Introduction to Numerical Relativity

1 Some numerical methods

The simplest example of an hyperbolic system is the advection equation.

$$\partial_t u + a\ \partial_x u = 0\,. \tag{1}$$

The Cauchy problem is defined by this equation in the domain $-\infty < x < \infty$, $t \geq 0$ together with the initial condition $u(0,x) = u_0(x)$. The solution is

$$u(t,x) = u_0(x - at)\ . \tag{2}$$

As time evolves, the initial data propagates unchanged to the right if $a > 0$ or left if $a < 0$ with velocity a. The solution is constant along each rays $x - at = x_0$, which are known as the characteristics of the problem. We will use the advection equation as our working example because in this case the finite differencing procedure is simpler and the resulting algorithms are easily extended to more complex equations

1.1 Von Newmann stability

The von Neumann stability analysis is a useful tool which allows a first simple validation of a given numerical scheme. The von Neumann analysis is considered as a local analysis because it does not take into account boundary effects and assumes that the coefficients of the finite difference equations are sufficiently slowly varying to be considered constant in time and space. Under this assuption, the solution can be seen as a sum of eigenmodes. Writting the eigenmode at each grid point u^n_j as

$$u^n_j = \xi^n e^{i\kappa x_j}\ , \quad \xi = \xi(\kappa) \in \mathbb{C}\ , \tag{3}$$

we can perform a spectral analysis of the finite difference equation. The number ξ in Eq. (3) is the amplification factor. We can see that $u^n_j = \xi u^{n-1}_j = \xi^2 u^{n-2}_j ... = \xi^n u^0_j$. The dependence of ξ on the wave number κ can be find by inserting (3) into the finite difference form of the differential equation. The lesson to be learnt here is that for the scheme to be stable, the magnitude of ξ must be smaller or equal to unity for all κ.

1.2 The upwind scheme

The upwind scheme is an *explicit scheme*, the solution at the new time level $n+1$ can be calculated explicitly from the quantities that are already known at the previous time level n.

Let us start making a Taylor expansion to derive a first order finite difference approximation to the space and time derivatives in the advection equation.

Depending on the direction in which the solution is translated, and hence on the value of the advection velocity, two different finite difference representations can be given of equation (1), these are

$$\frac{u_j^{n+1}-u_j^n}{k} = -a\left(\frac{u_j^n-u_{j-1}^n}{h}\right) + O(k,h) \ , \quad \text{if} \quad a>0 \ , \tag{4}$$

and

$$\frac{u_j^{n+1}-u_j^n}{k} = -a\left(\frac{u_{j+1}^n-u_j^n}{h}\right) + O(k,h) \ , \quad \text{if} \quad a<0 \ . \tag{5}$$

Solving for determining the solution at the next time level we arrive to

$$u_j^{n+1} = u_j^n - a\frac{k}{h}(u_j^n - u_{j-1}^n) + O(k^2,h) \ , \quad \text{if} \quad a>0 \ , \tag{6}$$

$$u_j^{n+1} = u_j^n - a\frac{k}{h}(u_{j+1}^n - u_j^n) + O(k^2,h) \ , \quad \text{if} \quad a<0 \ . \tag{7}$$

Applying the eigenmode decomposition (3) in (6), (7) the amplification factor can be written as

$$\xi = 1 - |a\frac{\Delta t}{\Delta x}|(1-\cos(\kappa h)) - ia\frac{k}{h}\sin(\kappa h) \ , \tag{8}$$

and its modulus

$$|\xi|^2 = 1 - 2|a\frac{k}{h}|(1-|a\frac{k}{h}|)(1-\cos(\kappa h)) \ , \tag{9}$$

which is less than one as long as the Courant-Friedrichs-Lewy condition (CFL condition)

$$a\frac{k}{h} = a\frac{\Delta t}{\Delta x} \leq 1 \ , \tag{10}$$

is satisfied. The quotient $a\frac{\Delta t}{\Delta x}$ is refered as the Courant number.

1.3 The Forward in Time Centered in Space scheme

The Forward in Time Centered in Space scheme consist in using a first order approximation for the time derivative and a second order aproximation for the spatial derivative. The FTCS is then expressed as

$$\frac{u_j^{n+1}-u_j^n}{k} = -a\left(\frac{u_{j+1}^n-u_{j-1}^n}{2h}\right) + O(k,h^2) \ , \tag{11}$$

or

$$u_j^{n+1} = u_j^n - a\frac{k}{2h}(u_{j+1}^n + u_{j-1}^n) + O(k^2,h^2) \ , \tag{12}$$

Unfortunately the FTCS scheme is unconditionally unstable, the numerical solution will be destroyed by numerical errors. Applying the mode decomposition to equation (12) and few algebraic steps lead to an amplification factor

$$\xi(\kappa) = 1 - ia\frac{k}{h}\sin(\kappa h) \ , \tag{13}$$

and its norm

$$|\xi|^2 = 1 + \left(a\frac{k}{h}\sin(\kappa h)\right)^2 \ , \tag{14}$$

then $|\xi|^2 > 1$ indepently of k and h that is FTCS is unconditionally unstable.

1.4 Lax-Friedrich scheme

This method arised as a proposal to cure the instability of the FTC scheme. The idea is based on replacing u_j^n in the FTCS formula (12) with its spatial average; $u_j^n = (u_{j+1}^n + u_{j-1}^n)/2$ then

$$u_j^{n+1} = \frac{(u_{j+1}^n + u_{j-1}^n)}{2} - a\frac{k}{2h}(u_{j+1}^n - u_{j-1}^n) + O(k^2, h^2) \ . \tag{15}$$

Almost surprisingly, the algorithm (15) is now conditionally stable as can be verified through a von Neumann stability analysis. Using the mode decomposition in (15) we obtain an amplification factor

$$|\xi|^2 = 1 - \sin^2(\kappa h)\left[1 - \left(a\frac{k}{h}\right)^2\right] \ , \tag{16}$$

then, as long as the CFL condition is satisfied ($a\frac{k}{h} < 1$) we have $|\xi| < 1$.

The correction introduced by the Lax-Friedrich scheme is equivalent to the introduction of a numerical dissipation. Adding and substracting u_j^n on the right hand side of equation (15) as

$$u_j^{n+1} = u_j^n + \frac{1}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) - \frac{a}{2}\frac{k}{h}(u_{j+1}^n - u_{j-1}^n) \ , \tag{17}$$

or

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\left(\frac{u_{j+1}^n - u_{j-1}^n}{2h}\right) + \frac{1}{2}\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{k}\right) \ , \tag{18}$$

but this is exactly the finite difference representation of the equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{1}{2}\varepsilon\frac{\partial^2 u}{\partial x^2} \ , \quad \varepsilon = \frac{h^2}{k} \ , \tag{19}$$

i.e. we have added a diffusion term.

One has to be careful while using this algorithm because, the scheme is stable but may suffers from a considerable dissipation.

1.5 Leap frog

This scheme consist in taking a second order finite difference in time and space of the form

$$\frac{u_j^{n+1} - u_j^{n-1}}{2k} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2h} + O(k^2, h^2) \ , \tag{20}$$

then the time advancing algorithm is

$$u_j^{n+1} = u_j^{n-1} - a\frac{k}{h}(u_{j+1}^n - u_{j-1}^n) + O(k^3, h^2) \ , \tag{21}$$

The Leapfrog scheme is von Neumann stable. After substituting the mode decomposition into (21) we find that amplification factor is

$$\xi = -i\alpha\sin(\kappa h) \pm \sqrt{1 - (\alpha\sin(\kappa h))^2} \quad \alpha = a\frac{k}{h} \ , \tag{22}$$

consequently

$$|\xi|^2 = \alpha^2\sin^2(\kappa h) + [1 - \alpha^2\sin^2(\kappa h)] = 1 \tag{23}$$

1.6 Lax-Wendroff

The Lax Wendroff method for the advection equation can be obtained from a Taylor expansion of the form

$$u(t+k,x) = u(t,x) + \frac{\partial u(t,x)}{\partial t}k + \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial t^2}k^2 + \dots \;, \tag{24}$$

and then using the equation of motion to substitute the time derivatives

$$u(t+k,x) = u(t,x) - a\frac{\partial u(t,x)}{\partial x}k + a^2\frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2}(k)^2 + \dots \;, \tag{25}$$

Approximating now the spatial derivatives using centered differences we get

$$u_j^{n+1} = u_j^n - a\frac{k}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2}{2}\left(\frac{k}{h}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{26}$$

1.7 Iterative Crank Nicholson

The idea behind the iterative Crank Nicholson scheme (ICN) is that of transforming a stable implicit method like the Crank-Nicholson scheme into an explicit one through a series of iterations.

The first iteration of ICN scheme starts by calculating an intermediate variable ${}^1\tilde{u}$ using

$$\frac{{}^{(1)}\tilde{u}_j^{n+1} - u_j^n}{k} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2h} \tag{27}$$

then another intermediate variable ${}^1\overline{u}$

$${}^1\overline{u}_j^{n+1/2} = \frac{1}{2}({}^{(1)}\tilde{u}_j^{n+1} + u_j^n) \tag{28}$$

The time step is completed using $\overline{u}$

$$\frac{u_j^{n+1} - u_j^n}{k} = -a\left(\frac{{}^{(1)}\overline{u}_{j+1}^{n+1/2} - {}^{(1)}\overline{u}_{j-1}^{n+1/2}}{2h}\right) \;, \tag{29}$$

Iterated Crank Nicholson with two iterations is carried following after calculating ${}^1\overline{u}_j^{n+1/2}$ in (28)

$$\frac{{}^2\tilde{u}_j^{n+1} - u_j^n}{k} = -a\left(\frac{{}^{(1)}\overline{u}_{j+1}^{n+1/2} - {}^{(1)}\overline{u}_{j-1}^{n+1/2}}{2h}\right) \tag{30}$$

$${}^2\overline{u}_j^{n+1/2} = \frac{1}{2}({}^{(2)}\tilde{u}_j^{n+1} + u_j^n) \;, \tag{31}$$

and the final step

$$\frac{u_j^{n+1} - u_j^n}{k} = -a\left(\frac{{}^{(2)}\overline{u}_{j+1}^{n+1/2} - {}^{(2)}\overline{u}_{j-1}^{n+1/2}}{2h}\right) \;, \tag{32}$$

While the magnitude of the amplification factor for iterated Crank-Nicholson does approach 1 as the number of iterations becomes infinite, the convergence is not monotonic. The magnitude oscillates above and below 1 with ever decreasing oscillations. In particular iterations 3 and 4 are stable.

It is important to notice that the truncation error is not modified by the number of iterations and is always $O(k^2, h^2)$ then, a number of iterations larger than two is not very useful and surely increase the amount of computational work.

1.8 Numerical dissipation

Let us consider the advection equation, but now discretize only in space using a centered difference approximation

$$\frac{\partial u}{\partial t} = -a\frac{u_{j+1} - u_{j-1}}{2h} \ , \tag{33}$$

and look for a normal mode solution of the form $u = e^{i\kappa(x-v't)}$ around x [1] where the discrete phase speed v' is to be determined. Subtituting this mode solution into the advection equation gives

$$i\kappa v'u = a\frac{2i\sin(\kappa h)}{2h}u \ , \tag{34}$$

and solving for v' gives

$$v' = a\frac{\sin\nu}{\nu} \tag{35}$$

where $\nu := \kappa h$ is a dimensionless wave number. In the low frequency limit, $\nu \to 0$ we have

$$v' = a\frac{\sin\nu}{\nu} \to a \tag{36}$$

so that low frequency components propagate with the phase speed v. However, in the high frequency limit, $\nu \to \pi$

$$v' = a\frac{\sin\nu}{\nu} \to 0 \ , \tag{37}$$

that is the high frequency components do not propagate at all. This is a typical behaviour of FDAs of hyperbolic type equations particularly for low-order schemes.

Some schemes are naturally dissipative as the Lax Wendroff scheme, while others, like leap frog are not. In order to ameliorate the troubles might arise because of the non propagating modes, one has to add dissipative terms in such a way as to mantain the original accuracy of the scheme. Let consider the Leap frog scheme applied to the advection equation.

$$u_j^{n+1} = u_j^{n-1} - a\frac{k}{h}(u_{j+1}^n - u_{j-1}^n) + O(k^3, h^2) \ , \tag{38}$$

We add dissipation to the scheme by modifying it as follows

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n) - \frac{\varepsilon}{16}(u_{j+2}^{n-1} - 4u_{j+1}^{n-1} + 6u_{j-1}^{n-1} + u_{j-2}^{n-1}) \tag{39}$$

where ε is a non-negative parameter.

What we have done is to add

$$(u_{j+2}^{n-1} - 4u_{j+1}^{n-1} + 6u_{j-1}^{n-1} + u_{j-2}^{n-1}) = h^4\left(\frac{\partial^4 u}{\partial x^4}\right)_j^{n-1} + O(h^6) \ , \sim O(h^4) \tag{40}$$

so that the term wich we added does not change the leading order truncation error.

[1] with $x_{j+1} = x + h$ and $x_{j-i} = x - h$.