STANDARD MODEL - II

LECTURE 1

① Group Theory elements

SU(N) groups $\longrightarrow$ Young Tableau

② Standard Model: Yang-Mills theories, QCD

EW, flavour, mass generation

Higgs mechanism

③ Brief intro to perturbation theory.

Feynman Rules: amplitudes, Xsecs and decay widths

$ee \longrightarrow ee$

④ BSM: Neutrino masses, seesaw mechanisms

- More Higgs bosons $\longrightarrow$ extended scalar sectors
- Leptoquarks $(g-2)_\mu$, $R_K^{(*)}$, $R_D^{(*)}$, $(g-2)_e$

• supersymmetry

For Group Theory

"Group Theory in a Nutshell for Physicists"

A. Zee

Princeton University Press . Princeton and Oxford

General properties of Lie Groups and Lie Algebras

Definition 1

A group G is a set with a map $G \times G \longrightarrow G$ known as group multiplication satisfying the following properties:

- $\boxed{\text{Associativity}}$: $(g \cdot h) \cdot \ell = g \cdot (h \cdot \ell)$, $\forall g, h, \ell \in G$,
- $\boxed{\text{Identity}}$: $\exists_{e \in G} : e \cdot g = g \quad \forall g \in G$,
- $\boxed{\text{Inverse}}$: $\forall_{g \in G} \exists_{g' \in G} : g' \cdot g = g \cdot g' = e$

In particle physics we are interested in $\boxed{\text{group representations}}$, which define a concrete realization of group transformations.

Eg: $\psi \longrightarrow \psi' = U \psi \qquad G$

$\psi \longrightarrow \psi' = e^{i\alpha(x)} \psi \qquad U(1)$

Definition 2

A group representation R is a map that associates to each group element a linear transformation acting on a particular vector space V (real or complex):

$$R : \quad G \longrightarrow \underbrace{GL(V)}_{\text{General Linear Group: Transformations acting on } V.}$$

where $\dim(R) = \dim(V)$

Reducible representation:

R is reducible if and only if there is a subspace $U \subset V$ left invariant by group transformations, i.e.

$$\boxed{R(g)\, u \in U \quad \forall u \in U \text{ and } \forall g \in G.}$$

If there is no UCV, the R is said to be IRREDUCIBLE, or an irrep. of G.

Irreps. are very relevant in particle physics.

Definition 3

A Lie group is a continuous group, i.e., in which all elements $g \in G$ depend continuously on a continuous set of parameters

$$\boxed{g = g(\alpha), \quad \alpha = \{\alpha_a\}, \quad a = 1, \dots, N}$$

The identity element e is taken to be

$$e = g(\alpha)\Big|_{\alpha=0}$$

$\alpha = 0 \longrightarrow$ origin of the space of parameters such that for any R

$$R(\alpha)\Big|_{\alpha=0} = \mathbb{1}$$

Let us Taylor expand $R(\alpha)$ in the vicinity of the identity element:

$$R(\underbrace{d\alpha}) = \mathbb{1} + \frac{\partial R(\alpha)}{\partial \alpha_a}\Big|_{\alpha=0} d\alpha_a + \cdots$$

Infinitesimal change in $\{\alpha_a\}$

$$\equiv \mathbb{1} + i X^a d\alpha_a + \cdots$$

$$\boxed{X^a = -i \frac{\partial}{\partial \alpha_a} R(\alpha)\Big|_{\alpha=0}} \longrightarrow \boxed{\text{GROUP GENERATORS}}$$

$R(d\alpha) \longrightarrow$ Representation of a group element arbitrarily close to identity

Included such that, for Unitary representations one has $\boxed{R^\dagger(\alpha) R(\alpha) = \mathbb{1}}$

Group multiplication allows one to obtain any other element of the group as:

$$\boxed{R(\alpha) = \lim_{K\to\infty}\left(1 + i\frac{X^a \alpha_a}{K}\right)^K = e^{iX^a\alpha_a}}$$

where $d\alpha_a = \lim_{K\to\infty} \frac{\alpha_a}{K}$

Exponential parametrization of group elements

$\boxed{\alpha_a \in \mathbb{R}}$

For unitary representations

$$R^\dagger(\alpha) R(\alpha) = \mathbb{1} \Leftrightarrow e^{-i(X^a)^\dagger \alpha_a} e^{iX^a\alpha_a} = \mathbb{1}$$

$\Rightarrow X^a = (X^a)^\dagger \Rightarrow$ GROUP GENERATORS ARE HERMITIAN OPERATORS

STRUCTURE CONSTANTS

Consider two distinc linear combinations of group generators

$$\alpha_a X^a \neq \beta_a X^a$$

Using the exponential parametrization

$$\left.\underbrace{\overset{R(\alpha)}{e^{i\alpha_a X^a}}}_{\text{group element}}\;\underbrace{\overset{R(\beta)}{e^{i\beta_b X^b}}}_{\text{group element}} = \underbrace{\overset{R(\delta)}{e^{i\delta_a X^a}}}_{\text{group element}}\right\}\ \text{group multiplication}$$

Continuity and differentiability of group elements allows one to find $\{\delta_a\}$ by expanding the above relation.

$$i\delta_a X^a = \log\Big[1 + \underbrace{e^{i\alpha_a X^a}\, e^{i\beta_b X^b} - 1}_{K}\Big] \equiv \log(1+K)$$

For small K one has:

$$\log(1+K) = K - \frac{K^2}{2} + \dots$$

Thus, up to <u>quadratic order</u>

$$K = e^{i\alpha_a X^a} e^{i\beta_b X^b} - 1$$

$$= \left(1 + i\alpha_a X^a - \tfrac{1}{2}(\alpha_a X^a)^2 + \dots\right)\left(1 + i\beta_b X^b - \tfrac{1}{2}(\beta_b X^b)^2 + \dots\right) - 1$$

$$= i\alpha_a X^a + i\beta_a X^a - \alpha_a X^a \beta_b X^b - \frac{1}{2}(\alpha_a X^a)^2 - \frac{1}{2}(\beta_b X^b)^2 + \dots$$

And hence:

$$i\delta_a X^a = i\alpha_a X^a + i\beta_b X^b - \alpha_a X^a \beta_b X^b - \frac{1}{2}(\alpha_a X^a)^2 - \frac{1}{2}(\beta_b X^b)^2$$
$$- \frac{1}{2}\left[(1 + i\alpha_a X^a)(1 + i\beta_b X^b) - 1\right]^2$$

$$= i\alpha_a X^a + i\beta_b X^b - \alpha_a X^a \beta_b X^b - \frac{1}{2}(\alpha_a X^a)^2 - \frac{1}{2}(\beta_b X^b)^2$$
$$- \frac{1}{2}\left[\cancel{1} + i\alpha_a X^a + i\beta_b X^b - \cancel{1} + \dots\right]^2$$

$$= i\alpha_a X^a + i\beta_b X^b \boxed{-\alpha_a X^a \beta_b X^b} - \frac{1}{2}\cancel{(\alpha_a X^a)^2} - \frac{1}{2}\cancel{(\beta_b X^b)^2}$$

$$+ \underbrace{\frac{1}{2}\left(\alpha_a X^a + \beta_b X^b\right)^2}_{\frac{1}{2}\cancel{(\alpha_a X^a)^2} + \frac{1}{2}\cancel{(\beta_b X^b)^2} \boxed{+\frac{1}{2}\alpha_a X^a \beta_b X^b} + \frac{1}{2}\beta_b X^b \alpha_a X^a} + \dots$$

$$= i(\alpha_a + \beta_a)X^a \boxed{-\frac{1}{2}\alpha_a X^a \beta_b X^b} + \frac{1}{2}\beta_b X^b \alpha_a X^a + \dots$$

$$= i(\alpha_a + \beta_a)X^a - \frac{1}{2}\underbrace{\left[\alpha_a X^a, \beta_b X^b\right]}_{\text{generators don't commute in general}} + \dots$$

Then:

$$i\delta_a X^a - i(\alpha_a + \beta_a)X^a = -\frac{1}{2}\left[\alpha_a X^a, \beta_b X^b\right] \iff$$

$$\left[\alpha_a X^a, \beta_b X^b\right] = -2i\left(\delta_c - \alpha_c - \beta_c\right)X^c \equiv i\gamma_c X^c$$

$$\Longrightarrow \quad \left[\alpha_a X^a, \beta_b X^b\right] = i\gamma_c X^c$$

which must hold for any choice of α_a, β_b and γ_c parameters we then note that, choosing the constants

$$\boxed{f^{abc}\alpha_a \beta_b = \gamma_c}$$

$$\alpha_a \beta_b [X^a, X^b] = i \gamma_c X^c \quad \Leftrightarrow$$

$$\cancel{\alpha_a \beta_b} [X^a, X^b] = i f^{abc} \cancel{\alpha_a \beta_b} X^c$$

$$\Rightarrow \boxed{[X^a, X^b] = i f^{abc} X^c}$$

The antisymmetry property of the commutator, i.e., $[A, B] = -[B, A]$

then $\boxed{f^{\boxed{ab}c} = -f^{\boxed{ba}c}}$

Antisymmetric in the first two indices

The f^{abc} are known as the group structure constants and define the Lie Algebra $\boxed{\mathfrak{g}(G)}$ of the group G described by the above commutator $[X^a, X^b] = i f^{abc} X^c$

$$[\sigma_i, \sigma_j] = \sum_k i \varepsilon_{ijk} \sigma_k$$

The structure constants are an intrinsic property of the Lie Algebra and are solely determined by the group multiplication and by continuity.

Proposition 1

For unitary group representations the structure constants are real.

Demonstration

We have seen above that, for unitary group representations the generators are hermitian, i.e,

$$(X^a)^\dagger = X^a.$$

Then, on one hand we have:

$$[X^a, X^b]^\dagger = (i f^{abc} X^c)^\dagger = -i (f^{abc})^* (X^c)^\dagger$$
$$= -i (f^{abc})^* X^c$$

on the other hand we know that

$$[\chi^a,\chi^b]^\dagger = (\chi^a\chi^b - \chi^b\chi^a)^\dagger$$

$$= (\chi^a\chi^b)^\dagger - (\chi^b\chi^a)^\dagger$$

$$= (\chi^b)^\dagger(\chi^a)^\dagger - (\chi^a)^\dagger(\chi^b)^\dagger$$

$$= \chi^b\chi^a - \chi^a\chi^b$$

$$= [\chi^b,\chi^a] = i f^{bac}\chi^c$$

$$= -i f^{abc}\chi^c$$

then :

$$-i(f^{abc})^*\chi^c = -i f^{abc}\chi^c \implies$$

$$(f^{abc})^* = f^{abc} \implies f^{abc} \in \mathbb{R}$$

$\boxed{///}$

Exercise: prove the Jacobi identity.

$$[X^a,[X^b,X^c]]+[X^b,[X^c,X^a]]+[X^c,[X^a,X^b]]=0$$

Adjoint Representation

The structure constants can be used to define the adjoint rep as

$(\sigma^i)^{jk}=i\,\varepsilon^{ijk}$

$$(T^a)^{bc}\equiv i f^{abc}$$

which are $N\times N$ matrices

Proposition 2

The T^a matrices satisfy the commutation relation of the corresponding Lie Algebra, i.e.

$$[T^a,T^b]=i f^{abc}T^c$$

Demonstration

Let's expand the commutator:

$$[T^a, T^b]^{cd} = (T^a T^b - T^b T^a)^{cd}$$

$$= (T^a)^{ce} (T^b)^{ed} - (T^b)^{ce} (T^a)^{ed}$$

$$= i f^{ace} \, i f^{bed} - i f^{bce} \, i f^{aed}$$

$$= - f^{ace} f^{bed} + f^{bce} f^{aed}$$

Now note that, for the generators

$$[X^a, [X^b, X^c]] = [X^a, i f^{bcd} X^d]$$

$$= i f^{bcd} [X^a, X^d]$$

$$= - f^{bcd} f^{ade} X^e$$

Using the Jacobi identity :

$$[X^a,[X^b,X^c]]+[X^b,[X^c,X^a]]+[X^c,[X^a,X^b]]=0$$

$$-\left(f^{bcd}f^{ade}+f^{cad}f^{bde}+f^{abd}f^{cde}\right)X^e=0$$

$$\Rightarrow f^{bcd}\,f^{\boxed{ade}}+f^{cad}\,f^{\boxed{bde}}+f^{abd}f^{cde}=0$$

$$d\longleftrightarrow e \qquad d\longleftrightarrow e \qquad d\longleftrightarrow e$$

$$\Rightarrow f^{bce}f^{aed}+f^{cae}f^{bed}+f^{abe}f^{ced}=0$$

$$\Rightarrow f^{bce}f^{aed}-f^{ace}f^{bed}=-f^{abe}f^{ced}$$

Then:

$$[T^a,T^b]^{cd}=-f^{abe}f^{ced}$$

$$=f^{abe}f^{ecd}$$

$$= i f^{abe} (T^e)^{cd}$$

$$\Rightarrow \boxed{[T^a, T^b] = i f^{abc} T^c}$$

$\blacksquare$

Definition 4

A sub-algebra $\mathcal{A} \subset \mathfrak{g}(G)$ is a linear space defined as

$$\forall X, Y \in \mathcal{A} \quad [X, Y] \in \mathcal{A}$$

- A sub-algebra is abelian if

$$\forall X, Y \in \mathcal{A} \quad [X, Y] = 0$$

(the same applies to the full $\mathfrak{g}(G)$)

- A sub-algebra is an ideal if:

$$\boxed{\forall x \in \mathfrak{h} \ \forall z \in \mathfrak{L}(G) \quad [X, Z] \in \mathfrak{h}}$$

and a proper ideal if, in addition

$$\mathfrak{h} \neq \mathfrak{L}(G), \{0\}$$

→ an ideal can be thought as a subgroup of G.

→ Proper group: A group is denoted improper if its subgroups are just itself and the identity. All other groups are called proper.

Definition 5

A Lie Algebra is simple if it does not contain any proper ideals and semi-simple if it does not contain abelian ideals exept $\{0\}$.

Theorem 1:

A Lie algebra is semi-simple if and only if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$$

where $\mathfrak{g}_i$, $i = 1, \dots, N$ are simple algebras

Proof:

This is not a full proof but instead a generic example to illustrate the general properties:

Consider the product group G

$$\boxed{G = G_1 \times G_2}$$

where the Lie algebras $\mathfrak{g}(G_1)$ and $\mathfrak{g}(G_2)$ are simple, i.e., they do not contain any proper ideal.

Using a matrix form, a generic element of G can be written as

$$g = \begin{pmatrix} \underset{}{\overset{\in G_1}{g_1}} & 0 \\ 0 & \underset{\in G_2}{g_2} \end{pmatrix} \in G$$

This is diagonal due to the linear independence of G_1 and G_2

such that the elements of the Lie algebra read as:

$$T = \begin{pmatrix} \overset{\in \mathfrak{L}_1(G_1)}{T_1} & 0 \\ 0 & \underset{\in \mathfrak{L}_2(G_2)}{T_2} \end{pmatrix} \in \mathfrak{L}(G) .$$

Consider now the sub-algebra $\mathfrak{L}_1(G_1)$ with elements:

$$T' = \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{L}_1(G_1)$$

The commutator of a generic element of $\mathfrak{L}(G)$ with a generic element of $\mathfrak{L}_1(G_1)$ is:

$$[T, T'] = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T_1' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

$$= \begin{pmatrix} T_1 T_1' & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T_1' T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} T_1'' & 0 \\ 0 & 0 \end{pmatrix},$$

with $T_1'' = [T_1, T_1'] \in \overbrace{\mathfrak{g}_1(G_1)}^{\text{plays the role of } \mathfrak{A}} \neq \mathfrak{g}(G)$

- A sub-algebra is <u>an ideal</u> if:

$$\boxed{\forall x \in \mathfrak{A}\ \forall z \in \mathfrak{g}(G) \quad [x, z] \in \mathfrak{A}}$$

and <u>a proper ideal</u> if, in addition $\mathfrak{A} \neq \mathfrak{g}(G), \{0\}$

- A Lie Algebra is simple if it does not contain any <u>proper ideals</u>
- and semi-simple if it does not contain abelian ideals

Thus, the elements of $\mathfrak{g}_1(G_1)$ constitute a proper ideal and therefore $\mathfrak{g}(G)$ cannot be simple.

Similar reasoning for $\mathfrak{g}_2(G_2)$ shows that it also forms a proper ideal.

Furthermore, since $\mathfrak{g}_1(G_1)$ and $\mathfrak{g}_2(G_2)$ are <u>simple</u> they <u>do not contain</u> proper ideals, which means that $\mathfrak{g}_1(G_1)$ and $\mathfrak{g}_2(G_2)$ are the only ideals of $\mathfrak{g}(G)$.

In conclusion, provided that $\mathfrak{g}_1(G_1)$ and $\mathfrak{g}_2(G_2)$ are not abelian, then

$$\boxed{\mathfrak{g}(G) = \mathfrak{g}_1(G_1) \oplus \mathfrak{g}_2(G_2)}$$

is semi-simple!

Definition 6

The Killing form or Killing metric associated with a Lie algebra $\mathfrak{g}(G)$ is a symmetric bilinear map:

$$\Gamma : \mathfrak{g}(G) \times \mathfrak{g}(G) \longrightarrow \mathbb{R}$$

defined by:

$$\Gamma(X, Y) = \mathrm{Tr}\left[ad(X) \cdot ad(Y) \right]$$

with $ad(X)$ the generators of $\mathfrak{g}(G)$ in the adjoint representation.

Considering a basis of generators $\{T^a\}$ for the Lie algebra one obtains the $\boxed{\text{Killing metric}}$ as:

$$\gamma^{ab} = \Gamma(T^a, T^b) = \mathrm{Tr}\left(T^a T^b\right) = (T^a)^{cd} (T^b)^{dc}$$

$$= i f^{acd}\, i f^{bdc} = - f^{acd} f^{bdc}$$

Theorem 2

If $\mathfrak{g}(G)$ is a compact Lie algebra, i.e., if the underlying Lie group is compact, the Killing metric is positive semi-definite

$$\Gamma(X, X) \geq 0, \quad \forall X \in \mathfrak{g}(G)$$

and $\Gamma(X,X) > 0, \; \forall X \in \mathfrak{g}(G)$ for simple algebras

The proof is out of the scope of this course.

It follows from Theorem 2 that if the Killing metric is positive definite then, one can choose an orthonormal basis for the Lie algebra such that

$$\boxed{\Gamma(T^a, T^b) = \delta^{ab}}$$

which means that the Killing metric can be diagonalized and an appropriate normalization for the generators can be chosen!

This leads to the following proposition!

Proposition 3

The structure constants are $\boxed{\text{completely antisymmetric}}$

Demonstration

$$\Gamma([T^a,T^b],T^c) + \Gamma(T^b,[T^a,T^c]) =$$

$$\mathrm{Tr}\left[(T^aT^b - T^bT^a)T^c\right] + \mathrm{Tr}\left[T^b(T^aT^c - T^cT^a)\right] =$$

$$\mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^bT^aT^c) + \mathrm{Tr}(T^bT^aT^c) - \mathrm{Tr}(T^bT^cT^a) =$$

using cyclic property of the trace

$$\mathrm{Tr}\cancel{(T^aT^bT^c)} - T\cancel{(T^aT^cT^b)} + \mathrm{Tr}\cancel{(T^aT^cT^b)} - \mathrm{Tr}\cancel{(T^aT^bT^c)} = 0$$

Then:

$$\Gamma(i f^{abe} T^e, T^c) + \Gamma(T^b, i f^{acd} T^d) =$$

$$i f^{abd}\,\underbrace{Tr(T^d T^c)}_{\delta^{dc}} + i f^{acd}\,\underbrace{Tr(T^b T^d)}_{\delta^{bd}} =$$

$$i f^{abd}\delta^{dc} + i f^{acd}\delta^{bd} =$$

$$i f^{abc} + i f^{acb} = 0 \implies$$

$$f^{a\boxed{bc}} = - f^{a\boxed{cb}}$$

which shows antisymmetry with respect to the last two indices.

Since, by definition, the structure constants are also antisymmetric with respect to the first two indices, this implies that they are <u>totally antisymmetric</u>

$\blacksquare$

Theorem 3

For a semi-simple Lie algebra the quadratic Casimir Operator

$$C \equiv \gamma^{ij} T_i T_j \quad , \quad \gamma^{ij} = \gamma_{ij}^{-1}$$

commutes with all generators:

$$[C, T_k] = 0 \quad , \quad \forall T_k \in \mathcal{L}(G)$$

Demonstration

$$[C, T_l] = [\gamma^{ij} T_i T_j, T_l]$$

$$= \gamma^{ij} [T_i T_j, T_l]$$

$$= \gamma^{ij} \left(T_i [T_j, T_l] + [T_i, T_l] T_j \right)$$

$$= i\gamma^{ij} \left(T_i f_{jlm} T_m + f_{ilm} T_m T_j \right)$$

$$= i\gamma^{ij} f_{jlm} T_i T_m + i\gamma^{ij} f_{ilm} T_m T_j$$

$$= i\gamma^{ij} f_{jlm} T_i T_m + i\underbrace{\gamma^{ji}}_{=\gamma^{ij} \to \text{symmetric}} f_{jlm} T_m T_i$$

$$= i \underbrace{f_{jlm}}_{\substack{\text{antisymmetric} \\ \text{under } j \leftrightarrow m \\ \text{interchange}}} \gamma^{ij} \underbrace{\left(T_i T_m + T_m T_i \right)}_{\substack{\text{symmetric under } j \leftrightarrow m \\ \text{interchange}}}$$

$$= 0$$

$\blacksquare$

The example of SU(2)

$\boxed{\text{The group of } 2 \times 2 \text{ unitary matrices with unit determinant.}}$