Introduction to Numerical Relativity

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1 Introduction to GR

In General Relativity, space-time is a 4-dimensional manifold of events. The coordinates x^α can be any smooth labeling of events in space-time, and we are free to make arbitrary transformations between coordinate systems,

$$x^\alpha \to x'^\alpha = x'^\alpha(x^\alpha) \ . \tag{1}$$

A vector $\vec{V}$ at any point in the manifold can be expressed in terms of its components in some basis:

$$\vec{V} = V^\alpha \vec{e}_\alpha \ . \tag{2}$$

Let us restrict ourselves to coordinate basis vectors for simplicity. These are tangent to the coordinate lines, so we can write them as the differential operators

$$\vec{e}_\alpha = \frac{\partial}{\partial x^\alpha} = \partial_\alpha \tag{3}$$

A linear, real valued function of one vector defines a 1 form. 1 forms act over vectors producing real numbers. The space of 1 forms define a vector space. The components of a 1-form are defined as the value of the form acting on the basis vector

$$q_\alpha = q(\vec{e}_\alpha) \ , \tag{4}$$

We can define a basis for the space of 1-forms $\tilde{\omega}^\alpha$ as those 1-forms such that, when acting on $\vec{e}_\alpha$ give us the identity matrix.

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \ . \tag{5}$$

In terms of the basis vectors and 1-forms, the action of an arbitrary 1-form $\tilde{q}$ on a vector $\vec{V}$ can be represented as

$$\begin{aligned}
\tilde{q}(\vec{V}) &= q_\alpha \tilde{\omega}^\alpha (V^\beta \vec{e}_\beta) \\
&= q_\alpha V^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) \\
&= q_\alpha V^\beta \delta^\alpha_\beta \\
&= q_\alpha V^\alpha \ .
\end{aligned}$$

This operation is called a contraction. In the literature, vectors and 1-forms are called contravariant vectors and covariant vectors

We can generalize this idea and think of real-valued functions of m 1-forms and n vectors that are linear in all their arguments. This defines a tensor of rank $\begin{pmatrix} m \\ n \end{pmatrix}$. The components of a tensor are the values of the tensor applied to the elements of the basis of vectors and 1-forms for example:

$$T^{\alpha\beta}{}_{\gamma\delta} \equiv T(\tilde{\omega}^\alpha, \tilde{\omega}^\beta, \vec{e}_\gamma, \vec{e}_\delta) \ . \tag{6}$$

In the given manifold M the notion of distance is given by a symmetric, non-degenerate $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor g. With this tensor we can calculate the magnitude of the displacement vector $\vec{dx}$ between two infinitesimally close points in the manifold

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \ . \tag{7}$$

Where the components of the metric tensor are

$$g(\vec{e}_\alpha, \vec{e}_\beta) = g_{\alpha\beta} \equiv \vec{e}_\alpha \cdot \vec{e}_\beta \ . \tag{8}$$

For two generic vectors, their scalar product is defined as

$$\begin{aligned} g(\vec{V}, \vec{U}) &= g(V^\alpha \vec{e}_\alpha, U^\beta \vec{e}_\beta) \\ &= V^\alpha U^\beta g_{\alpha\beta} \\ &\equiv \vec{V} \cdot \vec{U} \ . \end{aligned}$$

The metric tensor can be used to define a one to one mapping between vectors and 1-forms $g(\vec{V}, :)$ defines a 1-form. Since this form is associated to the vector $\vec{V}$ let us call it $\tilde{V}$ and its components are

$$\begin{aligned} V_\alpha &= g(\vec{V}, \vec{e}_\alpha) \\ &= g(V^\beta \vec{e}_\beta, \vec{e}_\alpha) \\ &= g_{\beta\alpha} V^\beta \ . \end{aligned}$$

If we multiply by the inverse of the metric defined such a way that $g^{\alpha\beta} = (g^{-1})^{\alpha\beta}$ we get

$$\begin{aligned} g^{\alpha\gamma} V_\alpha &= g^{\alpha\gamma} g_{\beta\alpha} V^\beta \\ &= \delta^\gamma_\beta V^\beta \\ &= V^\gamma \end{aligned}$$

These operations are known as lowering and rising the index of the vector.

The covariant derivative is represented by the operator ∇_α which denotes the α component of the derivative. The covariant derivative of a scalar is simply the usual partial derivative: If $f(x^\beta)$ is a scalar function over the manifold, then its covariant derivative is

$$\nabla_\alpha f = \frac{\partial}{\partial x^\alpha} f = \partial_\alpha f \ . \tag{9}$$

The covariant derivative of a vector field with components V^β is defined by

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + V^\gamma \Gamma^\beta_{\gamma\alpha} \ . \tag{10}$$

The Christoffel coefficients $\Gamma^\beta_{\gamma\alpha}$, can be calculated as

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\sigma} (\partial_\beta g_{\sigma\alpha} + \partial_\alpha g_{\beta\sigma} - \partial_\sigma g_{\alpha\beta}) \ . \tag{11}$$

The covariant derivative for a 1-form is

$$\nabla_\alpha V_\beta = \partial_\alpha V_\beta - V_\gamma \Gamma^\gamma_{\beta\alpha} \ . \tag{12}$$

The covariant derivative of the metric vanishes once we assume the Christoffel coefficients are symmetric in the two lower indices.

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - g_{\sigma\nu} \Gamma^\sigma_{\mu\alpha} - g_{\mu\sigma} \Gamma^\sigma_{\nu\alpha} = 0 \ . \tag{13}$$

Riemann tensor

Covariant derivatives do not commute in general. The no commutativity defines the Riemann tensor

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^\mu_{\nu\alpha\beta} V^\nu \ . \tag{14}$$

The components of the Riemann tensor can be written in terms of the connection and its derivatives as

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha} \ . \tag{15}$$

Several symmetries reduce the number of independent components of the Riemann tensor, in four dimensions for instance, instead of 4^4 it has 20.

There are contractions of the Riemann tensor that are very important in GR. For example the Ricci tensor

$$R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} \ , \tag{16}$$

and its contraction, the Ricci scalar,

$$R = R^\mu{}_\mu \tag{17}$$

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ , \tag{18}$$

One additional property of the Riemann tensor is that

$$R^\alpha{}_{\beta\mu\nu;\lambda} + R^\alpha{}_{\beta\lambda\mu;\nu} + R^\alpha{}_{\beta\nu\lambda;\mu} = 0 \ , \tag{19}$$

Where the ; denotes covariant derivative.

Einstein's equations

Newtonian gravitation can be described as a field theory for a scalar field ϕ satisfying Poisson's law: $\nabla^2\phi = 4\pi G\rho$ where ρ is the mass density and G is the Newton's gravitational constant. The gravitational acceleration of any object in the field is given by $-\nabla\phi$. Newtonian gravity is governed by an elliptic equation, changes in the distribution of matter instantaneously change the gravitational potential everywhere. Propagation of effects at speed greater than the speed of light leads to causality violation and as a consequence Newtonian gravity is not consistent with Special Relativity.

Einstein's equations are

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \ , \tag{20}$$

$T_{\mu\nu}$ is the stress energy tensor of matter and fields in the space-time. Einstein's equations say that matter and energy dictate how space-time is curved. To solve the equations we must find a metric that satisfies them at all space locations for all time. How to determine a good choice of coordinates is one of the major questions in Numerical Relativity.

Schwarzschild solution

To solve Einstein's equations for a static point source the metric should satisfy

- All the components of the metric are independent of t.
- The equations $g_{0,\alpha} = g_{\alpha,0} = 0$ must be satisfy.
- The solution is spatially symmetric with respect to the origin of coordinates.
- The metric, at infinity must be $g_{00} = -1$, $g_{11} = g_{22} = g_{33} = 1$.

Then we propose the following anzats

$$ds^2 = -Fdt^2 + G(dx^2 + dy^2 + dz^2) + H(xdx + ydy + zdz)^2 \ , \tag{21}$$

and using spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \ . \end{aligned}$$

then

$$\begin{aligned} ds^2 &= -Fdt^2 + G(dr^2 + r^2 d\theta^2 + r^2\sin^2 d\phi^2) + H(rdr)^2 \\ &= -Fdt^2 + (G + Hr^2)dr^2 + Gr^2(d\theta^2 + \sin^2 d\phi^2) \ , \end{aligned}$$

If we require that $g = det(g_{\mu\nu}) = -1$, it turns out that Einstein's Equations are

$$\partial_\alpha \Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} = 0 \ . \tag{22}$$

However, even the Minkowski space in spherical coordinates the determinant is not -1 but $r^2\sin\theta$. With the spatial volume element $r^2 dr \sin\theta d\theta d\phi$. If we define a set of new coordinates x and ψ as $x = \frac{r^3}{3}$, $\psi = -\cos\theta$

$$r^2 dr \sin\theta d\phi = dx d\psi d\phi \ , \tag{23}$$

In these coordinates, the element of line reads

$$ds^2 = -Fdt^2 + \left(\frac{G}{r^4} + \frac{H}{r^2}\right)dx^2 + Gr^2\left(\frac{d\psi^2}{1-\psi^2} + (1-\psi^2)d\phi^2\right) \ , \tag{24}$$

or

$$ds^2 = -f_0 dt^2 + f_1 dx^2 + f_2\left(\frac{d\psi^2}{1-\psi^2} + (1-\psi^2)d\phi^2\right) \ , \tag{25}$$

The Minkowski metric in this coordinates is simply

$$ds^2_{MK} = -dt^2 + \frac{1}{r^4}dx^2 + r^2\left(\frac{d\psi^2}{1-\psi^2} + (1-\psi^2)d\phi^2\right) \ , \tag{26}$$

Then, at $x \to \infty$

$$f_0 \to 1 \ , \quad f_1 \to \frac{1}{r^4} \ , \quad f_2 \to r^2 \ , \tag{27}$$

and $f_0 f_1 f_2^2 = 1$. After an integration of the Einstein equations we get

$$f_0 = 1 - \frac{2M}{(3x+b)^{1/3}} \quad f_1 = \frac{(3x+b)^{-4/3}}{1 - \frac{2M}{(3x+b)^{1/3}}}, \quad f_2 = (3x+b)^{2/3} \ . \tag{28}$$

If we define the variable

$$r_s = (3x+b)^{1/3} = (r^3+b)^{1/3} \ , \tag{29}$$

we obtain, replacing again $\psi = -\cos\theta$

$$ds^2 = -\left(1 - \frac{2M}{r_s}\right)dt^2 + \frac{1}{1-\frac{2M}{r_s}}dr_s^2 + r_s^2\left(d\theta^2 + \sin^2 d\phi^2\right) \ . \tag{30}$$

If we decide that the singular point is the location of the point particle $r = 0$, from f_1 we get

$$\frac{2M}{b^{1/3}} = 1 \Rightarrow b = (2M)^3 \ , \tag{31}$$

but from the definition of r_s this point corresponds to $r_s = 2M$. Nowadays we know that is just a coordinate singularity because there are other coordinate systems in which the metric is regular at that point. The unusual radial coordinate x was forced by the constraint $g = -1$, nevertheless it led to a quite simple derivation of the exact solution.

The high degree of non-linearity in the field equations means that a general solution for an arbitrary matter distribution is analytically intractable. The problem becomes easier if we look for special solutions, for instance those representing space-times possessing symmetries. As we shall see, the Schwarzschild solution represents the space-time geometry outside a spherically symmetric matter distribution.

I will introduce another set of coordinates that is frequently used in numerical relativity. It turns out that it is possible to rewrite the metric in such a way that the spatial part is conformally flat, that is, the spatial metric is just the Minkowski metric times a scalar function. In order to do this one must define a new radial coordinate r_c such that

$$r_s = r_c \left(1 + \frac{M}{2r_c}\right)^2 \ , \tag{32}$$

A transformation from Schwarzschild coordinates $\{t, r_s\}$ to the coordinates $\{t, r_c\}$

$$ds^2 = -\left(\frac{1 - M/2r_c}{1 + M/2r_c}\right)^2 dt^2 + \psi^4 (dr_c^2 + r_c^2 d\Omega^2) \tag{33}$$

with the conformal factor $\psi = 1 + \frac{M}{2r_c}$. In these coordinates the spatial metric is regular at the horizon, which now corresponds to $r_c = M/2$. Notice also that far away r_c and r_s approach each other. The coordinate r_c is usually called the isotropic radius since the spatial metric is just the flat metric(which is isotropic), times a conformal factor. The metric is singular at $r_c = 0$, however the transformation of coordinates shows that $r_c = 0$ corresponds to $r_s = \infty$, so this is in fact not the physical singularity at $r_s = 0$. It turns out that the region $r_s \in [0, M/2]$ represents the other side of the *Einstein-Rosen bridge*, or in other words the whole other universe has been compactified to this finite region.

The isotropic r_c does not reach the singularity at $r_s = 0$. For large r_c we see that $r_s \to \infty$, but for small r_c we see that once again $r_s \to \infty$. There is minimum of $r_s = 2M$ at $r_c = M/2$. We now have two copies of the space outside the event horizon, $r_s > 2M$, and the two spaces are connected by a wormhole with a throat at $r_s = 2M$. This wormhole picture of a black hole forms the basis of the initial data used in current black-hole simulations. The point $r_c = 0$, which represents the second asymptotically flat end, is refereed to as the puncture.

The singularity at $r_c = 0$ is then just a coordinate singularity associated with a compactification. Notice also that this metric has an isometry (i.e. it is invariant) with respect to the transformation $r_c \to M^2/4r_c$. This isometry corresponds to changing a point in our universe with the corresponding point on the other side of the throat.

Final remark

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