Towards the Einstein-Λ-scalar field system in spherical symmetry

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Weak Cosmic Censorship in Spherical Symmetry

Introduction

- Solutions to the EFEs may develop singularities. In particular gravitational collapse to a singularity must occur.
- What is the nature of the singularities resulting from gravitational collapse?
- Are the singularities hidden in a black hole? Or are they naked? Deterministic breakdown!
- *Weak Cosmic Censorship* is the conjecture that, generically, singularities arising from gravitational collapse are contained in black holes.
- WCC is a statement about, how assumptions on the initial data for the EFEs relates to properties of the resulting maximal global hyperbolic development of the data.
- Global results for large initial data are only known in $1+1$ dimensions.

Spherical Symmetry

- Spherical symmetry: the group $SO(3)$ acts by isometry, $\mathcal{Q} = \mathcal{M}/SO(3)$.
- Metric $\bar{g}$ of $\mathcal{Q}$ is related to the metric to $(\mathcal{M}, g)$ by

$$ds^2 = \bar{g}_{ab} dx^a dx^b + r^2 d\sigma^2_{\mathbb{S}^2}$$

where $d\sigma^2_{\mathbb{S}^2}$ is the metric on the round $\mathbb{S}^2$ and r is the *area-radius* function on $\mathcal{Q}$.

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Penrose diagrams

- The Penrose diagram is the image of a bounded conformal map

$$\mathcal{Q} \to \mathbb{R}^{1+1}$$

$$\bar{g}_{ab} dx^a dx^b = -\Omega^2 dudv$$

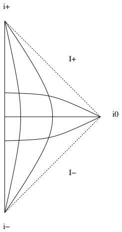
- Boundaries induced by the causal embebeding:

$\mathcal{I}^+$ are limit point of future-directed null rays as $r \to +\infty$

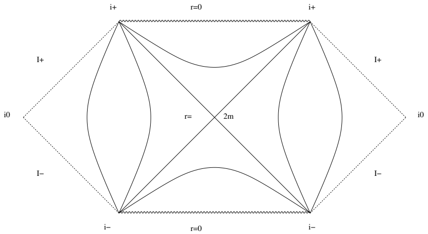
$\mathcal{I}^-$ are limit point of past-directed null rays as $r \to +\infty$

$(i^+ \, i^-$ future/past timelike infinity, $\quad i^0$ spacelike infinity$)$

Minkowski diagram



Schwarzschild diagram



The Black Hole Concept

- Using the causal structure of $\mathbb{R}^{1+1}$, the causal past $\mathcal{J}^-(p)$ of $p \in \mathcal{I}^+$ is well defined.
- Asyptotically flat: $\mathcal{I}^+$ is a null surface.
- $\mathcal{I}^+$ is complete: Consider the parallel transport of an ingoing null vector along a fixed outgoing null geodesic that has a limit point on $\mathcal{I}^+$. The affine length of its integral curves tends to ∞ as $\mathcal{I}^+$ is approached.
- Within this completness property we define the Black hole region: $\mathcal{Q} \setminus \mathcal{J}^-(\mathcal{I}^+)$
- The future boundary of $\mathcal{J}^-(\mathcal{I}^+) \in \mathcal{Q}$ is a null hypersurface: the *future event horizon* $\mathcal{H}^+$.
- In Minkowski: $\mathcal{J}^-(\mathcal{I}^+) \bigcap \in \mathcal{Q} = \mathcal{Q}$

Birkhoff Theorem

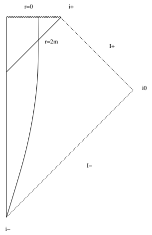
- Solutions of the Spherically symmetric Einstein Vacuum equations are isometrically the the Schwarzschild family. They are static.
- Produce dynamics by adding matter!

Oppenheimer-Snyder model

- Matter content: Homogenous dust ($\rho(t)$, $p = 0$)

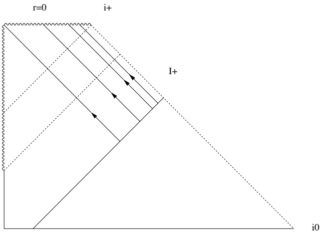
$$T_{\mu\nu} = \text{diag}(\rho, 0, 0, 0)$$

- Einstein equations reduces to an ODE.
- Collapses into a black-hole. Penrose diagram:



Christodoulou's inhomogeneous dust ball

- Inhomogeneous dust $\rho(T, R)$. Lead to a PDE.
- Exists a non-empty set of initial conditions which lead to Black Hole formation
- Exists a non-empty set of initial conditions giving rise to Naked Singularities



- $\mathcal{I}^+$ is incomplete. The affine length is finite.
- Dust Pathologies. Leads to singularities even in flat Minkowski spacetime.
- Well behaved (reasonable) matter models: Scalar field, Vlasov...
- *Weak Cosmic Censorship* is the conjecture that for generic asymptotically flat initial data for reasonable Einstein-matter system, the maximal globally hyperbolic development possesses a complete $\mathcal{I}^+$ (future null infinity).

Christodoulou's framework for the Einstein-scalar field system ($\Lambda = 0$)

- Scalar field

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\lambda\sigma} \left(\partial_\lambda \phi\right) \left(\partial_\sigma \phi\right)$$

- Sphericall symmetry. Bondi coordinates (u, r, θ, φ).
- Main variable:

$$h := \frac{\partial}{\partial r} \left(r\phi\right)$$

then

$$\phi = \bar{h} = \frac{1}{r} \int_0^r h\left(u, s\right) ds \quad \text{or} \quad \frac{\partial \phi}{\partial r} = \frac{\partial \bar{h}}{\partial r} = \frac{h - \bar{h}}{r}$$

where the bar denotes the mean value between 0 and r.

- Small data: dispersive solution, asymptotically Minkowski (CMP86)

$$|h(u, r)| \lesssim \frac{1}{(1 + r + u)^3} \quad , \quad |\partial_r h(u, r)| \lesssim \frac{1}{(1 + r + u)^4}$$

- Large Data: Collapse into a black hole [] *If the final Bondi mass is finite then the exterior is asymptotically schwarzschild*

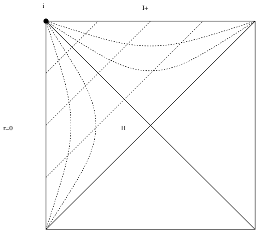
$$r \leq 2M_f$$

Spherical linear waves in de Sitter spacetime (arxiv:1107.0802v1[gr-qc])

- De Sitter metric in Bondi coordinates (u, r, θ, φ):

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right)du^2 - 2dudr + r^2 d\sigma^2$$

- *Cosmological horizon* : $g_{uu} = 0 \Rightarrow r_c = \sqrt{\frac{3}{\Lambda}}$
- *Local region*: $0 \leq r < r_c$
- *Cosmological region*: $r_c < r < \infty$
- Penrose diagram:



The evolution equation

- Wave equation: First-order PDE

$$\left[\frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right)\frac{\partial}{\partial r}\right] h(u,r) = -\frac{\Lambda}{3}(h - \bar{h})$$

- With initial data given in a future light cone

$$h(u = 0, r) = h_0(r)$$

- The integral lines of D (infalling light rays) are the characteristics, i.e, the solutions to

$$\frac{dr}{du} = -\frac{1}{2}\left(1 - \frac{\Lambda}{3}r^2\right)$$

- In the *local region* $(r < r_c)$:

$$r(u) = \sqrt{\frac{3}{\Lambda}}\tanh\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c - u)\right)$$

- In the *cosmological region* $(r > r_c)$:

$$r(u) = \sqrt{\frac{3}{\Lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}(c - u)\right)$$

Existence and uniqueness

- The solution is the fixed point of the operator

$$\mathcal{F}(h)(u_1, r_1) = h_0(r_0) e^{-\frac{\Lambda}{3}\int_0^{u_1} r(v)dv} - \frac{\Lambda}{3}\int_0^{u_1} r(u)\bar{h}(u, r(u)) e^{-\frac{\Lambda}{3}\int_u^{u_1} r(v)dv} du$$

- Given $U, R > 0$, let $\mathcal{C}^0_{U,R}$ denote t$\left(\mathcal{C}^0([0,U]\times[0,R]), \|\cdot\|_{\mathcal{C}^0_{U,R}} < \infty\right)$, where

$$\|f\|_{\mathcal{C}^0_{U,R}} := \sup_{(u,r)\in[0,U]\times[0,R]} |f(u,r)|,$$

- Existence and uniqueness of a fixed point is given by the *contraction mapping theorem*:

$$\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\|_{\mathcal{C}^0_{U,R}} \le \underbrace{\sup_{(u_1,r_1)\in[0,U]\times[0,R]} \left\{1 - e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}\right\}}_{:=\sigma} \|h_1 - h_2\|_{\mathcal{C}^0_{U,R}}$$

  if $0 \le \sigma < 1$.

- $\sigma_{loc}(U) = 1 - \frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{4}, \quad \sigma_{hor}(U) = 1 - e^{-\sqrt{\frac{\Lambda}{3}}U}, \quad \sigma_{cosm}(U,R) = 1 - \frac{3}{\Lambda}\frac{e^{-\sqrt{\frac{\Lambda}{3}}U}}{R^2}$
- Uniqueness garantees the existence of a unique global solution.
- If the data is $\mathcal{C}^k$ then the solution is $\mathcal{C}^k$.

Boundedness in terms of the data

- If $\|h\|_{\mathcal{C}^0} \leq \|h_0\|_{\mathcal{C}^0}$, then

$$|\mathcal{F}(h)(u_1,r_1)| \leq \|h_0\|_{\mathcal{C}^0}\ e^{-\frac{\Lambda}{3}\int_0^{u_1} r(s)ds} + \|\bar{h}\|_{\mathcal{C}^0}\ \frac{\Lambda}{3}\int_0^{u_1} r(v)e^{-\frac{\Lambda}{3}\int_v^{u_1} r(s)ds}dv$$

$$\leq \|h_0\|_{\mathcal{C}^0} \underbrace{\left(e^{-\frac{\Lambda}{3}\int_0^{u_1} r(v)dv} + \frac{\Lambda}{3}\int_0^{u_1} r(u)e^{-\frac{\Lambda}{3}\int_u^{u_1} r(v)dv}du\right)}_{\equiv 1} = \|h_0\|_{\mathcal{C}^0}\ .$$

- Now consider the sequence

$$\begin{cases} h_0(u,r) = h_0(r) \\ h_{n+1} = \mathcal{F}(h_n) \end{cases} .$$

We have already established that, for any $U > 0$ and $R \geq r_c$, h_n converges in $\mathcal{C}^0_{U,R}$ to h, the solution of our problem. Then the above result tells us that

$$\|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_n\|_{\mathcal{C}^0} \leq \|h_0\|_{\mathcal{C}^0}$$

and so

$$\|h\|_{\mathcal{C}^0_{U,R}} = \lim_{n\to\infty} \|h_n\|_{\mathcal{C}^0_{U,R}} \leq \|h_0\|_{\mathcal{C}^0}\ .$$

- We can differentiate

$$D(\partial_r h) = -2\frac{\Lambda}{3} r\,(\partial_r h)\ .$$

- Integrating the last equation along the (ingoing) characteristics,

$$\partial_r h(u_1, r_1) = \partial_r h_0(r_0) e^{-2\frac{\Lambda}{3}\int_0^{u_1} r(s)ds}. \quad (1)$$

  It is then clear that initial data controls the supremum norm of $\partial_r h$.

- Let $d_0 = \|(1+r)^p \partial_r h_0\|_{\mathcal{C}^0}$. If $0 \leq p \leq 4$ and $H \leq 2\sqrt{\Lambda/3}$, then
- Cosmological region:

$$\sup_{(u_1,r_1)\in[0,U]\times[r_c,R]} \left|(1+r_1)^p e^{Hu_1} \partial_r h(u_1, r_1)\right| \leq 2^p (2\alpha+1)^p d_0\ .$$

- local region:

$$\sup_{(u_1,r_1)\in[0,U]\times[0,r_c]} \left|(1+r_1)^p e^{Hu_1} \partial_r h(u_1, r_1)\right| \leq 16 d_0$$

A Price-law: Uniform exponential decay in Bondi time

- Basic estimate:

$$
\begin{aligned}
\left|h(u,r)-\bar{h}(u,r)\right| &\leq \frac{1}{r}\int_0^r |h(u,r)-h(u,s)|\,ds \\
&\leq \frac{1}{r}\int_0^r\int_s^r |\partial_\rho h(u,\rho)|\,d\rho\,ds \\
&\lesssim \frac{1}{r}\int_0^r\int_s^r \frac{e^{-Hu}}{(1+\rho)^p}\,d\rho\,ds \lesssim \begin{cases} \frac{e^{-Hu}}{1+r} & , \quad 2<p\leq 4 \\ re^{-Hu} & , \quad 0\leq p\leq 2 \end{cases} .
\end{aligned}
$$

- From the main evolution equation

$$
\begin{aligned}
|\partial_u h| &= \left|Dh+\frac{1}{2}\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \\
&\leq \left|-\frac{\Lambda}{3}r\left(h-\bar{h}\right)\right| + \frac{1}{2}\left|\left(1-\frac{\Lambda}{3}r^2\right)\partial_r h\right| \lesssim (1+r)^{n(p)}e^{-Hu} ,
\end{aligned}
$$

with

$$
n(p)=\begin{cases} 0 & , \quad 2<p\leq 4 \\ 2 & , \quad 0\leq p\leq 2 \end{cases} .
$$

- Now since $\partial_u h$ is integrable with respect to u, by the fundamental theorem of calculus, the limit exists

$$\lim_{u\to\infty} h(u,r) = \underline{h}(r)\ .$$

- On the other hand

$$\begin{aligned} |\underline{h}(r_2) - \underline{h}(r_1)| &= \lim_{u\to\infty} |h(u,r_2) - h(u,r_1)| \\ &\leq \lim_{u\to\infty} \left| \int_{r_1}^{r_2} |\partial_r h(u,r)| dr \right| \\ &\lesssim \lim_{u\to\infty} |r_2 - r_1| e^{-Hu} = 0\ , \end{aligned}$$

and, consequently, there exists $\underline{h} \in \mathbb{R}$ such that

$$\underline{h}(r) \equiv \underline{h}\ .$$

- Finally

$$\begin{aligned} |h(u,r) - \underline{h}| &\leq \int_u^\infty |\partial_v h(v,r)|\, dv \\ &\lesssim \int_u^\infty (1+r)^{n(p)} e^{-Hv} dv \\ &\lesssim (1+r)^{n(p)} e^{-Hu}\ . \end{aligned}$$

Einstein-Λ scalar field system in Bondi coordinates

- Sphericallly symmetric metric in Bondi coordinates

$$ds^2 = -g(u,r)\tilde{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Omega^2$$

- Cosmological radius r_c when $\tilde{g} = 0$ (it is not a null surface!!)
- Einstein Field Equations, $G_{\mu\nu} = \kappa T_{\mu\nu}$:

$$(rr) \qquad \frac{1}{g}\frac{\partial g}{\partial r} = \frac{\kappa}{2}\frac{\left(h - \bar{h}\right)^2}{r}$$

$$(\theta\theta) \qquad \frac{\partial}{\partial r}(r\tilde{g}) = g\left(1 - \Lambda r^2\right)$$

- Matter equation: $\nabla^\alpha \nabla_\alpha \phi = 0$

$$\left[\frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r}\right] h = \left[\frac{g - \tilde{g}}{2r} - g\frac{\Lambda}{2} r\right]\left(h - \bar{h}\right)$$

- Using the the boundary condition $g(u, r=0) = 1$ we integrate the ODE (rr)

$$g(u,r) = e^{\frac{\kappa}{2}\int_0^r \frac{(h-\bar{h})^2}{s}ds}$$

- Moreover, integrating equation $(\theta\theta)$, which is also an ODE at each u,

$$\tilde{g}(u,r) = \bar{g} - \frac{\Lambda}{r}\int_0^r gs^2 ds$$

assuming $\tilde{g}$ finite. For $\tilde{g} = 0$ this eq. gives a lower bound for $1/\sqrt{\Lambda} < r_c$.

- Then g is a monotonically nondecreasing function of r at each u, and

$$\tilde{g} \leq \bar{g} \leq g, \quad \text{and} \quad 1 \leq g.$$

Also from continuity of g and the fact that $\lim_{r\to 0} r^2 g(\bar{u}, r) = 0$, it follows that

$$\lim_{r\to 0}\tilde{g}(u,r) = \lim_{r\to 0}\bar{g}(u,r) = g(u,0) = 1.$$

The mass equation

- We introduce the local mass function defined by: (Renormalized Hawking mass)

$$m(u,r) = \frac{r}{2}\left(1 - \frac{\tilde{g}}{g} - \frac{\Lambda}{3}r^2\right) \quad (2)$$

which is the mass contained within the sphere of radius r at retarded time u. It is zero at $r = 0$ and positive or zero $m(u,r) \geq 0$, for $r > 0$ if and only if

$$\frac{\tilde{g}}{g} \leq 1 - \frac{\Lambda}{3}r^2 \quad (3)$$

Thus for all u, the cosmological radius, r_c, is bounded by

$$\sqrt{\frac{1}{\Lambda}} < r_c(u) \leq \sqrt{\frac{3}{\Lambda}} \quad (4)$$

in particular $r_c = \sqrt{3/\Lambda}$ if and only if the mass vanishes. So for $0 < r \leq r_c < \sqrt{3/\Lambda}$ the mass is nonnegative and $m < r/2$.

- Differentiating the mass function

$$\frac{\partial m}{\partial r} = \frac{\kappa}{2}\frac{\tilde{g}}{2g}(h-\bar{h})^2$$

$$\frac{\partial m}{\partial u} = -\frac{r}{2}\frac{\partial}{\partial u}\left(\frac{\tilde{g}}{g}\right)$$

and the mass derivative along the incoming light rays is

$$Dm = -\frac{\kappa}{2g}r^2(D\bar{h})^2$$

Thus for $r \leq r_c(u)$, we have $m(u,r)$ is a monotonically nondecreasing function of r at each u, and $m(u,r)$ is monotonically nonincreasing function along the incoming light rays.

- Now, let $\mathbf{v}$ be a vector-field tangent to the curve $r_c(u)$, i.e.

$$\mathbf{v}[\tilde{g}] = 0 \quad \Rightarrow \quad (\partial_u\tilde{g})v^u + (\partial_r\tilde{g})v^r = 0$$

We have that $\partial_r\tilde{g} < 0$ at r_c, then in order to $\mathbf{v}$ be future-oriented we write

$$\mathbf{v} := -(\partial_r\tilde{g})\frac{\partial}{\partial u} + (\partial_u\tilde{g})\frac{\partial}{\partial r}$$

it follows that its norm is given by

$$\mathbf{g}(\mathbf{v},\mathbf{v}) = 2g(\partial_r\tilde{g})(\partial_u\tilde{g}) \leq 0$$

and $\mathbf{v}$ is time-like (or null if $\partial_u\tilde{g} = 0$).

Also

$$\mathbf{v}[r] = \partial_u \tilde{g} = \kappa r (\partial_u \phi)^2 \geq 0$$

and

$$\mathbf{v}[m] = \frac{\kappa}{2} r^2 \frac{(D\bar{h})^2 (\partial_r \tilde{g})}{g} < 0$$

Thus

$$\frac{d}{du} r_c(u) \geq 0$$

and the cosmological horizon is a nondecreasing function of u. This implies that the limit

$$\lim_{u \to \infty} r_c(u) := r_1$$

exists, and $r_1 \leq \sqrt{3/\Lambda}$, while

$$\frac{d}{du} m(u, r_c(u)) \leq 0$$

and $m(u, r_c)$ is monotonically nonincreasing function of u. Then since $m(u, r)$ is nonnegative for $r \leq r_c$, the limit

$$\lim_{u \to \infty} m(u, r_c(u)) := M_1 = \frac{r_1}{2} \left(1 - \frac{\Lambda}{3} r_1^2 \right)$$

exists, with $M_1 < 1/(3\sqrt{\Lambda})$. If $r_1 = \sqrt{3/\Lambda}$ then $M_1 = 0$ and we have asymptotically the trivial solution (de Sitter), otherwise we get a Schwarzschild-de Sitter solution.