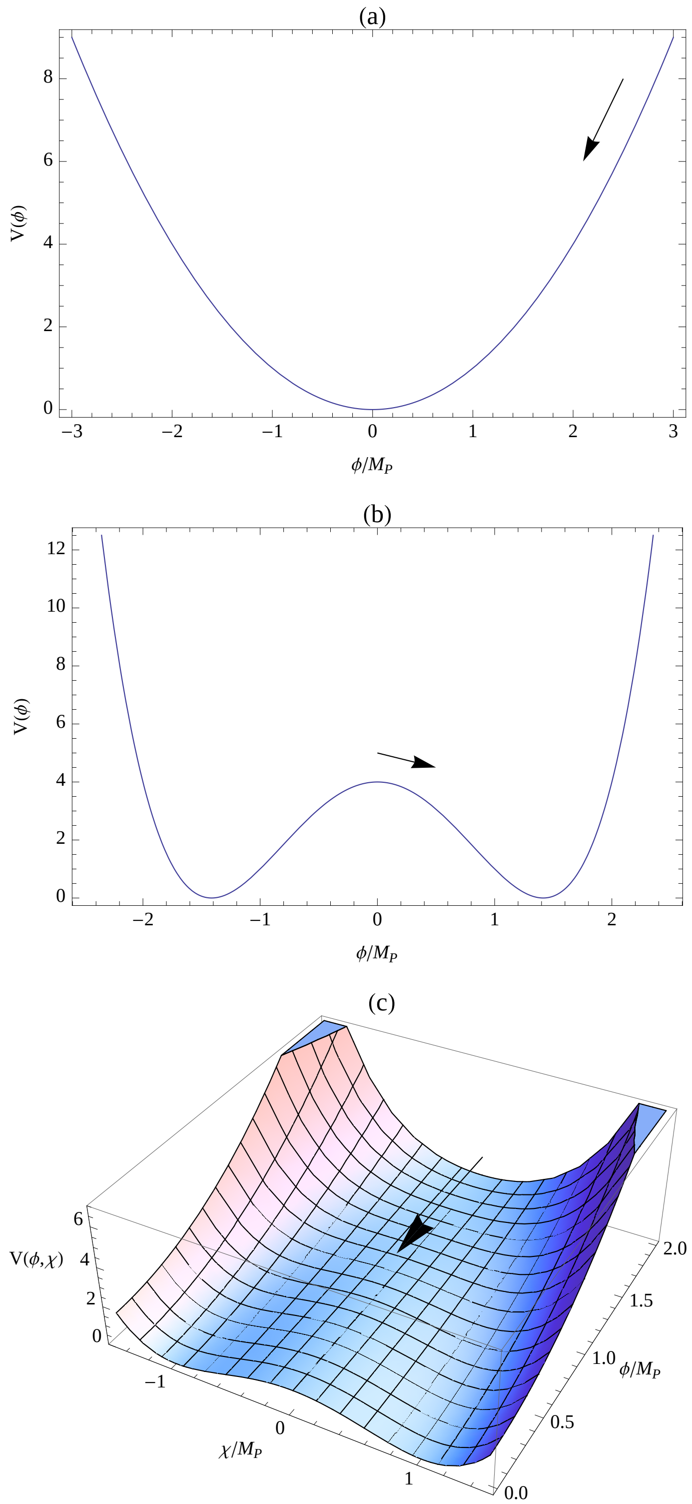

Figure 1: Potential curves for (a) large field models, in particular a quadratic potential; (b) small field models, in particular a hilltop/Higgs potential; and (c) hybrid models, with $f(\phi) = 0$ for illustrative purposes. In all cases the motion of the inflaton field during inflation is indicated by an arrow.

Reheating

Once the slow-roll conditions are violated, the kinetic energy of the inflaton will increase and may eventually dominate, so that the equation of state parameter may evolve form $w_\phi \simeq -1$ during inflation to $w_\phi \simeq 1$. This kinetic energy-dominated phase is known as *kination*, and it is easy to see that $\rho_\phi \propto a^{-6}$, so that the energy density of the inflaton redshifts away quite quickly.

During the kination phase the field generically field evolves towards the minimum of its potential, e.g. at $\phi = 0$ for the quadratic potential, and executes damped oscillations about this minimum. So far we have neglected any interactions of the inflaton field with other degrees of freedom, but these must be included in order to transfer the energy of the inflaton field to ordinary particles and exit into the standard cosmology dominated first by radiation and then by matter. Couplings between different fields are a feature of the Standard Model of particle physics, and simple interactions with other scalar fields, e.g. $g^2\phi^2\chi^2$ as in the hybrid model, or Yukawa interactions with fermions, $g\phi\bar{\psi}\psi$, should in principle be included. These couplings make the inflaton field unstable and allow them to decay into other degrees of freedom. When the field is oscillating about its minimum, the equation of motion is then modified by the addition of a friction term proportional to the decay width Γ_ϕ, which we will take as an arbitrary parameter but can be computed for each particular interaction:

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi\dot{\phi} + V'(\phi) = 0 \ . \tag{31}$$

This term is typically negligible during the slow-roll phase if $\Gamma_\phi \ll H$, as the inflaton is moving slowly, although one may consider models where it may have a significant effect (see problem 10). Moreover, although the derivation of this contribution in the context of quantum field theory is outside the scope of these lectures, we should note that in the general case it corresponds to a dissipation coefficient $\Upsilon\dot{\phi}$ that may take several different forms depending on the structure of the interactions and dynamics of the inflaton field, and only in the final phase where the inflaton oscillates about the minimum of the potential do we obtain $\Upsilon = \Gamma_\phi$.

Multiplying Eq. (31) by $\dot{\phi}$, we recover the standard energy-momentum conservation modified by the decay term:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\Gamma_\phi\dot{\phi}^2 \ . \tag{32}$$

Recall that the right hand side of this equation vanished for a perfect fluid, i.e. when no entropy was being created or destroyed, $dS = 0$. This means that by decaying into other particles the inflaton will create the entropy required to exit into the standard cosmology. This period is known as *reheating* and may be a complex process. Typically, one assumes that the inflaton decays either directly or indirectly into the Standard Model degrees of freedom through interactions of the form given above and that the decay products scatter off each other sufficiently quickly (within a Hubble time) to reach a state of thermal equilibrium. As the dominant contribution to the entropy in thermal equilibrium corresponds to relativistic degrees of freedom, the energy lost by the inflaton will lead to the formation of a thermal radiation bath at a temperature T, and the equation for the radiation energy density reads:

$$\dot{\rho}_r + 3H(\rho_r + p_r) = \Gamma_\phi\dot{\phi}^2 \ , \tag{33}$$

where $p_r = \rho_r/3$ and $\rho_r = (\pi^2/30)g_*T^4$ as we have seen previously. The total energy density $\rho = \rho_\phi + \rho_r$ and pressure $p = p_\phi + p_r$ satisfy the standard energy-momentum conservation $\dot{\rho} + 3H(\rho + p) = 0$, since the total entropy of the coupled inflaton-radiation system is conserved and entropy is simply being transferred between these two components.

Close to the minimum at $\phi = \phi_m$ such that $V'(\phi_m) = 0$, we can expand the potential as:

$$V(\phi) = V(\phi_m) + \frac{1}{2}V''(\phi_m)(\phi - \phi_m)^2 + \dots \ , \tag{34}$$

so that to leading the field behaves as a damped harmonic oscillator whatever the form of the potential. Recall that for a harmonic oscillator we have $\langle V\rangle = \langle\dot{\phi}^2/2\rangle = \rho_\phi/2$, so that $p_\phi \simeq 0$ and the oscillating inflaton behaves as non-relativistic pressureless matter, a period that may be preceded by the above mentioned kination phase. During the oscillating phase, the coupled inflaton-radiation system thus obeys the following set of coupled differential equations:

$$\begin{aligned}\dot{\rho}_\phi + 3H\rho_\phi &= -\Gamma_\phi \rho_\phi \\ \dot{\rho}_r + 4H\rho_r &= \Gamma_\phi \rho_\phi \\ H^2 &= \frac{\rho_\phi + \rho_r}{3M_P^2} \ .\end{aligned} \tag{35}$$

For simplicity, let us assume the kination phase can be neglected and oscillations begin immediately after inflation ends. Then, for a constant decay width the inflaton energy density evolves like:

$$\rho_\phi(t) = \rho_\phi(t_e)\left(\frac{a}{a_e}\right)^{-3} e^{-\Gamma_\phi(t-t_e)} \ , \tag{36}$$

where $\rho_\phi(t_e) \simeq V(\phi_e)$ if the inflaton energy density is not too diluted during the kinetic energy-dominated phase. This is exactly the behaviour expected for unstable non-relativistic matter redshifting with the volume and decaying with the usual exponential law. While the oscillating inflaton is still the dominant component and has not decayed significantly, $a(t) \propto a^{2/3}$ as for non-relativistic matter and $H(t) = 2/3t$. Using this we can solve the radiation equation, giving:

$$\rho_r(t) \simeq \frac{3}{5}\rho_\phi\left(\frac{t}{\tau_\phi}\right)\left(1 - \left(\frac{t_e}{t}\right)^{5/3}\right) \ , \tag{37}$$

where $\tau_\phi = \Gamma_\phi^{-1}$ is the inflaton decay time and we used that $\rho_r \simeq 0$ at the end of inflation, since any initial component would have been exponentially diluted. This implies that the radiation energy density starts growing and then decreases, reaching a maximum temperature:

$$T_{max} \sim g_*^{-1/4} V(\phi_e)^{1/8} (M_P \Gamma_\phi)^{1/4} \ . \tag{38}$$

This is illustrated in Figure 2, where one can also see that radiation overcomes the inflaton contribution to the energy density at $t \sim \tau_\phi$, i.e. by the time the inflaton has effectively decayed and $\Gamma_\phi \gtrsim H$, and we enter the radiation era.

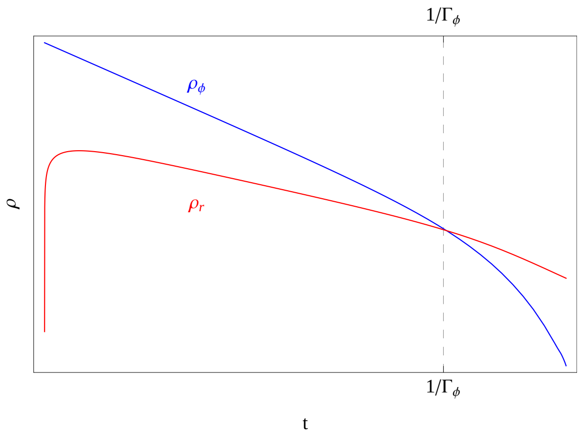

Figure 2: Evolution of the inflaton and radiation energy densities during the oscillating phase at the end of inflation, showing that radiation is created and eventually overcomes the inflaton contribution after the latter has decayed at $t \simeq \Gamma_\phi^{-1}$.

We can then estimate the temperature at the start of the radiation era by equating the Hubble parameter at this stage $H \simeq \pi/\sqrt{90}T^2/M_P$ with the inflaton decay width, which yields the *reheating* temperature:

$$T_R \sim 1.7 g_*^{-1/2} (M_P \Gamma_\phi)^{1/2} \ . \tag{39}$$

Note that while the maximum temperature depends on the energy density at the end of inflation, the reheating temperature only depends on the duration of the reheating period, $\tau_\phi = \Gamma_\phi^{-1}$, with a very long decay time leading to a lower reheating temperature. While we have so far kept the scale of inflation arbitrary, we will see in the next lecture how the temperature anisotropies in the CMB can be used to determine this. If the inflaton decays quickly, most of its energy will be converted into radiation and we have $T_R \sim \rho_\phi(t_e)^{1/4}$. On the other hand we must require that reheating is complete before the synthesis of light elements begins, so that in general we have:

$$10 \text{ MeV} \lesssim T_R \lesssim \rho_\phi(t_e)^{1/4} \ . \tag{40}$$

Besides conventional decay, the inflaton also excites Fourier modes of the fields it couples to and that have a frequency comparable to its oscillating frequency. Although we will not describe the details of this process in this course, it shares similar features to many other resonant mechanisms found in nature and leads to an exponentially fast production of particles. This is particularly important in the early stages of the oscillating phase, when the amplitude of the oscillations is still large, whereas conventional decay becomes the dominant process at later times. For this reason, this is known as *preheating*.

In summary, slow-roll inflation provides a simple an elegant mechanism to solve the problems of the standard Big Bang cosmology. During an initial phase dominated by the potential energy of the inflaton scalar field, the universe expands in an accelerated fashion, mimicking the effect of a cosmological constant. This period lasts for $50 - 60$ e-folds until the potential steepens and the slow-roll conditions are no longer valid. The kinetic energy of the inflaton becomes comparable and may overcome its potential energy, and the field evolves quickly towards the minimum of its potential. Before stabilizing at the minimum, the field executes damped oscillations and on average behaves as non-relativistic matter. While it oscillates, the inflaton decays into relativistic degrees of freedom that eventually thermalize and become the dominant component, thus reheating the universe and leading to the standard cosmological evolution.

Problem 10

In *warm inflation* models, interactions between the inflaton and other fields lead to an effective friction term in the equation of motion and that acts a source of radiation during inflation:

$$\begin{gathered} \ddot{\phi} + 3H\dot{\phi} + \Upsilon\dot{\phi} + V'(\phi) = 0 \\ \dot{\rho}_r + 4H\rho_r = \Upsilon\dot{\phi}^2 \ , \end{gathered} \tag{41}$$

where $\Upsilon = \Upsilon(\phi, T)$ is in general distinct from the inflaton decay width Γ_ϕ responsible for reheating. For simplicity, let us consider a constant dissipation coefficient.

(a) Determine the slow-roll equation for the inflaton field and show that the slow-roll conditions are given by:

$$\epsilon_\phi \ll (1+Q) \ , \qquad |\eta_\phi| \ll 1 + Q \ , \tag{42}$$

where $Q = \Upsilon/3H$, thus alleviating the need for a very flat potential with strong dissipation.

(b) Obtain an expression for the number of e-folds of inflation with a quadratic potential and compare with Eq. (22).

(c) Show that the radiation energy density approaches an almost constant value such that

$$\frac{\rho_r}{V(\phi)} \simeq \frac{\epsilon_\phi}{2} \frac{Q}{(1+Q)^2} \ , \tag{43}$$

so that it is always subdominant in the slow-roll regime but becomes significant at the end of inflation for strong dissipation.

References

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