Palatini theories of gravity and applications to black holes

D. Rubiera-Garcia
Oviedo U. (Spain)
a collaboration with G. J. Olmo at Valencia U.

March 24, 2012

To read more (and better)

- Sotiriou-Faraoni arXiv:0805.1726[gr-qc]
- **Capozziello-De Laurentis, arXiv: 1108.6266[gr-qc]**
- Olmo, arXiv:1101.3864[gr-qc]
- Zanelli, arXiv:hep-th/0502193.

Palatini method dates back to Einstein-Cartan's discussions on the twentieth century.

Outline

Modified gravity theories

- General Relativity (GR) accounts for a huge number of phenomena on different scales up to measured precision. However:
- Black Hole (BH) interiors should be modified by Quantum Gravity effects at the Planck scale to avoid the formation of singularities.
- Huge discrepancy between gravitationally estimated amounts of matter and energy and the direct measurements via electromagnetic radiation motivates the search for alternative theories of gravity accounting for large scale structures without dark matter and/or dark energy.
- Planck scale physics should act as well to avoid the Big Bang singularity.
- Since a quantum theory of gravity is unknown it is justified to explore phenomenological approaches in order to gain insights on quantum gravity effects.
- Classical gravity theories with Planck scale (corrections may provide a glimpse of those changes and of the elements required to find an effective description of the quantum gravity dynamics.

Metric formalism

- Extensions of GR with higher-order corrections are motivated by study of quantum fields in curved spacetimes and by approaches to quantum gravity based on string theory.
- They naturally appear as modifications of GR in quantum gravity approaches.
- But:
  - Such theories generically lead to higher-order partial derivative equations.
  - They suffer from ghost and other perturbative instabilities.
  - Despite depending on more independent parameters regular solutions are scarce in the literature.
- The first two problems can be avoided if the curvature invariants appear in appropriate combinations

Lovelock gravities

- The Lovelock lagrangian consists of a sum of dimensionally extended Euler densities as

$$L_{LOV} = \sum_{k=0}^{[n/2]} c_k L_k$$

where [z] is the integral part of the number z, n is the number of spatial dimensions, c_k is the k-th order Lovelock parameter and L_k are given by

$$L_k = \frac{1}{2^k} \delta^{\rho_1\sigma_1\cdots\rho_k\sigma_k}_{\mu_1\nu_1\cdots\mu_k\nu_k} R^{\mu_1\nu_1}_{\rho_1\sigma_1} \cdots R^{\mu_k\nu_k}_{\rho_k\sigma_k}$$

- For a given n only terms with $k < n/2$ contribute to the equations of motion, terms with $k > n/2$ do not and $k = n/2$ is a topological term.
- Gauss-Bonnet: simplest case, the action picks up three new terms

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

- Although static charged black hole solutions are known (e.g. Maxwell, Born-Infeld) they still contain singularities or are ill defined.

Warming up

- An alternative route to provide modified dynamics beyond GR is to assume Palatini formalism.
- The key idea lies in assuming that metric and affine structures of the theory are independent.
- When the connection is not constrained *a priori* to be the Christoffel symbols of the metric even $f(R)$ theories yield second-order field equations, which contrasts with the usual (metric) formalism.
- In the Palatini approach metric and connection are regarded as independent entities and the field equations are obtained by independent variation with respect to both of them.
- Though this does not affect the dynamics of GR, it does have important consequences for the extensions of it.
- In Palatini theories the matter plays an active role in the construction of the independent connection, to produce modified dynamics.
- When there is no matter the field equations boil down to those of GR with possibly an effective cosmological "constant" term.

- The unusual role played by the matter in the construction of the geometry together with the second-order equations of motion make Palatini theories of gravity specially suitable to explore the effects of new gravitational physics on cosmological models and on the structure of black holes
- Palatini theories usually are considered in the ultraviolet regimen.
- In particular they are appropriate to study Planck-scale effects on the very internal structure of black holes.
- A suitable theory at this regard is provided by

  $$f(R, Q) = R + aR^2/R_P + bQ/R_P$$

  where $R = g^{\mu\nu}R_{\mu\nu}$, $Q = R_{\mu\nu}R^{\mu\nu}$ and a and b are constants.
- The presence of a Ricci-squared terms is very important because it leads for modified dynamics even for traceless sources, as opposed to $f(R)$ where the dynamics differ from that of GR only for tracelesness sources.
- Theories of this type has been considering in the metric formalism in looking for ghost and singularity-free theories of gravity and in Palatini formalism within cosmological models, being able to replace the Big Bang singularity by a cosmic bounce in several scenarios.

Dynamics of Palatini theories

- To fix the ideas take as the lagrangian density

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(g_{\mu\nu}, R^\alpha_{\beta\mu\nu}) + S_m[g_{\mu\nu}, \Psi]$$

  with S_m: matter action, Ψ: matter fields and
  $R^\alpha_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\nu\beta} - \partial_\nu\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\beta} + \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\beta}$
- Note that:
  - The independent connection $\Gamma^\alpha_{\beta\gamma}$ *does not* need to be symmetric $\Gamma^\alpha_{\beta\gamma} \neq \Gamma^\alpha_{\gamma\beta} \rightarrow$ possibility of nonvanishing torsion $S^\lambda_{\mu\nu} \equiv (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})/2$.
  - The Ricci $R_{\mu\nu}$, in general, *does not* need to be symmetric.
  - Moreover, the Ricci *does not* need to by symmetric even though the connection is symmetric.
  - This defines a non-metricity

$$Q_{\lambda\alpha\beta} = -\nabla_\lambda g_{\alpha\beta}; Q_\mu = \frac{1}{4}Q^\nu_{\mu\nu}$$

    in such a way that the antisymmetric part of the Ricci is written as

$$R_{[\alpha\beta]} = -2\nabla_{[\beta}Q_{\alpha]}$$

- Variation of the action with respect to the connection yields the general result

$$\begin{aligned}\frac{1}{\sqrt{-g}}\nabla_\alpha[\sqrt{-g}M^{(\beta\nu)}] &= (Object(torsion))^{\nu\beta\kappa}_{\alpha\lambda}g^{\lambda\rho}M_{[\kappa\rho]}\\ \frac{1}{\sqrt{-g}}\nabla_\alpha[\sqrt{-g}M^{[\beta\nu]}] &= (Object(torsion))^{\nu\beta\kappa}_{\alpha\lambda}g^{\lambda\rho}M_{(\kappa\rho)}\end{aligned}$$

where $M^{(\beta\nu)} = f_R g^{\beta\nu} + 2f_Q R^{(\beta\nu)}(\Gamma)$ and $M^{[\beta\nu]} = 2f_Q R^{[\beta\nu]}(\Gamma)$

- When the the torsion vanishes the symmetric and antisymmetric parts of $M^{\beta\nu}$ decouple
- When the Ricci is symmetric $R_{[\mu\nu]} = 0$ the connection turns out to be the gradient of a scalar function i.e. $\Gamma^\sigma_{\sigma\nu} = \partial_\nu\phi$. Moreover, the field equations in this case boil down to those of GR with a cosmological constant term, guaranteeing the absence of new propagating degrees of freedom and not to be affected by ghosts and other dynamical instabilities
- When $R_{[\mu\nu]} \neq 0$ then $T^\sigma_{\sigma\nu} = \partial_\nu\phi + B_\mu$ has a purely vectorial component B_μ which adds new dynamical degrees of freedom.

- Assuming only the symmetric parts for the connection and the Ricci we obtain the field equations

$$\begin{aligned} f_R R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} + 2 f_Q R_{\mu\alpha} R^{\alpha}{}_{\nu} &= \kappa^2 T_{\mu\nu} \\ \nabla_\beta \left[\sqrt{-g} \left(f_R g^{\mu\nu} + 2 f_Q R^{\mu\nu} \right) \right] &= 0 \end{aligned}$$

- This can be seen as an algebraic equation for the connection in which, besides the metric $g_{\mu\nu}$, there is an explicit dependence on the matter energy-momentum tensor.
- This equation defines a new metric $h_{\mu\nu}$ such that

$$\nabla_\beta [\sqrt{-h} h^{\mu\nu}] = 0$$

and the mismatch between the "physical" $g_{\mu\nu}$ and "effective" $h_{\mu\nu}$ metrics is due to the matter through the object $\Sigma = f_R g^{\mu\nu} + 2 f_Q R^{\mu\nu}$.
- And the field equations become

$$R_\mu{}^\nu(h) = \frac{1}{\sqrt{\det \hat{\Sigma}}} \left(\frac{f}{2} \delta_\mu{}^\nu + \kappa^2 T_\mu{}^\nu \right)$$

Static charged black hole solutions

- Electrovacuum scenarios

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + \frac{1}{8\pi}\int d^4x\sqrt{-g}\varphi(X,Y)$$

where $X \equiv -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} = \vec{E}^2 - \vec{B}^2$ and $Y \equiv -\frac{1}{2}F_{\alpha\beta}F^{*\alpha\beta} = 2\vec{E}\cdot\vec{B}$, are the two field invariants with $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

- Energy-momentum tensor

$$T_\mu{}^\nu = \frac{1}{4\pi}\begin{pmatrix} \left(\frac{\varphi}{2} - X\varphi_X\right)\hat{I} & \hat{0} \\ \hat{0} & \frac{\varphi}{2}\hat{I} \end{pmatrix}$$

- Trace

$$T = \frac{1}{2\pi}\left[\varphi - X\varphi_X\right]$$

The $f(R)$ case

- The connection equation now tell us

$$\Sigma_{\nu}{}^{\alpha} = f_R \delta_{\nu}{}^{\alpha}$$

$\rightarrow$ physical and effective metric are conformally related.

- The metric equation is

$$f_R R_{\mu\nu}(h) - \frac{f}{2} g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

- Taking the trace of this equation with $g^{\mu\nu}$, we obtain equation

$$R f_R - 2f = \kappa^2 T$$

From this equation it follows that $R \equiv g^{\mu\nu} R_{\mu\nu}(h) = R(T)$. Consequently, in order to obtain deviations of the dynamics from GR, one is lead to consider NEDs.

- The field equations...

$$\begin{aligned} G_{\mu\nu}(g) &= \frac{\kappa^2}{f_R}T_{\mu\nu} - \frac{Rf_R - f}{2f_R}g_{\mu\nu} - \frac{3}{2f_R^2}(\partial_\mu f_R \partial_\nu f_R - \frac{1}{2}g_{\mu\nu}(\partial f_R)^2) + \\ &+ \frac{1}{f_R}(\nabla_\mu \nabla_\nu f_R - g_{\mu\nu}\nabla_\alpha \nabla^\alpha f_R) \end{aligned}$$

- From the trace equation, it follows that all the R-dependent terms on the right-hand side of (1) are functions of the trace T.
- For $f(R)$ theories Maxwell's electrodynamics ($T = 0$) yields the same solutions as GR+Λ.
- Nonlinear theories of electrodynamics (NED) are able to yield $T \neq 0$ they can be used to probe the Palatini dynamics on $f(R)$ black holes, producing deviations from their GR counterparts.
- Instead of solving this system by "brute force" is better to take advantage of the effective geometry $h_{\mu\nu}$.

- Take a line element $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2$, associated to the physical geometry

$$ds^2 = \frac{1}{f_R}\left(-A(\tilde{r})e^{\Psi(\tilde{r})}dt^2 + \frac{d\tilde{r}^2}{A(\tilde{r})} + \tilde{r}^2 d\Omega^2\right)$$

- This is related to the effective geometry $ds^2 = \frac{1}{f_R}d\tilde{s}^2$ with $d\tilde{s}^2 = h_{\mu\nu}dx^\mu dx^\nu = h_{tt}dt^2 + h_{\tilde{r}\tilde{r}}d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$.
- Working with the effective geometry one gets several relations $\tilde{r}^2 = f_R r^2$, $g_{tt} = -f_R^{-1}A(\tilde{r})e^{\Psi(\tilde{r})}$ and $g_{rr} = f_R^{-1}A(\tilde{r})^{-1}(d\tilde{r}/dr)^2$.
- Solving with $h_{\mu\nu}$ is much easier!

$$A(\tilde{r}) = 1 - \frac{2M(\tilde{r})}{\tilde{r}}$$

where the mass function satisfies

$$M_{\tilde{r}} = \frac{\left(f + \frac{\kappa^2}{4\pi}\varphi\right)r^2}{4f_R} \leftrightarrow M_r = \frac{\left(f + \frac{\kappa^2}{4\pi}\varphi\right)r^2}{4f_R^{3/2}}\left(f_R + \frac{r}{2}f_{R,r}\right)$$

Given $f(R)$ and $\varphi(X,Y)$ the mass functionprovides a complete solution. The GR limit is recovered when $f_R \to 1$ and gives $M_r = -\frac{\kappa^2}{2}r^2 T_t^t$.

The $f(R, Q)$ case

- Now the relation between the effective and physical metrics is nonconformal

$$h_{\mu\nu} = \sqrt{\det \Sigma}[\Sigma^{-1}]_{\mu}{}^{\alpha} g_{\alpha\nu} \ ; \ \Sigma_{\nu}{}^{\alpha} = f_R \delta_{\nu}{}^{\alpha} + 2 f_Q P_{\nu}{}^{\alpha}$$

where $P_{\mu}{}^{\nu} = R_{\mu\alpha} g^{\alpha\nu}$.

- The field equations are again easily solved in terms of the effective geometry

$$R_{\mu}{}^{\nu}(h) = \frac{1}{\sqrt{\det \Sigma}} \left(\frac{f}{2} \delta_{\mu}{}^{\nu} + \kappa T_{\mu}{}^{\nu} \right)$$

- In order to do this one must compute the object Σ:

$$\hat{\Sigma} = \begin{pmatrix} \sigma_- I & 0 \\ 0 & \sigma_+ I \end{pmatrix} \ , \ \sigma_{\pm} = \left(\frac{f_R}{2} + \sqrt{2 f_Q} \lambda_{\pm} \right) \ ,$$

$$\lambda_+ = \frac{1}{2} \left(f + \frac{f_R^2}{4 f_Q} + \frac{\kappa^2}{4\pi} \varphi \right) \ , \ \lambda_- = \frac{1}{2} \left(f + \frac{f_R^2}{4 f_Q} + \frac{\kappa^2}{4\pi} (\varphi - 2 X \varphi_X) \right)$$

- Playing again with the effective and physical metrics we define a line element

$$ds^2 = \frac{1}{\sigma_+}\left(-A(\hat{r})e^{\Psi(\hat{r})}dt^2 + \frac{1}{A(\hat{r})}d\hat{r}^2\right) + \frac{\hat{r}^2}{\sigma_-}d\Omega^2$$

which follows from defining the auxiliary line element
$d\hat{s}^2 = h_{\mu\nu}dx^\mu dx^\nu = h_{tt}dt^2 + h_{\hat{r}\hat{r}}d\hat{r}^2 + \hat{r}^2 d\Omega^2$ with $\hat{r}^2 = r^2\sigma_-$.

- Inserting this metric in the field equations we find again that $\Psi(\hat{r}) = constant$ and $A(\hat{r}) = 1 - \frac{2M(\hat{r})}{\hat{r}}$ with

$$M_r = \frac{\left(f + \frac{\kappa^2}{4\pi}\varphi\right) r^2 \sigma_-^{1/2}}{4\sigma_+}\left(1 + \frac{r\sigma_{-,r}}{2\sigma_-}\right)$$

- This is the general expression for the derivative of the mass function M_r in $f(R,Q)$ theories coupled to an arbitrary NED $\varphi(X, Y=0)$.
- Consequently, black holes with electric charge in $f(R,Q)$ theories are completely characterized by a single function $M(r)$ that, once the gravity Lagrangian is chosen, allows to fully determine the metric.

Discussion

- Metric and connection are physically independent entities and should not be constrained *a priori*.
- (Palatini) theories of gravity based on this assumption lacks any problem related to higher-order derivative equations or dynamical instabilities.
- For the case of GR this modification is harmless as the connection equations restricts Γ to be the Levi-Civita connection of the metric.
- For higher-curvature gravity theories Γ turns out to be the Levi-Civita connection of an effective metric.
- Cosmological and black hole models based on $f(R)$ and $f(R,Q)$ theories are very successful at avoiding or at least ameliorating singularities.
- However, the particular model has to be chosen carefully in order not to enter in conflict with observations (e.g. solar system experiments).
- Model such as $f(R,Q) = R + R^2/R_P + (R_{\mu\nu}R^{\mu\nu})^2/R_P$ are very satisfactory: natural interpretation from a effective field theory perspective and contains interesting physics (e.g. nonsingular solutions).
- Several results coming soon...