Palatini $f(R)$ and $f(R, Q)$ black holes with electromagnetic charge

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Motivation

- Black Hole (BH) interiors should be modified by Quantum Gravity effects at the Planck scale to avoid the formation of singularities.
- Classical Gravity theories with Planck scale corrections may provide a glimpse of those changes and of the elements required to find an effective description of the Quantum Gravity dynamics.
- Examples of Classical Gravity Theories containing higher curvature invariants naturally arise in various scenarios such as low-energy effective actions of string theories and perturbative approaches to quantum gravity.
- If those theories are formulated a la Palatini, i.e. assuming that metric and connection are independent objects, the resulting theory and solutions are completely different from the usual (metric) approach.
- In the usual metric approach, where the connection is the Levi-Civita connection, higher-order curvature invariants induce new dynamical degrees of freedom such as higher-order derivative equations and/or scalar and vector degrees of freedom.
- In Palatini theories of the form $f(R, Q)$ (with $Q = R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu} = R_{\nu\mu}$), however, no new degrees of freedom appear. The vacuum equations coincide with the eqs. of GR and the modified dynamics only arises in regions containing matter/energy sources.
- This is due to the active role played by the matter in the construction of the connection.

- In $f(R)$ the gravitational dynamics is modified by terms depending only on the trace of the energy-momentum tensor
- In $f(R, Q)$ there is a dependence on $T_{\mu\nu}$ so even for $Trace = 0$ new effects are found.
- In Palatini, this family of theories yields the usual Einstein-de Sitter equations in vacuum (GR+Λ) but exhibits modified dynamics even for traceless matter sources. For this reason, the simplest extension of the usual black hole solutions of GR could be found in this family of theories.
- Electrically charged black holes provide the simplest and best tractable models to explore Planck-scale effects.
- Nonlinear electrodynamics for the $f(R)$ case and standard Maxwell electrodynamics for $f(R, Q)$ theories are the natural choices to obtain BHs with electric charge.

Main elements

- Let be the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R,Q) + S_m[g_{\mu\nu},\psi]$$

where $R_{\mu\nu}(\Gamma) = -\partial_\mu \Gamma^\lambda_{\lambda\nu} + \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda}$

- Variation w.r.t. metric and connection yields:
- Metric:

$$f_R R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} + 2f_Q R_{\mu\alpha}R^\alpha{}_\nu = \kappa^2 T_{\mu\nu}$$

- Connection:

$$\nabla_\beta\left[\sqrt{-g}\left(f_R g^{\mu\nu} + 2f_Q R^{\mu\nu}\right)\right] = 0$$

- Defining $P_\mu{}^\nu \equiv R_{\mu\alpha}g^{\alpha\nu}$ the metric field equation becomes

$$f_R P_\mu{}^\nu - \frac{f}{2}\delta_\mu{}^\nu + 2f_Q P_\mu{}^\alpha P_\alpha{}^\nu = \kappa^2 T_\mu{}^\nu$$

- Using matrix notation this is equivalent to:

$$2f_Q\hat{P}^2 + f_R\hat{P} - \frac{f}{2}\hat{I} = \kappa^2\hat{T}$$

which implies that $R = Tr[\hat{P}]$ and $Q = Tr[\hat{P}^2]$.

- Using this information, we find $\Gamma^{\alpha}_{\mu\nu}$ = Levi-Civita of $h_{\mu\nu}$, with

$$h^{\mu\nu} = \frac{g^{\mu\alpha}{\Sigma_\alpha}^\nu}{\sqrt{\det\Sigma}}; {\Sigma_\alpha}^\nu = f_R\delta^\nu_\alpha + 2f_Q{P_\alpha}^\nu$$

- Now expressing $g_{\mu\nu} = {\Sigma_\mu}^\alpha h_{\alpha\nu}/\sqrt{\det\Sigma}$ the Ricci tensor $R_{\mu\nu}(\Gamma) \equiv R_{\mu\nu}(h)$ is

$$R^\nu_\mu(h) = \frac{1}{\sqrt{\det\hat{\Sigma}}}(\frac{f}{2}\delta^\nu_\mu + \kappa^2(T^\nu_\mu))$$

where ${R_\mu}^\nu(h) \equiv R_{\mu\alpha}(h)h^{\alpha\nu}$

- This equation determines the dynamics of Palatini $f(R, Q)$ theories
- The RHS is completely determined by the matter.
- In vacuum, ${R_\mu}^\nu(h) = \tilde{\Lambda}{\delta_\mu}^\nu$ yields the same solutions as GR.

Palatini f(R) black holes with nonlinear electrodynamics

- Action
$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + \frac{1}{8\pi}\int d^4x\sqrt{-g}\varphi(X,Y)$$
where $X \equiv -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} = \vec{E}^2 - \vec{B}^2$ and $Y \equiv -\frac{1}{2}F_{\alpha\beta}F^{*\alpha\beta} = 2\vec{E}\cdot\vec{B}$, with $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $\vec{E}$ and $\vec{B}$ the electric and magnetic fields.
- Now $f_R R_{\mu\nu}(\Gamma) - \frac{f}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu}$ implies
$$Rf_R - 2f = \kappa^2 T$$
so $R \equiv g^{\mu\nu}R_{\mu\nu} = R(T)$. Using $h_{\mu\nu} = f_R g_{\mu\nu}$.
- Since the $f(R)$ modified dynamics is due to the $T-$dependent terms, Maxwell's electrodynamics yields the same solutions as GR+Λ.
  - Non-linear theories of electrodynamics (NEDs) can yield $T \neq 0$.
  - NEDs with $T \neq 0$ can be used to probe Palatini $f(R)$ BHs dynamics.
- **NEDs provide the simplest non-standard solutions to spherically symmetric spacetimes in Palatini $f(R)$ theories.**

- The electromagnetic field equation is obtained as

$$\nabla_\mu \left(\varphi_X F^{\mu\nu} + \varphi_Y F^{*\mu\nu}\right) = 0 \rightarrow \varphi_X^2 X = \frac{q^2}{r^4}$$

for a purely electric radial field ($X = -g_{tt}g_{rr}(F^{tr})^2$)

- The energy-momentum tensor may be written as

$$T_\mu{}^\nu = \frac{1}{4\pi}\begin{pmatrix} \left(\frac{\varphi}{2} - X\varphi_X\right)\hat{I} & \hat{0} \\ \hat{0} & \frac{\varphi}{2}\hat{I} \end{pmatrix}$$

and the Einstein equations to be solved are written as

$$R_\mu{}^\nu(h) = \frac{1}{f_R^2}\begin{pmatrix} \left[\frac{f}{2} + \frac{\kappa^2}{4\pi}\left(\frac{\varphi}{2} - X\varphi_X\right)\right]\hat{I} & \hat{0} \\ \hat{0} & \left[\frac{f}{2} + \frac{\kappa^2}{8\pi}\varphi\right]\hat{I} \end{pmatrix}$$

- Trace of the energy-momentum tensor

$$T = \frac{1}{2\pi}\left[\varphi - X\varphi_X\right]$$

must be non vanishing.

- There are two choices for the system of coordinates
  - For $g_{\mu\nu}$: $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2$ (used to obtain X).
  - For $h_{\mu\nu}$ (which is the Levi-Civita connection of the metric):

$$ds^2 = \frac{1}{f_R}\left(-A(\tilde{r})e^{\psi(\tilde{r})}dt^2 + \frac{d\tilde{r}^2}{A(\tilde{r})} + \tilde{r}^2 d\Omega^2\right)$$

  - Where $r^2 = \frac{\tilde{r}^2}{f_R}$, $g_{tt} = -\frac{A(\tilde{r})e^{\psi(\tilde{r})}}{f_R}$, $g_{rr} = \frac{(d\tilde{r}/dr)^2}{A(\tilde{r})f_R}$

- We solve for the metric $h_{\mu\nu}$ and then use the conformal transformation back to the physical metric $g_{\mu\nu}$. Determine two functions ψ and A.
- $R_t^t - R_{\tilde{r}}^{\tilde{r}}$ gives $\psi = constant$.
- Defining $A(\tilde{r}) = 1 - 2M(\tilde{r})/\tilde{r}$ and using $\tilde{r}^2 = r^2 f_R$ we find in terms of the coordinate r:

$$M_r = \frac{\left(f + \frac{\kappa^2}{4\pi}\varphi\right) r^2}{4 f_R^{3/2}}\left(f_R + \frac{r}{2} f_{R,r}\right)$$

  Given gravity ($f(R)$) and matter ($\varphi(X)$) this provides a complete solution.
- Note that the GR limit is recovered when $f_R \to 1 \to M_r = -\frac{k^2}{2} r^2 T_t^t$

- Gravity:

$$f(R) = R \pm \frac{R^2}{R_P}, f_R = 1 \pm \frac{2R}{R_P}, R = -\kappa^2 T$$

$\rightarrow f_R$ may vanish at some curvature radius R_P.

- Matter: We take the Born-Infeld lagrangian ($r^4 = \frac{q^2 z^4}{\beta^2}$)

$$\varphi(X) = 2\beta^2 \left(1 - \sqrt{1 - \frac{X}{\beta^2} - \frac{Y^2}{4\beta^4}}\right)$$

as the NED source, which has a non-vanishing trace:

$$T = -\frac{\beta^2}{2\pi\sqrt{1+\frac{1}{z^4}}}\left[\frac{1}{z^4} + 2\left(1 - \sqrt{1+\frac{1}{z^4}}\right)\right]$$

and energy

$$T_t^t = \frac{\beta^2}{4\pi}\left(\frac{\sqrt{z^4+1}}{z^2} - 1\right) \rightarrow \varepsilon(q) = 4\pi\int_0^\infty drr^2 T_t^t(r,q) = \frac{\pi^{3/2}\beta^{1/2}}{3\Gamma(3/4)^2} q^{3/2}$$

Study of $f(R) = R + R^2/R_P$

- Writing the metric as

$$A(\tilde{r}) = 1 - \frac{2M(\tilde{r})}{\tilde{r}}$$

$M(\tilde{r})$ can be obtained as (remember that $r^2 = \frac{\tilde{r}^2}{f_R}$)

$$\hat{M}(z) \equiv \frac{M(z)}{M_0} = 1 + \frac{\gamma^3\kappa^2\beta^2}{M_0} G(z)$$

$$G(z) = -\int_z^\infty dz' \tilde{M}_{z'}$$

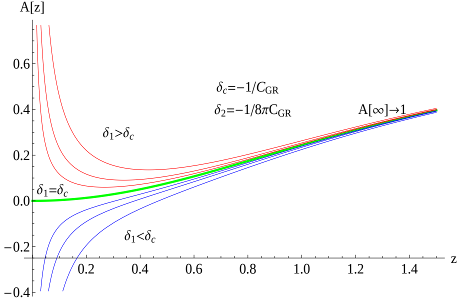
- The function A is expressed in terms of the physical metric as

$$A(z) = \frac{1}{f_R}\left[1 - \frac{(1+\delta_1 G(z))}{\delta_2 z f_R^{1/2}}\right]$$

($z = r/\gamma, \gamma = \sqrt{\beta/q}$) where the only adjustable constants are

$$\delta_1 = 2(4\pi)^{3/4}\left(\frac{r_q}{r_S}\right)\sqrt{\frac{r_q}{l_\beta}}; \delta_2 = (4\pi)^{1/4}\left(\frac{r_q}{r_S}\right)\sqrt{\frac{l_\beta}{r_q}}$$

- The Born-Infeld-GR black holes



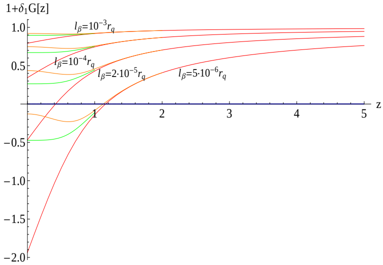
where C_{GR} is the energy of the BI field times a constant.

- The Kretchsmann scalar Kret $= R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is obtained when $1 + C_{GR}\delta_1 = 0$ as

$$\text{Kret}(z) = \frac{1}{16\pi^2\delta_2^2 C_{GR}^2}\left(\frac{1}{z^4} - \frac{2}{3z^2} + \frac{13}{15} + \ldots\right)$$

so it goes as $\sim 1/z^4$ as compared to the general BI case $\sim 1/z^6$ and the GR-RN case $\sim 1/z^8$.

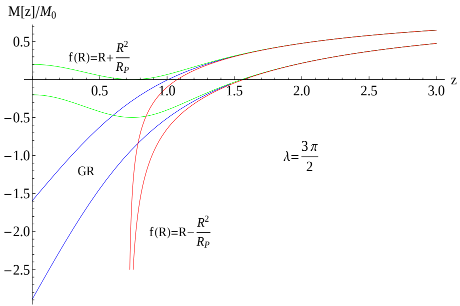
- For astrophysical black holes with low charge to mass ratio ($\delta_1 \ll 1$), the existence and location of the outer horizon is only slightly modified as compared with the GR case.
- Horizons obtained through the cuts between the curves $1 + \delta_1 G(z)$ and $\delta_2 z f_R^{1/2}$.



- There exist a critical value $l_\beta^{crit} = \frac{32\pi^{3/2} C^2 r_q^3}{r_S^2}$ where C is an integration constant, that in the GR case $C_{GR} = -\frac{1}{8\pi}\varepsilon!$.
  - $\lambda > \pi/2$ and $l_\beta > l_\beta^{crit}$ two horizons,
  - $\lambda = \pi/2$ and $l_\beta > l_\beta^{crit}$ two horizons (innermost approaches the center).
  - $\lambda < \pi/2$ up to one horizon.
- The Kretchsmann invariant diverges as $\sim \frac{1}{z^4}$. Same as minimum divergence as in BI-GR.

Study of $f(R) = R - R^2/R_P$

- A similar analysis as in the $f(R) = R + R^2/R_P$ case leads to several conclusions:
  - f_R vanishes at some small $z_+ \longrightarrow$ the metric is only defined beyond z_+.
  - There are always two horizons: inner + outer.
  - The strength of the singularity through the Kretschmann invariant is softened: it diverges as $\sim \frac{1}{(z-z_+)^2}$.



Conclusions:

- No extra degrees of freedom are introduced: Palatini $f(R)$ only *deforms* existing solutions.
- External horizons are almost placed at the same locations as in GR, at least for macroscopic black holes.
- In the $f(R) = R + R^2/R_P$ case additional inner horizons may arise. Singularities not softened.
- In the $f(R) = R - R^2/R_P$ case the singularity is shifted to a finite radius r_+, diverging as $\sim 1/(z - z_+)^2$ and two horizons arise in all cases.
- **This motivates us to search for more extensions of the gravity sector in order to study quantum effects at Planck's scale.**

Palatini f(R,Q) black holes with linear electrodynamics

- Basic equations

$$R_\mu^\nu(h) = \frac{1}{\sqrt{\det \hat{\Sigma}}}\left(\frac{f}{2}(\delta_\mu^\nu + \kappa^2(T_\mu^\nu))\right); \hat{\Sigma} = \begin{pmatrix} \sigma_- \hat{I} & \hat{0} \\ \hat{0} & \sigma_+ \hat{I} \end{pmatrix}$$

where we have defined $\sigma_\pm = \left(\frac{f_R}{2} + \sqrt{2f_Q}\lambda_\pm\right)$, $\lambda_+^2 = \frac{1}{2}\left(f + \frac{f_R^2}{8f_Q} + \frac{\kappa^2}{4\pi}\varphi\right)$, and $\lambda_-^2 = \frac{1}{2}\left(f + \frac{f_R^2}{8f_Q} + \frac{\kappa^2}{4\pi}[\varphi - 2X\varphi_X]\right)$

- and defining

$$ds^2 = \frac{1}{\sigma_+}\left(-A(\tilde{r})e^{\psi(\tilde{r})}dt^2 + \frac{1}{A(\tilde{r})}d\tilde{r}^2\right) + \frac{\tilde{r}^2}{\sigma_-}d\Omega^2$$

- One finds that $\psi(\tilde{r})$ =constant and $A(\tilde{r}) = 1 - \frac{2M(\tilde{r})}{\tilde{r}}$ satisfies:

$$M_r = \frac{\left(f + \frac{\kappa^2}{4\pi}\varphi\right) r^2 \sigma_-^{1/2}}{4\sigma_+}\left(1 + \frac{r\ \sigma_{-,r}}{2\sigma_-^{1/2}}\right)$$

- This is the general expression for the mass function $M(\tilde{r})$ in $f(R, Q)$ theories coupled to an arbitrary NED $\varphi(X)$.

- Gravity:

$$f(R,Q) = R + aR^2/R_P + bR_{\mu\nu}R^{\mu\nu}/R_P$$

- Matter: Maxwell lagrangian $\varphi = X = F_{\mu\nu}F^{\mu\nu}$.
- Metric components:

$$\begin{aligned} A(z) &= \left[1 - \frac{1+\delta_1 G(z)}{\delta_2 z \sigma_-^{1/2}}\right] \\ g_{tt} &= -\frac{A(z)}{\sigma_+}; g_{rr} = \frac{1}{A\sigma_+}\left(\frac{d\tilde{z}}{dz}\right)^2 = \frac{\sigma_+}{\sigma_-}\frac{1}{A(z)} \end{aligned}$$

- Expansion around the center:

$$g_{tt} \approx \frac{(1+\beta\delta_1)}{4\delta_2}\left(\frac{1}{\sqrt{z-1}} + \frac{9}{4}\sqrt{z-1} - \dots\right) - \frac{1}{2}\left(1 - \frac{\delta_1}{\delta_2}\right) - \left(1 - \frac{2\delta_1}{3\delta_2}\right)(z-1) + \dots$$

- But good news, for $1 + \delta_1\beta = 0$ ($\beta \sim -1.748$)

$$g_{tt} \approx -\frac{1}{2}\left(1 + \frac{1}{\delta_2\beta}\right) - \left(1 + \frac{2}{3\delta_2\beta}\right)(z-1) + \dots$$

finite!!!.

- Kretchsmann scalar in the case $(1+\delta_1\beta=0)$

$$\gamma^4 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=\left(16+\frac{88}{9\beta^2\delta_2^2}+\frac{64}{3\beta\delta_2}\right)-\frac{64\left(68+147\beta\delta_2+90\beta^2\delta_2^2\right)(z-1)}{45\left(\beta^2\delta_2^2\right)}+\ldots$$

finite!!!

- Several gravitational configurations depending on δ_1 and δ_2.
- Singular solutions $(1+\delta_1\beta\neq 0)$:
    - Black holes with two horizons (RN).
    - Extreme black holes (RN).
    - Naked singularities (RN).
    - Black holes with a single horizon.
- Nonsingular solutions $(1+\delta_1\beta=0)$:
    - Black holes with a single horizon, without singularities and metric finite at the center.
    - Solutions without horizons, but without singularities.
    - The turning point is found at $(1+\delta_2\beta=0)$ (very small values of charge).

Nonsingular Black Holes (arXiv:1112.0475 [gr-qc])

- The charge-to-mass ratio is fixed.
- No curvature divergence and the geometry near $z = 1$ (core) is Minkowskian.
- All the charge is accumulated in the surface of a core of radius $\sim \sqrt{r_q l_P}$.
- Mass is quantized in Planck's unit (times the number of electric charge).
- Area of the core and of the horizon are quantized as well.
- The density of the core is constant and of order Planck's density.

Conclusions

- Palatini theories generate modified dynamics without adding new d.o.f.
  - In $f(R)$ there is modified dynamics for sources with $T \neq 0$.
  - In $f(R, Q)$ the modified dynamics arises even if $T = 0$.
- To probe the effects of the Palatini $f(R)$ dynamics on BH structure Non-linear Electrodynamics (NEDs) with $T \neq 0$ is necessary.
- The form of the metric for arbitrary $f(R)$ and $\varphi(X)$ has been found.
- A particular solution for a given $f(R)$ and $\varphi(X)$ (Born-Infeld) was obtained and its properties characterized.
- The extension of the formalism to $f(R, Q)$ theories is also obtained.