1 Particle on a circle

Consider the following Lagrangian on the 3D configuration space (q_1, q_2, q_3):

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}q_3(q_1^2 + q_2^2 - r^2)\,. \tag{1}$$

1. Obtain the equations of motion, write down the general solution and interpret it. What is the meaning of q_3?
2. Write down the Hamiltonian. Are there any primary constraints?
3. Obtain the equations of motion from the Hamiltonian. Are there any secondary constraints?
4. Show that the total Hamiltonian can be written as

$$H_T = \frac{1}{2}r^2 q_3 + u_\alpha \psi^\alpha\,, \tag{2}$$

where $\psi_\alpha \approx 0$, for some u^α that you must specify. Hence conclude that $H_T \approx E$.

2 An example with too many coordinates

Consider the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1 + \dot{q}_2)^2 - V(q_1 + q_2)\,. \tag{3}$$

1. Obtain the conjugate momenta. Is there any primary constraint?
2. Write down the most general Hamiltonian and show that there are no secondary constraints.
3. What is the meaning of the Hamiltonian evaluated on the constraint surface?
4. Show that this first-class constraint generates a symmetry

$$\delta q_1 = \varepsilon\,, \qquad \delta q_2 = -\varepsilon\,, \qquad \delta p_1 = \delta p_2 = 0\,, \tag{4}$$

which leaves the physical state unchanged.

3 The electromagnetic field

Consider the action for the Maxwell field A_μ

$$S = \int d^4x \, -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{5}$$

1. By choosing x^0 as the time coordinate, show that the Lagrangian can be written as

$$L = \frac{1}{2}F_{0i}F_{0i} - \frac{1}{4}F_{ij}F_{ij}\,, \tag{6}$$

where $F_{0i} = V_i - \partial_i A_0$ and $V_i \equiv \partial_0 A_i$ is the 'velocity' field.

2. Obtain the conjugate momenta obeying $\{A_\mu(x), E_\nu(y)\} = \eta_{\mu\nu}\delta^{(3)}(x-y)$. Is there any primary constraint?

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3. Show that the Hamiltonian is

$$H = \int d^3x \, \frac{1}{2} E_i E_i + \frac{1}{4} F_{ij} F_{ij} + E_i \partial_i A_0 + u E_0 \,, \tag{7}$$

where u is arbitrary, and obtain the equations of motion.

4. Show that there is only one secondary constraint, and interpret it.

5. Show that both constraint are first-class. Hence they should generate gauge transformations. Consider a general linear combination

$$\phi[u] = \int d^3x \, u_1(x)\phi_1(x) + u_2(x)\phi_2(x) \,, \tag{8}$$

and compute the variations it induces on the phase space variables. Is this what you expected?

6. Write the total Hamiltonian as

$$H_T = \int d^3x \, \frac{1}{2} E_i E_i + \frac{1}{4} F_{ij} F_{ij} + u_1 E_0 - u_2 \partial_i E_i \,. \tag{9}$$

Show that the electric and magnetic fields are the non-trivial observables, and that the energy is given by the value of the Hamiltonian on the constraint surface.

4 Relativistic point particle

The action for a relativistic point particle moving in flat Minkowski space coupled to an external electromagnetic field is

$$S = \int d\tau \, m \sqrt{-\dot{q}_\mu(\tau)\dot{q}^\mu(\tau)} - e\dot{q}^\mu(\tau) A_\mu(q(\tau)) \,, \tag{10}$$

which is invariant under $\tau \to f(\tau)$.

1. Compute the conjugate momentum p_μ. Is there any primary constraint?

2. Show that the Hamiltonian vanishes on the constraint surface. Why is this not surprising?

3. Consider a general action $S = \int d\tau L(q, \dot{q})$ with this property. Show that both the Lagrangian and the Hamiltonian are homogeneous in $\dot{q}$, and conclude that $H \approx 0$ is a general property of these systems.

4. Obtain the equations of motion for the canonical variables.

5. Take $A_\mu = 0$ from now on. Show that the class of equivalence of the gauge transformation induced by the constraint is actually the particle's worldline.

6. Show that P_i and C, defined by

$$P_i = p_i \,, \qquad p^0 = C\sqrt{p_i p^i + m^2} \,, \tag{11}$$

are observables and conserved charges. Conclude that another property of such systems is that every observable is automatically a conserved charge.

7. We should also be able to measure the particles's position. If we make a gauge transformation such that $q'^0 = t$, then q'^i gives the space point we want. The function

$$Q_i(t) \approx q_i + \frac{t - q_0}{p^0} p_i \tag{12}$$

gives a different phase space function for each value of t. For a state on the constraint surface, it gives the space point where the particle is, was or will be when $q^0 = t$. Check that it is an observable.

5 General Relativity

Consider the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g}\, R\,. \tag{13}$$

After ADM decomposition

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{14}$$

the Lagrangian density is

$$\mathcal{L} = N\sqrt{h}\left(R + K_{ij}K^{ij} - K^2\right)\,, \tag{15}$$

where R is the 3-dimensional Ricci scalar and K_{ij} is the extrinsic curvature defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\,. \tag{16}$$

1. Compute the conjugate momenta obeying

$$\begin{aligned} \{h_{kl}(x), \pi^{ij}(y)\} &= \frac{1}{2}\left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right)\delta^{(3)}(x-y)\,, & (17)\\ \{N(x), \pi(y)\} &= \delta^{(3)}(x-y)\,, & (18)\\ \{N_i(x), p^i(y)\} &= \delta^i_j \delta^{(3)}(x-y)\,. & (19) \end{aligned}$$

   Are there any primary constraints?

2. Show that, up to a total divergence, the Hamiltonian density can be written as

$$\mathscr{H} = -2\sqrt{h}\left(N\mathcal{H} + N_i\mathcal{M}^i\right)\,, \tag{20}$$

   where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\left(R + K^2 - K_{ij}K^{ij}\right)\,, & (21)\\ \mathcal{M}^i &= \nabla_j\left(K^{ij} - h^{ij}K\right)\,. & (22) \end{aligned}$$

3. Add the primary constraints to the Hamiltonian and compute all secondary constraints. Show that they are all first class. How many degrees of freedom has the graviton?

4. Show that $\sqrt{h}\mathcal{M}^i\xi_i(x)$ induces spatial diffeomorphisms.