Introduction to Numerical Relativity

The analysis of the initial value problem is based on a particular way of separating the initial data into freely specifiable and determined from constraints pieces. Part of the motivation is to identify the dynamical gravitational variables, as those which correspond to the radiative degrees of freedom in a covariant way. From the point of view of the construction of the initial data the greatest achievement is that the equations for the initial values can be solved in a more or less straightforward way.

1 Conformal transformations

Our goal is to construct a physical metric γ_{ij}, extrinsic curvature K_{ij} which satisfy the constraint equations. Let us begin considering a conformal transformation applied to the spatial metric γ_{ij}

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij} \ , \tag{1}$$

$\bar{\gamma}_{ij}$ will be a freely specifiable in the initial data construction. We call ψ the conformal factor and the conformally related metric. The connection coefficients are related as

$$\Gamma^i_{jk} = \bar{\Gamma}^i_{jk} + 2(\delta^i{}_j \bar{D}_k \ln\psi + \delta^i{}_k \bar{D}_j \ln\psi - \bar{\gamma}_{jk}\bar{\gamma}^{il}\bar{D}_l \ln\psi) \ , \tag{2}$$

and the Ricci tensor

$$R_{ij} = \bar{R}_{ij} - 2(\bar{D}_i D_j \ln\psi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m \ln\psi) + 4((\bar{D}_i \ln\psi)(\bar{D}_j \ln\psi) - \bar{\gamma}_{ij}\bar{\gamma}^{lm}(\bar{D}_l \ln\psi)(\bar{D}_m \ln\psi)) \ . \tag{3}$$

For the Ricci scalar we get

$$R = \psi^{-4}\bar{R} - 8\psi^{-5}\bar{D}^2\psi \ , \quad \bar{D}^2 = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j \ . \tag{4}$$

The Hamiltonian constraint can be written as

$$8\bar{D}^2\psi - \psi\bar{R} - \psi^5 K^2 + \psi^5 K_{ij}K^{ij} = -16\pi\psi^5\rho \ , \tag{5}$$

which for a given choice of the conformally related metric $\bar{\gamma}_{ij}$ can be interpreted as an equation for the conformal factor. The extrinsic curvature K_{ij} has to satisfy the momentum constraint and will be useful to rescale it as well. We first split the extrinsic curvature into its trace K, which will be freely specifiable and a trace free part A_{ij}.

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K \ , \tag{6}$$

and then conformally transform the trace-free part

$$A^{ij} = \psi^{-10}\bar{A}^{ij} \qquad A_{ij} = \psi^{-2}\bar{A}_{ij} \ , \tag{7}$$

with this choice the tensor $\bar{A}^{ij}$ has zero divergence if and only if A^{ij} does. This is not the only possible choice. Inserting this decomposition into the Hamiltonian constraint we get.

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho \ , \tag{8}$$

and the momentum constraint becomes

$$\bar{D}_j \bar{A}^{ij} - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi\psi^{10} j^i \ , \qquad (9)$$

Conformal transverse traceless decomposition

Any symmetric, traceless tensor can be split into a transverse-traceless part that is divergence-less and a longitudinal part that can be written as a symmetric, traceless gradient of a vector. In other words we can decompose $\bar{A}^{ij}$ as

$$\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{TT}} + \bar{A}^{ij}_{\mathrm{L}} \ , \qquad (10)$$

where the transverse part $\bar{A}^{ij}_{\mathrm{TT}}$ is divergence-less

$$\bar{D}_j \bar{A}^{ij}_{\mathrm{TT}} = 0 \ , \qquad (11)$$

and the longitudinal part $\bar{A}^{ij}_{\mathrm{L}}$ satisfies

$$\bar{A}^{ij}_{\mathrm{L}} = (\bar{L}W)^{ij} \equiv \bar{D}^i W^j + \bar{D}^j W^i - \frac{2}{3}\bar{\gamma}^{ij}\bar{D}_k W^k \ , \qquad (12)$$

where W^k is a vector potential.

We can write the divergence of $\bar{A}^{ij}$ as

$$\begin{aligned} \bar{D}_j \bar{A}^{ij} &= \bar{D}_j \bar{A}^{ij}_L = \bar{D}_j (\bar{L}W)^{ij} & (13) \\ &= \bar{D}^2 W^i + \frac{1}{3}\bar{D}^i(\bar{D}_j W^j) + \bar{R}^i{}_j W^j & (14) \\ &= (\bar{\Delta}_L W)^i & (15) \end{aligned}$$

$\bar{\Delta}_L$ is known as the vector laplacian.

The momentum constraint in terms of the potential becomes

$$(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi\psi^{10} S^i \ , \qquad (16)$$

We now see that we can freely choose the conformally related metric $\bar{\gamma}^{ij}$, the mean curvature K and the transverse-traceless part of the conformally related extrinsic curvature $\bar{A}^{ij}_{TT}$. Given these choices, we can solve the Hamiltonian constraint to find ψ and the momentum constraint to find W^i. Although the system of 4 coupled non-linear equations can be solved numerically, in practice there are several important special cases in which the equations simplify:

If the the slice is maximal ($K = 0$) then the 4-vector non-linear elliptic constraints decouple into separate linear 3-vector (momentum constraint) and non-linear (Hamiltonian) equations, which may be solved separately.

If the slice is vacuum, time-symmetric ($K_{ij} = 0$), and 3-conformally flat, then the full constraint equations have an analytical solution, that represents an arbitrary number of momentarily stationary Schwarzschild black holes.

If the slice is vacuum ($\rho = S^i = 0$), maximal ($K = 0$), and 3-conformally flat ($\bar{\gamma}_{ij}$ is a flat metric), the equations simplify still further. In particular, for this case has found the analytical general solution for the linear momentum constraint equation. This means that only the Hamiltonian constraint need be solved numerically.

Black hole solutions

Let us consider vacuum solutions $\rho = 0, S^i = 0$ and focus on a moment of time symmetry, at which all the derivatives of γ_{ij} are zero, $\beta^i = 0, K_{ij} = 0$ and $K = 0$. On such a time slice the momentum constraint is satisfied trivially and the Hamiltonian constraint reduces to

$$\bar{D}^2\psi = \frac{1}{8}\psi\bar{R} \ , \tag{17}$$

Let us choose the conformally related metric to be flat $\bar{\gamma}_{ij} = \eta_{ij}$ then $\bar{R}_{ij} = 0$ and $\bar{R} = 0$ hence

$$\bar{D}^2\psi = 0 \ . \tag{18}$$

We will be interested in asymptotically flat solutions that satisfy

$$\psi \to 1 + O\left(\frac{1}{r}\right) \quad r \to \infty \tag{19}$$

the solution is then $\psi = 1 + \frac{M}{2r}$ that is the spatial part of the Schwarzschild solution in isotropic coordinates.

$$dl^2 = \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \ . \tag{20}$$

Furthermore, since $\bar{D}^2\psi$ is linear we can obtain an arbitrary number of black holes at a moment of time symmetry.

$$\psi = 1 + \sum_\alpha \frac{\mathcal{M}}{2r_\alpha} \ , \quad r_\alpha = |x^i - c^i_\alpha| \ , \tag{21}$$

r_α is the coordinate separation from the centre c^i_α of the αth black hole.

The solution to the constraint equations for two black holes instantaneously at rest at a moment of time symmetry can be used as initial data for head-on collisions of black holes.

Let us focus on vacuum solutions and assume a maximal slicing $K = 0$ which amounts to assuming that the initial slice Σ has a certain shape that maximizes its volume. Then the momentum constraint decouples from the Hamiltonian constraint and becomes

$$(\bar{\Delta}_L W)^i = 0 \tag{22}$$

that can be solved independently. If we assume conformal flatness $\bar{\gamma}_{ij} = \eta_{ij}$ the vector laplacian in Cartesian coordinates reduces to

$$\partial^j\partial_j W^i + \frac{1}{3}\partial^i\partial_j W^j = 0 \ , \tag{23}$$

The solutions of this equation are called Bowen-York solutions. In the following we will present two different solutions, one representing a spinning black hole and other representing a boosted black hole.

Spinning black hole

Let us start writing the vector W^i as

$$W_i = V_i + \partial_i U \ , \tag{24}$$

then the equation for W^i splits into a coupled set of Poisson equations

$$\partial^j\partial_j V_i + \frac{1}{3}\partial_i\partial_j V_j + \partial^j\partial_j\partial_i U + \frac{1}{3}\partial_i\partial^j\partial_j U = S_i \tag{25}$$

We can choose U in such a way that the two terms involving U satisfy

$$\partial^j\partial_j U = -\frac{1}{4}\partial_j V^j \tag{26}$$

$$\tag{27}$$

and then

$$\partial^j \partial_j V_i = S_i \ , \tag{28}$$

If we assume $S_i = 0$ and $V_i = 0$ the general spherically symmetric for U is given by

$$U = a - \frac{b}{r} \tag{29}$$

with a and b two constants and $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$W^i = \eta^{ij} \partial_j U = b\frac{x^i}{r^3} = b\frac{l^i}{r^2} = bX^i \ , \tag{30}$$

where

$$l^i = \frac{x^i}{r} \ , \quad X^i = \frac{l^i}{r^2} \ . \tag{31}$$

We can generalize the solution to

$$W^i = \epsilon^{ijk} X_j J_k \tag{32}$$

assuming that J^k is a vector with constant coefficients when expressed in Cartesian coordinates. In spherical polar coordinates the non vanishing component of W^i is $W^\phi = -\frac{J}{r^3}$ where J is the magnitude of J^i aligned with the polar axis. Furthermore, we have

$$\bar{A}^L_{r\phi} = \frac{3J}{r^2} \sin^2\theta \tag{33}$$

and

$$\bar{A}_{ij}\bar{A}^{ij} = \frac{18J^2}{r^2} \sin^2\theta \ , \tag{34}$$

Boosted black hole

In an alternative approach we can decompose W^i as

$$W_i = \frac{7}{8} V_i - \frac{1}{8}(\partial_i U + x^k \partial_i V_k) \tag{35}$$

with $\partial^j \partial_j U = 0$ and $\partial^j \partial_j V_i = 0$.

Now by assuming $U = 0$ and writing the solution for V_i as $V_i = -2\frac{P_i}{r}$ with P_i an arbitrary vector with constant coefficients when expressed in Cartesian coordinates we get

$$W^i = -\frac{1}{4r}(7P^i + l^i l_k P^k) \ , \tag{36}$$

and consequently

$$\bar{A}^{ij}_L = (\bar{L}W)^{ij} = \frac{3}{2r^2}(P^i l^j + P^j l^i - (\eta^{ij} - l^i l^j) l_k P^k) \ . \tag{37}$$

By virtue of the linearity of the momentum constraint we can add several terms of this form to obtain a solution describing multiple, boosted black holes and in fact we can allow these black holes to have spin.