Introduction to Numerical Relativity

1 ADM Mass

It is well known that no local expression for the energy density of the gravitational field can exist in general relativity. However there does exist a well-defined notion of the total energy of an isolated system as measured by a distant observer. One sees this intuitively; it can be measured by observing the relative accelerations of geodesics (freely falling masses) in the laboratory, a direct measurement of the Riemann tensor. The total mass energy of an isolated source was originally calculated in the Hamiltonian formulation of general relativity by Arnowitt, Deser and Misner (ADM)

The ADM mass measures the total mass-energy of an isolated gravitating system at any instant of time measured within a spatial surface enclosing the system at infinity. The ADM mass is defined by an integral over the 2-dimensional surface at infinity S_t of a spatial slice Σ_t

$$M_{ADM} = \lim_{r\to\infty} \frac{1}{16\pi} \int [D^j \gamma_{ij} - D_i(f^{kl}\gamma_{kl})] S^i \sqrt{q} d^2 y \ . \quad (1)$$

For this definition it is assumed that the induced metric on Σ_t is such that $\gamma_{ij} = f_{ij} + O(r^{-1})$. f_{ij} is the flat metric and D is the connection associated with f. S^i are the components of the unit normal to S_t within Σ_t. $\vec{S}$ is the unit normal to S_t, q is the induced metric on it and $y^a = (y^1, y^2)$ are some coordinates on S_t.

Let us consider Schwarzschild in Boyer-Lindquist coordinates $x^\alpha = (t, r, \theta, \phi)$. Σ_t is the hypersurface of constant time and the components of the induced metric on Σ_t are $\gamma_{ij} = \mathrm{diag}[(1 - \frac{2M}{r})^{-1}, r^2, r^2 \sin^2\theta]$ $f_{ij} = \mathrm{diag}[(1, r^2, r^2 \sin^2\theta]$. Let be S_t the sphere $r = \mathrm{const}$ in the hypersurface Σ_t, then $y^a = (\theta, \phi)$, $\sqrt{q} = r^2 \sin\theta$, $S^i \sqrt{q} dy^2 = r^2 \sin\theta d\theta d\phi (\partial_r)^i$, $(\partial_r)^i = (r, 0, 0)$. Then the integral (1) becomes

$$M_{ADM} = \lim_{r\to\infty} \frac{1}{16\pi} \int [D^j \gamma_{ij} - D_i(f^{kl}\gamma_{kl})] r^2 \sin\theta d\theta d\phi \ , \quad (2)$$

where

$$D^j \gamma_{ij} - D_i(f^{kl}\gamma_{kl}) = \frac{4M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \sim \frac{4M}{r^2} \ . \quad (3)$$

As a result we get

$$M_{ADM} = \lim_{r\to\infty} \frac{1}{16\pi} \int \frac{4M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} r^2 \sin\theta d\theta d\phi \sim M \ . \quad (4)$$

Conformal form

Let us introduce the conformal metric $\bar{\gamma}$ and the conformal factor ψ associated to γ

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \ , \quad \det\,(\bar{\gamma}_{ij}) = 1 \ . \quad (5)$$

Then the ADM mass can be expressed as

$$M_{ADM} = -\lim_{r\to\infty} \frac{1}{2\pi} \int S^i (D_i - \frac{1}{8} D^j \bar{\gamma}_{ij}) \sqrt{q} d^2 y \ . \quad (6)$$

For Schwarzschild in isotropic coordinates $\psi = \left(1 + \frac{M}{2r}\right)$, $\gamma_{ij} = f_{ij}$ and $D^j \bar{\gamma}_{ij} = 0$. The ADM mass is then

$$M_{ADM} = -\frac{1}{2\pi} \int \frac{\partial \psi}{\partial r} r^2 \sin\theta d\theta d\phi \ . \tag{7}$$

Now, $\frac{\partial \psi}{\partial r} = -\frac{M}{2r^2}$. Consequently $M_{ADM} = M$ as expected.

In numerical applications, the integral is evaluated on a large surface at a finite distance from the source. In this case M_{ADM} will change in time whenever there is a flux of matter or gravitational radiation passing across the surface. The rate of change of M_{ADM} will reflect the rate at which mass-energy is carried across the surface by these fluxes.

Sometimes is useful to work in Cartesian type coordinates. In particular if we use Cartesian coordinates in the definition of asymptotic flatness in (1) then $D_i = \frac{\partial}{\partial x^i}$ and $f^{kl} = \delta^{kl}$. The ADM mass can be written as

$$M_{ADM} = \frac{1}{16\pi} \int \left(\frac{\partial \gamma_{ij}}{\partial x^j} - \frac{\partial \gamma_{ji}}{\partial x^i} \right) S^i \sqrt{q} d^2 y \ . \tag{8}$$

Apparent horizons

The concept of apparent horizon allow us to locate black holes during evolution. The apparent horizon is defined as the outermost smooth 2-surface embedded in the spatial slice Σ_t, whose future null geodesics have zero expansion everywhere. The singularity theorems of GR tell us that if an apparent horizon exist on a given time slice, it must be inside a black hole event horizon. However the the absence of an apparent horizon does not necessarily imply that a black hole is absent. If fact it is possible to construct slicing in Schwarzschild spacetime with no apparent horizons. Also it is possible to show that apparent horizons do not form during spherical collapse in polar slicings. Thus the apparent horizon has a strong gauge dependence nature. The usual expectation when performing a black hole simulation is that an apparent horizon will eventually appear on the slice whenever a black hole is present. Let us consider a closed 2-dimensional surface S on Σ_t, S is spatial by construction. Let s^a be its outward pointing unit normal vector lying in Σ_t, then $s^a s_a = 1$ and $n^a s_a = 0$. The metric on Σ_t, γ_{ab} induces a 2-dimensional metric m_{ab} on s given by

$$m_{ab} = \gamma_{ab} - s_a s_b = g_{ab} + n_a n_b - s_a s_b \ . \tag{9}$$

For each point on S we can construct 2 future pointing null geodesics whose projection on Σ_t is orthogonal to S. Tangents k^a and ℓ^a to these geodesics on S are

$$k^a = \frac{1}{\sqrt{2}}(n^a + s^a) \ , \quad \ell^a = \frac{1}{\sqrt{2}}(n^a - s^a) \ . \tag{10}$$

Then we have; $k_a k^a = 0$, $\ell_a \ell^a = 0$ and $m_{ab} k^a = 0$, $m_{ab} \ell^a = 0$. We have chosen the normalization such that $k^a \ell_a = -1$. We can express m_{ab} in terms of k^a and ℓ^a as

$$m_{ab} = g_{ab} + k_a \ell_b + \ell_a k_b. \tag{11}$$

The expansion of the outgoing null geodesics orthogonal to S is defined as

$$\Theta = m^{ab} \nabla_a k_b. \tag{12}$$

The projection with m ensures that only the derivatives tangents to S appear in this expression. Now we define an outer trapped surface as a 2-dimensional surface S embedded in Σ_t on which the expansion Θ of the outgoing null geodesics orthogonal to S is negative everywhere. A trapped region is any region of Σ_t that contains outer trapped surfaces. An apparent horizon is the outer boundary of any connected trapped region. This definition makes the apparent horizon a marginally trapped surface on which the expansion of outgoing null geodesics vanishes $\Theta = 0$.

Example in spherical symmetry

The most general metric in spherical symmetry can be written as

$$ds^2 = -(\alpha^2 - A^2\beta^2)dt^2 + 2A^2\beta dr dt + A^2 dr^2 + B^2 r^2 (d\theta^2 + \sin\theta^2 d\phi^2) \ . \tag{13}$$

Let us consider a spherical surface S centred on the origin. The spatial normal vector s^a to S is then

$$s^a = (0, \frac{1}{A}, 0, 0) \quad s_a = (A\beta, A, 0, 0) \ . \tag{14}$$

The outgoing null normal is

$$k_a = \frac{1}{\sqrt{2}} (A\beta - \alpha, A, 0, 0) \quad k^a = \frac{1}{\sqrt{2}} \left(\frac{1}{\alpha}, \frac{1}{A} - \frac{\beta}{\alpha}, 0, 0 \right) \ , \tag{15}$$

and the metric

$$m_{ab} = \mathrm{diag}(0, B^2 r^2, B^2 r^2 \sin\theta^2) \ . \tag{16}$$

The expansion becomes

$$\Theta = \frac{\sqrt{2}}{rB} \left(\frac{1}{\alpha} \partial_t (Br) + \left(\frac{1}{A} - \frac{\beta}{\alpha} \right) \partial_r (Br) \right) \ . \tag{17}$$

As an example take Schwarzschild spacetime in isotropic coordinates, then

$$A = B = \psi^2 = \left(1 + \frac{M}{2r} \right)^2 \ , \qquad \beta = 0 \qquad \partial_t (Br) = 0 \ , \tag{18}$$

then the expansion vanishes at

$$\partial_r (Br) = 0 \ , \tag{19}$$

which happens at $r = \frac{M}{2r}$.