Introduction to Numerical Relativity

Given a manifold $\mathcal{M}$ describing a spacetime with 4-metric we want to foliate it via space-like, three-dimensional hypersurfaces: $\{\Sigma_i\}$. We label such hypersurfaces with the time coordinate t, Σ_t. We define the 1-form

$$\Omega_a = \nabla_a t \ , \tag{1}$$

with the normalization

$$|\Omega|^2 = g^{ab}\nabla_a t \nabla_b t = -\frac{1}{\alpha^2} \ , \tag{2}$$

The function α is called the lapse function and it is strictly positive for spacelike hypersurfaces $\alpha(t, x^i) > 0$. We define also the unit normal vector to the hypersurface Σ_t

$$n^a = -\alpha g^{ab}\Omega_b = -\alpha g^{ab}\nabla_b t \ . \tag{3}$$

The sign is chosen so that the related unit vector is future directed. i.e. $n^a n_a = -1$, n^a can be viewed as the four velocity of an observer moving orthogonally to the hypersurfaces of Σ_t Such an observer will have a four-acceleration, a^b

$$a^b = n^a \nabla_a n^b \ , \tag{4}$$

With the above vector we can construct the spatial metric induced on the 3 dimensional hypersurface

$$\gamma_{ab} = g_{ab} + n_a n_b \ , \tag{5}$$

The inverse can be found by raising the indices with g^{ab}

$$\gamma_{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b \ , \tag{6}$$

The metric γ_{ab} is purely spatial, lives within the hypersurface. Its contraction with the normal vector is zero

$$n^a \gamma_{ab} = n^a g_{ab} + n^a n_a n_b = n_b - n_b = 0 \ . \tag{7}$$

The projector operator projects a 4-dimensional tensor into a spatial slice,

$$\gamma^a{}_b = g^a{}_b + n^a n_b \ , \tag{8}$$

in particular the projection of an arbitrary 4-vector v^a, $\gamma^a{}_b v^b$ is purely spatial. In order to project tensors of higher rank, each free index has to be contracted with the projection operator. Sometimes the notation

$$\perp T_{ab} = \gamma_a{}^c \gamma_b{}^d T_{cd} \ , \tag{9}$$

is used in the literature.

Any tensor can be decomposed into its spatial and timelike parts,

$$v^a = \delta^a_b v^b = (\gamma^a{}_b - n^a n_b) v^b = \perp v^a - n^a n_b v^b \ , \tag{10}$$

$$T_{ab} = \perp T_{ab} - n_a n^c \perp T_{cb} - n_b n^c \perp T_{ac} + n_a n_b n^c n^d T_{cd} \ . \tag{11}$$

We will need a 3-dimensional covariant derivative D_a that maps spatial tensors into spatial tensors. We also require that this covariant derivate must be compatible with the induced metric, $D_c\gamma_{ab} = 0$. For a scalar function this derivate is defined as

$$D_a f = \gamma_a^b \nabla_b f \ , \tag{12}$$

and for a spacetime tensor:

$$D_a T^b{}_c = \gamma^d{}_a \gamma^b{}_e \gamma^f{}_c \nabla_d T^e{}_f \ , \tag{13}$$

then

$$\begin{aligned}
D_c\gamma_{ab} &= \gamma^d{}_c \gamma^f{}_a \gamma^g{}_b \nabla_d \gamma_{fg} \\
&= \gamma^d{}_c \gamma^f{}_a \gamma^g{}_b \nabla_d (g_{fg} + n_f n_g) \\
&= \gamma^d{}_c \gamma^f{}_a \gamma^g{}_b (n_f \nabla_d n_g + n_g \nabla_d n_f) \\
&= 0 \ ,
\end{aligned}$$

where we have used property (7) and the compatibility of g_{ab} with the 4-covariant derivative.

Extrinsic curvature

The extrinsic curvature, can be found projecting gradients of the normal vector on Σ_t.

$$K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d = -\gamma^c{}_a \gamma^d{}_b \nabla_{(c} n_{d)} \tag{14}$$

also, it can be writen as

$$\begin{aligned}
K_{ab} &= -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d \\
&= -(\delta^c{}_a + n^c n_a)(\delta^d{}_b + n^d n_b) \nabla_c n_d \\
&= -(\delta^c{}_a + n_a n^c) \delta^d{}_b \nabla_c n_d \\
&= -\nabla_a n_b - n_a a_b \ .
\end{aligned}$$

Perhaps the most useful expression is

$$K_{ab} = -\frac{1}{2} \mathcal{L}_n \gamma_{ab} = (\nabla_a n_b + n_a n^d \nabla_d n_b). \tag{15}$$

Gauss-Codazzi equations

Let us consider the definition of 3d Riemann tensor, acting on a pure spatial vector

$$(D_i D_j - D_j D_i)\omega_k = {}^3R_{ijk}{}^l \omega_l. \tag{16}$$

Using the defintion of 3d covariant derivative

$$\begin{aligned}
D_i D_j \omega_k &= \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a (\gamma_b^l \gamma_c^m \nabla_l \omega_m) \\
&= \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a \nabla_b \omega_c + \gamma_i^a \gamma_j^b \gamma_k^m (\nabla_a \gamma_b^l)(\nabla_l \omega_m) + \gamma_i^a \gamma_j^l \gamma_k^c (\nabla_a \gamma_c^m)(\nabla_l \omega_m) \\
&= \gamma_i^a \gamma_j^b \gamma_k^c \nabla_a \nabla_b \omega_c - \gamma_k^m (n^l \nabla_l \omega_m) K_{ij} - K_{ik} K_j^c \omega_c \ ,
\end{aligned}$$

To prove the last equality, let us consider first

$$\begin{aligned}
\nabla_c \gamma_a^b = \nabla_c (g_a^b + n_a n^b) &= \nabla_c (n_a n^b) = n_a \nabla_c n^b + n^b \nabla_c n_a \\
&= -n_a (K_c^b + n_c D^b \ln \alpha) - n^b (K_{ac} + n_c D_a \ln \alpha)
\end{aligned}$$

Consequently

$$\gamma_i^c \gamma_j^a \nabla_c \gamma_a^b = -n^b K_{ij} \ , \tag{17}$$

and the contraction in second term of the right hand side $\gamma_i^a \gamma_j^b (\nabla_a \gamma_b^l) = -n^l K_{ij}$.

Secondly, since ω_a is purely spatial $n^a \omega_a = 0$ we rewrote the third term in the second line as

$$\gamma_j^l n^m \nabla_l \omega_m = -\gamma_j^l \omega_m \nabla_l n^m = \omega_m K_j^m \ . \tag{18}$$

Finally $(D_i D_j - D_j D_i)\omega_k$ gives

$$^3R_{ijk}{}^l \omega_l = \gamma^a{}_i \gamma^b{}_j \gamma^c{}_k R_{abc}{}^l \omega_l - \omega_l K_j^l K_{ik} + \omega_l K_{il} K_{jk} \ . \tag{19}$$

Then we have the Gauss Codazzi equations

$$^3R_{ijkl} = \gamma^a{}_i \gamma^b{}_j \gamma^c{}_k \gamma^d{}_l R_{abcd} + K_{il} K_{jk} - K_{ik} K_{jl} \ , \tag{20}$$

Hamiltonian constraint

if we contract Eq. (20) with γ^{ik}

$$^3R_{jl} = \gamma_j^b \gamma_l^d (R_{bd} + n^a n^c R_{abcd}) + K_{jk} K_l^k - K K_{jl} \ , \tag{21}$$

where we have used $\gamma^{ac} = g^{ac} + n^a n^c$. Multipliying again with γ^{jl}

$$^3R = (R + 2R_{ad} n^a n^d) + K_{il} K^{il} - K^2 \ , \tag{22}$$

Notice that if we use Einstein's equations

$$R + 2R_{ad} n^a n^d = 2G_{ad} n^a n^d = 16\pi T_{ad} n^a n^d \ , \tag{23}$$

in the previous formula we get $\rho = T_{ad} n^a n^d$

$$^3R + K^2 - K_{il} K^{il} = 16\pi\rho \ , \tag{24}$$

which is known as the Hamiltonian constraint.

Momentum constraint

Let us consider other projections

$$\begin{aligned}
D_i D_j n^k = \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a D_b n^c &= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a (\gamma_b^d \gamma_e^c \nabla_d n^e) \\
&= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c + \gamma_i^a \gamma_j^b \gamma_c^k (\nabla_a \gamma_b^d) \nabla_d n^c + \gamma_i^a \gamma_j^b \gamma_c^k (\nabla_a \gamma_e^c) \nabla_b n^e \\
&= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - \gamma_c^k K_{ij} n^d \nabla_d n^c - \gamma_j^b K_i^k n_e \nabla_b n^e \\
&= \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - (D^k \ln\alpha) K_{ij}
\end{aligned}$$

where we have used $D^c \ln\alpha = n^d \nabla_d n^c$ and $n_e \nabla_b n^e = 0$. Then

$$D_i D_j n^k = \gamma_i^a \gamma_j^b \gamma_c^k \nabla_a \nabla_b n^c - (D^k \ln\alpha) K_{ij} \tag{25}$$

if we rename the index $k \to i$

$$D_i D_j n^i = \gamma_c^a \gamma_j^b \nabla_a \nabla_b n^c - (D^i \ln\alpha) K_{ij} \ , \tag{26}$$

or $k \to j$ and $i \leftrightarrow j$ to keep the index i on n^i

$$D_j D_i n^i = \gamma^a_j \gamma^b_c \nabla_a \nabla_b n^c - (D^i \ln \alpha) K_{ij} \ , \tag{27}$$

we can construct the Riemann tensor using the nonconmutativity of $(\nabla_a \nabla_b - \nabla_b \nabla_a)$ as

$$D_i D_j n^i - D_j D_i n^i = \gamma^a_c \gamma^b_j (\nabla_a \nabla_b - \nabla_b \nabla_a) n^c = -\gamma^a_c \gamma^b_j R_{abd}{}^c n^d = R_{bd} \gamma^b_j n^d \tag{28}$$

Taking into account that $D_i n^i = -K$ and $D_j n^i = -K^i_j$ we get

$$D_i K^i_j - D_j K = -R_{bd} \gamma^b_j n^d, \tag{29}$$

Using the projected Eintein's equation $R_{bd}\gamma^b_j n^d = 8\pi T_{bd} \gamma^b_j n^d$. We get the Momentum constraint.

$$D_i K^i_j - D_j K = 8\pi S_j \tag{30}$$

where we have defined $S_j = -T_{bd}\gamma^b_j$.

Evolution of the extrinsic curvature

From the contaction of the Gauss-Codazzi equations (21)

$${}^3R_{jl} = \gamma^b_j \gamma^d_l (R_{bd} + n^a n^c R_{abcd}) + K_{il} K^i_j - K K_{jl} \ , \tag{31}$$

the term that involves $n^a n^c R_{abcd}$ contains second time derivatives of the metric. Let us consider

$$R_{abcd} n^d = (\nabla_a \nabla_b - \nabla_b \nabla_a) n_c \tag{32}$$

and

$$\begin{aligned} \nabla_a \nabla_b n_c &= \nabla_a (-K_{bc} - n_b D_c \ln\alpha) && (33) \\ &= -\nabla_a K_{bc} + (K_{ab} + n_a D_b \ln \alpha) D_c \ln\alpha - n_b \nabla_a D_c \ln\alpha && (34) \end{aligned}$$

where we have used $\nabla_b n_c = -K_{bc} - n_b D_c \ln \alpha$. Thus the projection of $R_{abcd} n^d = (\nabla_a \nabla_b - \nabla_b \nabla_a) n_c$

$$\gamma^b_j \gamma^d_l n^a n^c R_{abcd} = \gamma^a_j \gamma^c_l n^b n^d R_{abcd} = \gamma^a_j \gamma^c_l n^b (-\nabla_a K_{bc} + \nabla_b K_{ac}) + (D_j \ln \alpha) D_l \ln \alpha + D_j D_l \ln \alpha \ . \tag{35}$$

The first term of the right hand side of the last expression can be rewritten as

$$-\gamma^a_j \gamma^c_l n^b \nabla_a K_{bc} = \gamma^a_j K_{bl} \nabla_a n^b = -\gamma^a_j K_{bl} (K^b_a + n_a D^b \ln \alpha) = -K_{kl} K^k_j \ , \tag{36}$$

and the second term

$$\gamma^a_j \gamma^c_l n^b \nabla_b K_{ac} = -\gamma^a_j \gamma^c_l (\mathcal{L}_n K_{bc} - K_{ab} \nabla_c n^b - K_{bc} \nabla_a n^b) = \mathcal{L}_n K_{jl} + 2 K_{jk} K^k_l \ , \tag{37}$$

hence we can write the contraction as

$$\gamma^a_j \gamma^c_l R_{abcd} n^b n^d = \mathcal{L}_n K_{jl} + K_{jk} K^k_l + \frac{1}{\alpha} D_j D_l \alpha \ . \tag{38}$$

Let us go back to the term $\gamma^b_i \gamma^d_j R_{bd}$. Using the Einstein's equations

$$\gamma^b_i \gamma^d_j R_{bd} = 8\pi \gamma^b_i \gamma^d_j (T_{bd} - \frac{1}{2} g_{bd} T) \ . \tag{39}$$

Let us consider the second term

$$
\begin{aligned}
g_{bd} g^{ef} T_{ef} \gamma^b_i \gamma^d_j &= (\gamma^{ef} - n^e n^f) \gamma_{di} \gamma^d_j T_{ef} &(40)\\
&= (\gamma^{ef} - n^e n^f) \gamma_{ij} T_{ef} &(41)\\
&= \gamma^{ef} \gamma_{ij} T_{ef} - \gamma_{ij} \rho &(42)\\
&= \gamma_{ij} (S - \rho) \ , &(43)
\end{aligned}
$$

with

$$
S = \gamma^{ij} S_{ij} = \gamma^{ij} \gamma^e_i \gamma^f_j T_{ef} = \gamma^{ef} T_{ef} \ , \quad S_{ij} = \gamma^c_i \gamma^d_j T_{cd} \ , \tag{44}
$$

From the Gauss Codazzi equation (21)

$$
{}^3R_{jl} = \gamma^b_j \gamma^d_l R_{bd} + \frac{1}{\alpha} D_j D_l \alpha + \mathcal{L}_n K_{jl} + 2 K_{kl} K^k_j - K K_{jl} \tag{45}
$$

and using

$$
\gamma^b_j \gamma^d_l R_{bd} = 8\pi [S_{jl} - \frac{1}{2} \gamma_{jl} (S - \rho)] \ , \tag{46}
$$

we have the evolution for K_{jl}

$$
\mathcal{L}_n K_{jl} = {}^3R_{jl} - \frac{1}{\alpha} D_j D_l \alpha + 2 K_{kl} K^k_j - K K_{jl} - 8\pi [S_{jl} - \frac{1}{2} \gamma_{jl} (S - \rho)] \ . \tag{47}
$$