Centro de Matemática
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A five dimensional Newman-Penrose and Geroch-Held-Penrose formalism

Alfonso García-Parrado Gómez-Lobo

Spinors in a 5-dimensional Lorentzian vector space

The curvature spinors

Newman-Penrose and G.H.P. formalisms in 5 dimensions

Five dimensional type D space-times

The class $\mathcal{A}$

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Outline

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3. Newman-Penrose and G.H.P. formalisms in 5 dimensions
4. Five dimensional type D space-times
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The spacetime tensor algebra

Let $\mathbf{L}$ be a 5-dimensional real vector space endowed with a real scalar product $g(\ ,\ )$ of Lorentzian signature. Use the vector space $\mathbf{L}$ and its dual $\mathbf{L}^*$ to build a tensor algebra $\mathfrak{T}(\mathbf{L})$

The spinorial tensor algebra

Let $\mathbf{S}$ be a complex vector space whose dimension is for the moment left unspecified. Use the vector space $\mathbf{S}$ and its dual $\mathbf{S}^*$ to build a tensor algebra $\mathfrak{T}(\mathbf{S})$

Notation

We use abstract indices to denote tensorial quantities: small Latin letters $a, b, \dots$ are indices for elements in $\mathfrak{T}(\mathbf{L})$ and uppercase Latin letters $A, B, \dots$ are indices for elements in $\mathfrak{T}(\mathbf{S})$.

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We introduce now a mixed quantity ${\gamma_{aB}}^C$ which fulfils the following algebraic property

$$ {\gamma_{aA}}^B {\gamma_{bB}}^C + {\gamma_{bA}}^B {\gamma_{aB}}^C = -{\delta_A}^C g_{ab}, $$

This relation means that ${\gamma_{aB}}^C$ can be regarded as belonging to a representation on the vector space $\mathbf{S}$ of the *Clifford algebra* $Cl(\mathbf{L}, g)$.

Theorem

If the quantity $\gamma_{aB}{}^{C}$ *belongs to an irreducible representation of* $Cl(\mathbf{L}, g)$, *then:*

- *the dimension of* $\mathbf{S}$ *is 4.*
- *There exist two antisymmetric spinors* ϵ_{AB}, $\widehat{\epsilon}^{AB}$, *unique up to a constant, such that*

$$\epsilon_{AB}\widehat{\epsilon}^{CB} = \delta_{A}{}^{C}\,,$$
$$\gamma_{aD}{}^{A}\gamma^{a}{}_{C}{}^{B} = \frac{1}{2}\delta_{D}{}^{A}\delta_{C}{}^{B} - \delta_{C}{}^{A}\delta_{D}{}^{B} + \epsilon_{CD}\widehat{\epsilon}^{AB}.$$

- The quantities $\widehat{\epsilon}^{AB}$, ϵ_{AB} can be used to raise and lower spinor indices. Example:

$$\xi^A\epsilon_{AB} = \xi_B\ , \quad \xi^A = \widehat{\epsilon}^{AB}\xi_B.$$

From now on we set $\widehat{\epsilon}^{AB} = \epsilon^{AB}$.

- Previous theorem entails the relations

$$\gamma_a{}^{AB}\gamma_{bAB} = -2g_{ab}\ , \gamma^a{}_{[AB]} = \gamma^a{}_{AB}\ , \quad \epsilon^{AB}\gamma^a{}_{AB} = 0\ ,$$

$$\gamma_{aCD}\gamma^a{}_{AB} = \epsilon_{AD}\epsilon_{BC} - \epsilon_{AC}\epsilon_{BD} + \frac{1}{2}\epsilon_{AB}\epsilon_{CD}.$$

- Use these properties to transform tensors into spinors and back. Example:

$$v^{AB} = \gamma_a{}^{AB}v^a\ , \quad v^a = -\frac{1}{2}\gamma^a{}_{AB}v^{AB}.$$

The spin tetrad and the semi-null pentad

Let $\{o^A, \iota^A, \tilde{o}^A, \tilde{\iota}^A\}$ be a basis in in $\mathbf{S}$ such that

$$\epsilon_{AB} = 2o_{[A}\iota_{B]} + 2\tilde{o}_{[A}\tilde{\iota}_{B]}.$$

This entails

$$o^A\iota_A = -1 = \tilde{o}^A\tilde{\iota}_A \ , \quad o^A\tilde{o}_A = \iota^A\tilde{\iota}_A = o^A\tilde{\iota}_A = \iota^A\tilde{o}_A = 0$$

The basis $\{o^A, \iota^A, \tilde{o}^A, \tilde{\iota}^A\}$ is a spin tetrad. From it we construct a semi-null pentad $\{l^a, n^a, m^a, \bar{m}^a, u^a\}$ as follows

$$l^a \equiv \gamma^a{}_{AB} o^A \tilde{o}^B \ , \quad n^a \equiv \gamma^a{}_{AB}\iota^A\tilde{\iota}^B \ ,$$
$$m^a \equiv -o^A\gamma^a{}_{AB}\tilde{\iota}^B \ , \bar{m}^a \equiv \tilde{o}^A\gamma^a{}_{AB}\iota^B,$$
$$u^a \equiv 2o^B\gamma^a{}_{AB}\iota^A = -2\tilde{o}^B\gamma^a{}_{AB}\tilde{\iota}^A.$$

Hence

$$l^a n_a = 1 \ , \quad m^a\bar{m}_a = -1 \ , \quad u^a u_a = -2.$$

The curvature spinors

Theorem

The Riemann tensor R_{abcf} of the covariant derivative ∇_a can be decomposed in the form

$$R_{abcf} = \Lambda(g_{af}g_{bc} - g_{ac}g_{bf}) - \\ -\frac{1}{2}G_{ab}{}^{AB}G_{cf}{}^{CD}\Psi_{ABCD} - G_{ab}{}^{AB}G_{cf}{}^{CD}\Omega_{ACBD},$$

where

$$G^{ab}{}_{AC} \equiv -\gamma^{a}{}_{(A}{}^{B}\,\gamma^{b}{}_{C)B}.$$

The quantities Λ, Ω_{ABCD} and Ψ_{ABCD} are known as the *curvature spinors.*

The curvature spinors fulfill the identities

$$\Psi_{(ABCD)} = \Psi_{ABCD}\ , \quad \Omega_{ABCD} = \Omega_{[AB]CD} = \Omega_{CDAB}\ ,$$
$$\Omega_{AB}{}^{C}{}_{C} = \Omega_{A}{}^{C}{}_{CD} = 0\ , \quad \Omega_{ABCD} + \Omega_{BCAD} + \Omega_{CABD} = 0.$$

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Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- The frame derivations:

$$D \equiv l^a\nabla_a\ , \quad \Delta \equiv n^a\nabla_a\ , \quad \delta \equiv m^a\nabla_a\ , \quad \bar\delta \equiv \bar m^a\nabla_a\ , \quad \mathcal{D} \equiv u^a\nabla_a .$$

- The spin coefficients:

$$D o_A = \epsilon o_A + \mathfrak{d}\tilde\iota_A - \kappa\iota_A + \chi\tilde o_A\ , \quad \Delta o_A = \gamma o_A + \mathfrak{e}\tilde\iota_A - \tau\iota_A + \omega\tilde o_A\ ,$$
$$\delta o_A = \beta o_A - \sigma\iota_A + \varsigma\tilde\iota_A + \phi\tilde o_A\ , \quad \bar\delta o_A = \alpha o_A - \rho\iota_A + \bar\varsigma\tilde\iota_A + \upsilon\tilde o_A\ ,$$
$$\mathcal{D} o_A = -\theta o_A + \eta\iota_A + \mathfrak{f}\,\tilde\iota_A + \psi\tilde o_A\ , \quad D\iota_A = \mathfrak{a}\tilde o_A - \epsilon\iota_A + \pi o_A - \bar\chi\tilde\iota_A\ ,$$
$$\Delta\iota_A = \mathfrak{b}\tilde o_A - \gamma\iota_A + \nu o_A - \bar\omega\tilde\iota_A\ , \quad \delta\iota_A = -\beta\iota_A + \mu o_A + \xi\tilde o_A - \bar\upsilon\tilde\iota_A\ ,$$
$$\bar\delta\iota_A = -\alpha\iota_A + \lambda o_A + \bar\xi\tilde o_A - \bar\phi\tilde\iota_A\ , \quad \mathcal{D}\iota_A = \mathfrak{c}\tilde o_A + \zeta o_A + \theta\iota_A - \bar\psi\tilde\iota_A .$$

Twelve Newman-Penrose 4-D spin coefficients

$\alpha,\ \beta,\ \gamma,\ \epsilon,\ \kappa,\ \lambda,\ \mu,\ \nu,\ \pi,\ \rho,\ \sigma,\ \tau.$

Ten complex 5-D spin coefficients

$\zeta,\ \eta,\ \theta,\ \chi,\ \omega,\ \phi,\ \xi,\ \upsilon,\ \psi,\ \varsigma.$

Six real 5-D spin coefficients

$\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{d},\ \mathfrak{e},\ \mathfrak{f}.$

$2\times 12 + 2\times 10 + 6 = 50$ real Ricci rotation coefficients.

Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- Components of the Weyl spinor:

$$\Psi_0 \equiv \Psi_{ABCD} o^A o^B o^C o^D \ , \ {}^*\Psi_0 \equiv \Psi_{ABCD} o^A o^B o^C \tilde{o}^D \ ,$$

$$\Psi_1 \equiv \Psi_{ABCD} o^A o^B o^C \iota^D \ , \ {}^*\Psi_1 \equiv \Psi_{ABCD} o^A o^B \iota^C \tilde{o}^D \ , \Psi^*_1 \equiv \Psi_{ABCD} o^A o^B o^C \tilde{\iota}^D \ ,$$

$$\Psi_2 \equiv \Psi_{ABCD} o^A o^B \iota^C \iota^D \ , \ {}^*\Psi_2 \equiv \Psi_{ABCD} o^A \iota^B \iota^C \tilde{o}^D \ , \Psi^*_2 \equiv \Psi_{ABCD} o^A o^B \iota^C \tilde{\iota}^D \ ,$$

$$\Psi_3 \equiv \Psi_{ABCD} o^A \iota^B \iota^C \iota^D \ , \ {}^*\Psi_3 \equiv \Psi_{ABCD} \tilde{o}^A \iota^B \iota^C \iota^D \ , \Psi^*_3 \equiv \Psi_{ABCD} o^A \iota^B \iota^C \tilde{\iota}^D \ ,$$

$$\Psi_4 \equiv \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D \ , \Psi^*_4 \equiv \Psi_{ABCD} \iota^A \iota^B \iota^C \tilde{\iota}^D ,$$

$$\Psi_{01} \equiv \Psi_{ABCD} o^A o^B \tilde{o}^C \tilde{\iota}^D \ , \Psi_{02} \equiv \Psi_{ABCD} o^A o^B \tilde{\iota}^C \tilde{\iota}^D \ , \Psi_{12} \equiv \Psi_{ABCD} o^A \iota^B \tilde{\iota}^C \tilde{\iota}^D .$$

$$\Psi_{00} \equiv \Psi_{ABCD} o^A o^B \tilde{o}^C \tilde{o}^D \ , \Psi_{11} \equiv \Psi_{ABCD} o^A \iota^B \tilde{o}^C \tilde{\iota}^D \ , \Psi_{22} \equiv \Psi_{ABCD} \iota^A \iota^B \tilde{\iota}^C \tilde{\iota}^D \ ,$$

Five 4-D Newman-Penrose components

$\Psi_0, \ \Psi_1, \ \Psi_2, \ \Psi_3, \ \Psi_4.$

Eleven 5-D complex components

${}^*\Psi_0 \ , \ {}^*\Psi_1 \ , \Psi^*_1 \ , \ {}^*\Psi_2 \ , \Psi^*_2 \ , \ {}^*\Psi_3 \ , \Psi^*_3 \ , \Psi^*_4 \ , \Psi_{01} \ , \Psi_{02} \ , \Psi_{12}.$

Three 5-D real components

$\Psi_{00}, \ \Psi_{11}, \ \Psi_{22}.$

$2\times$(16 complex components)+3 real components=35.

Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- Components of the Ricci spinor:

$$\Phi_{00} \equiv \Omega_{ABCD} o^A \tilde{o}^B o^C \tilde{o}^D \,, \Phi_{11} \equiv \Omega_{ABCD} o^A \tilde{o}^B \iota^C \tilde{\iota}^D \,, \Phi_{22} \equiv \Omega_{ABCD} \iota^A \tilde{\iota}^B \iota^C \tilde{\iota}^D \,,$$
$$\Omega \equiv \Omega_{ABCD} \tilde{o}^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D \,, \ {}^*\Phi_{01} \equiv \Omega_{ABCD} o^A \tilde{o}^B \tilde{o}^C \tilde{\iota}^D \,, \ {}^*\Phi_{12} \equiv \Omega_{ABCD} \iota^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D \,,$$
$$\Phi_{01} \equiv \Omega_{ABCD} o^A \tilde{o}^B o^C \tilde{\iota}^D \,, \Phi_{02} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B o^C \tilde{\iota}^D \,, \Phi_{12} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B \iota^C \tilde{\iota}^D \,,$$
$${}^*\Phi_{02} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D .$$

4-D Newman-Penrose components

Real: Φ_{00}, Φ_{11}, Φ_{22}, Complex: Φ_{01}, Φ_{02}, Φ_{12}.

5-D components

Real: Ω, ${}^*\Phi_{01}$, ${}^*\Phi_{12}$, Complex: ${}^*\Phi_{02}$.

2$\times$(4 complex components)+6 real components=14

Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- Commutation relations:

$$
\begin{aligned}
&D\Delta - \Delta D = -(\gamma + \bar{\gamma})D - (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\tau})\delta - (\bar{\pi} + \tau)\bar{\delta} - (\mathfrak{a} + \mathfrak{e})\mathcal{D}\,,\\
&D\delta - \delta D = -(\bar{\alpha} + \beta + \bar{\pi})D + \kappa\Delta + (\epsilon - \bar{\epsilon} - \bar{\rho})\delta - \sigma\bar{\delta} - (\varsigma - \chi)\mathcal{D}\,,\\
&D\mathcal{D} - \mathcal{D}D = (-2\mathfrak{a} + \theta + \bar{\theta})D + 2\mathfrak{d}\Delta + (\bar{\eta} - 2\bar{\chi})\delta + (\eta - 2\chi)\bar{\delta} - \mathfrak{f}\mathcal{D}\,,\\
&\Delta\delta - \delta\Delta = -\bar{\nu}D + (\bar{\alpha} + \beta + \tau)\Delta + (\gamma - \bar{\gamma} + \mu)\delta + \bar{\lambda}\bar{\delta} + (\xi + \omega)\mathcal{D}\,,\\
&\Delta\mathcal{D} - \mathcal{D}\Delta = -2\mathfrak{b}D + (2\mathfrak{e} - \theta - \bar{\theta})\Delta + (\zeta - 2\bar{\omega})\delta + (\bar{\zeta} - 2\omega)\bar{\delta} + \mathfrak{c}\mathcal{D}\,,\\
&\delta\bar{\delta} - \bar{\delta}\delta = (-\mu + \bar{\mu})D + (-\rho + \bar{\rho})\Delta + (-\alpha + \bar{\beta})\delta + (\bar{\alpha} - \beta)\bar{\delta} - (\upsilon - \bar{\upsilon})\mathcal{D}\,,\\
&\delta\mathcal{D} - \mathcal{D}\delta = (\bar{\zeta} - 2\xi)D + (\eta + 2\varsigma)\Delta + (\theta - \bar{\theta} - 2\bar{\upsilon})\delta - 2\phi\bar{\delta} - \psi\mathcal{D}\,.
\end{aligned}
$$

- Ricci (Newman-Penrose equations) and Bianchi identities.

Remark

The spin coefficients and curvature scalars were defined in terms of a spin tetrad, but one does not need spinors to introduce them and indeed they can be defined with respect to any semi-null Pentad.

Extension of the G.H.P. formalism to 5 dimensions

Let $L \in \mathbb{C}$ and consider

$$o^A \to L o^A \;,\; \iota^A \to \frac{\iota^A}{L} \;,\; \tilde{o}^A \to \bar{L}\tilde{o}^A \;,\; \tilde{\iota}^A \to \frac{\tilde{\iota}^A}{\bar{L}} .$$

Some quantities behave as weighted quantities under this transformation.

$$Z \mapsto L^p \bar{L}^q Z \Rightarrow Z \to Z^{\{p,q\}} .$$

Weighted spin coefficients

$$\mathfrak{a} \to \{0,0\} \;,\; \mathfrak{e} \to \{0,0\} \;,\; \mathfrak{d} \to \{2,2\} \;,\; \mathfrak{f} \to \{1,1\} \;,\; \mathfrak{c} \to \{-1,-1\} \;,$$
$$\mathfrak{b} \to \{-2,-2\} \;,\; \zeta \to \{-2,0\} \;,\; \eta \to \{2,0\} \;,\; \kappa \to \{3,1\} \;,\; \lambda \to \{-3,1\} \;,$$
$$\mu \to \{-1,-1\} \;,\; \nu \to \{-3,-1\} \;,\; \xi \to \{0,-2\} \;,\; \rho \to \{1,1\} \;,$$
$$\varsigma \to \{2,0\} \;,\; \tau \to \{1,-1\} \;,\; \upsilon \to \{0,0\} \;,\; \phi \to \{2,-2\} \;,$$
$$\chi \to \{2,0\} \;,\; \psi \to \{1,-1\} \;,\; \omega \to \{0,-2\} \;,\; \pi \to \{-1,1\} \;,\; \sigma \to \{3,-1\} .$$

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Weights of the Weyl spinor components

$$\Psi_0 \to \{4,0\}\,,\Psi_1 \to \{2,0\}\,,\ {}^*\Psi_0 \to \{3,1\}\,,\Psi^*_1 \to \{3,-1\}\,,$$
$$\Psi_2 \to \{0,0\}\,,\ {}^*\Psi_1 \to \{1,1\}\,,\Psi^*_2 \to \{1,-1\}\,,\Psi_{00} \to \{2,2\}\,,$$
$$\Psi_{01} \to \{2,0\}\,,\Psi_{02} \to \{2,-2\}\,,\Psi_3 \to \{-2,0\}\,,\ {}^*\Psi_2 \to \{-1,1\}\,,$$
$$\Psi^*_3 \to \{-1,-1\}\,,\Psi_{11} \to \{0,0\}\,,\Psi_{12} \to \{0,-2\}\,,$$
$$\Psi_4 \to \{-4,0\}\,,\ {}^*\Psi_3 \to \{-3,1\}\,,\Psi^*_4 \to \{-3,-1\}\,,$$
$$\Psi_{22} \to \{-2,-2\}\,,$$

Weights of the Ricci spinor components

$$\Phi_{01} \to \{2,0\}\,,\Phi_{02} \to \{2,-2\}\,,\Phi_{12} \to \{0,-2\},\ {}^*\Phi_{02} \to \{1,-1\}\,,$$
$$\Phi_{00} \to \{2,2\}\,,\Phi_{11} \to \{0,0\}\,,\ {}^*\Phi_{01} \to \{1,1\}\,,$$
$$\Phi_{22} \to \{-2,-2\}\,,\ {}^*\Phi_{12} \to \{-1,-1\}\,,\Omega \to \{0,0\}$$

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The G.H.P. operators

$$\begin{aligned}
\text{þ} Z^{\{p,q\}} &= -(p\epsilon + q\bar{\epsilon})Z^{\{p,q\}} + DZ^{\{p,q\}} \ ,\\
\text{þ}' Z^{\{p,q\}} &= -(p\gamma + q\bar{\gamma})Z^{\{p,q\}} + \Delta Z^{\{p,q\}} \ ,\\
\eth Z^{\{p,q\}} &= -(q\bar{\alpha} + p\beta)Z^{\{p,q\}} + \delta Z^{\{p,q\}} \ ,\\
\eth' Z^{\{p,q\}} &= -(p\alpha + q\bar{\beta})Z^{\{p,q\}} + \bar{\delta} Z^{\{p,q\}} \ ,\\
\widehat{\mathcal{D}} Z^{\{p,q\}} &= (p\theta + q\bar{\theta})Z^{\{p,q\}} + \mathcal{D}Z^{\{p,q\}} \ .
\end{aligned}$$

Weights of the G.H.P. operators

$$\text{þ} Z^{\{p,q\}} \mapsto L^{1+p}\bar{L}^{1+q}\text{þ} Z^{\{p,q\}} \ , \text{þ}' Z^{\{p,q\}} \mapsto L^{p-1}\bar{L}^{q-1}\text{þ}' Z^{\{p,q\}} \ ,$$
$$\eth Z^{\{p,q\}} \mapsto L^{1+p}\bar{L}^{q-1}\eth Z^{\{p,q\}} \ , \eth' Z^{\{p,q\}} \mapsto L^{p-1}\bar{L}^{q+1}\eth' Z^{\{p,q\}} .$$
$$\widehat{\mathcal{D}} Z^{\{p,q\}} \mapsto \widehat{\mathcal{D}} Z^{\{p,q\}} .$$

Remark

Again the GHP formalism can be formulated purely in tensor terms.

Five dimensional type D space-times

Definition (Coley, Milson, Pravda, Pravdová)

A spacetime of dimension five is of Petrov type D if there is a semi-null pentad $N \equiv \{l^a, n^a, m^a, \bar{m}^a, u^a\}$ in which the only non-vanishing components of Weyl tensor are the zero boost components. The null directions $\langle\vec{l}\rangle$, $\langle\vec{n}\rangle$ are the Weyl alligned null directions (WANDS).

In N the only nonvanishing Weyl scalars are then

$$\Psi_{11}\ ,\Psi_2\ ,\Psi_{02}\ ,\Psi_2^*\ ,\ {}^*\Psi_2.$$

The class $\mathcal{A}$ (Collaboration with Lode Wylleman)

Definition (the class $\mathcal{A}$)

Consider a five dimensional Einstein space-time

$$R_{ab} = -4\Lambda g_{ab}\ ,$$

and assume further that its Weyl tensor is of type D and invariant under rotations on a 2-dimensional space-like plane orthogonal to the WANDS (spatial spin isotropy). Then we say that such a space-time belongs to the class $\mathcal{A}$.

Under the assumptions of the previous definition one can choose a semi-null pentad such that only the Weyl scalars

$$\Psi_{11}\ , \Psi_2\ ,$$

are non-zero.

Remark

In dimension 4, $\mathcal{A} = D$ but in dimension five $\mathcal{A} \subset D$

The classification of $\mathcal{A}$

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For the class $\mathcal{A}$ the only non-vanishing curvature scalars are Ψ_{11}, Ψ_2 and Λ (which is a constant). Our formalism yields

Class $\mathcal{A}$ sub-types.

Sub-type	$\mathcal{A}$-I	$\mathcal{A}$-II	$\mathcal{A}$-III	$\mathcal{A}^-$	$\mathcal{A}^+$
Relation	$\Psi_{11}=0$	h_1	h_2	$\Psi_2=-2\Psi_{11}$	$\Psi_2=2\Psi_{11}$

$$h_1 \equiv \overline{\Psi}_2 = \Psi_2\,, \quad \Psi_2\overline{\Psi}_2 \neq 4\Psi_{11}^2\,, \quad \Psi_{11}\neq 0\,,$$
$$h_2 \equiv \Psi_2\overline{\Psi}_2 = 4\Psi_{11}^2\,, \quad \overline{\Psi}_2 \neq \Psi_2.$$

- $\mathcal{A}$-I: $ds^2=(dx^5)^2+f(x^5)\tilde{g}_{\mu\nu}dx^\mu dx^\nu$, $\tilde{g}$ 4-dim type D Einstein space.
- $\mathcal{A}$-II: $ds^2=(dx^5)^2+\frac{ds^2_\Sigma}{f(x^5)^2}+\frac{ds^2_\Theta}{g(x^5)^2}$, ds^2_Σ, ds^2_Θ constant curvature 2-spaces.
- $\mathcal{A}^-$ and $\mathcal{A}^+$ (A, B, C, constants):

$$(\mathcal{A}^-):\quad ds^2=-(1+Br^{-2}+Cr^2)dt^2+(1+Br^{-2}+Cr^2)^{-1}Adr^2+r^2d\Omega^2_-\,,$$
$$(\mathcal{A}^+):\quad ds^2=(1+Br^{-2}+Cr^2)dt^2+(1+Br^{-2}+Cr^2)^{-1}Adr^2+r^2d\Omega^2_+\,,$$

$d\Omega^2_-$ ($d\Omega^2_+$) Riemannian (Lorentzian) constant curvature 3-spaces.

The sub-type $\mathcal{A}$-III ($\Psi_2\overline{\Psi}_2 = 4\Psi_{11}^2$, $\overline{\Psi}_2 \neq \Psi_2$)

- This is a completely integrable system of differential equations in the variables Ψ_{11}, Ψ_2, $\mathfrak{c}$, ψ, $\bar{\psi}$, $\mathfrak{f}$, and $\mathfrak{e}$.
- This is a complex class which splits into different sub-types. The full integration of all of them is a difficult problem. The Kerr-NUT-AdS 5-dim black hole is in this sub-type (Taghavi-Chavert 2011). What else can be found ? Type D black rings ?
- The particular sub-type $\mathfrak{e} = \mathfrak{f} = \Lambda = 0$ has been studied and reduced to a system of PDE's. This is a family of metrics having 3 commuting space-like Killing vectors $\vec{\xi}_1$, $\vec{\xi}_2$, $\vec{\xi}_3$ generating an everywhere null distribution.

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- Our NP and GHP formalisms in dimension five enable us to find new exact solutions of the Einstein Field Equations. Explicit integration methods are, however, required.
- Similar studies can be attempted for other five dimensional type D Einstein spaces (for example analysing different invariance groups for the Weyl tensor compatible with its algebraic type).