STANDARD MODEL - I

LECTURE 6

$$\partial_\mu j^\mu = 0 \quad , \quad \partial_0 j^0 = 0 \Rightarrow$$

$$\Rightarrow \frac{dQ}{dt} = \int d^3x \, \frac{dj^0}{dt} = 0$$

internal symmetries

② $\boxed{\text{Space-time symmetries}}$

start with pure translations:

$$\Psi'(x') = S\,\Psi(x)\,, \qquad S = e^{s} = 1 + s + O(s^2)$$

$$\Psi'(x') \simeq \Psi(x) + s\,\Psi(x)$$

$$= \underbrace{\Psi(x') - \Psi(x')}_{\delta} + \Psi(x) + s\,\Psi(x)$$

$$= \Psi(x') - \left[\Psi(x') - \Psi(x)\right] + s\,\Psi(x)$$

$$= *$$

Definition of derivative

$$\partial_\nu \Psi(x) = \frac{\Psi(x') - \Psi(x)}{\delta x^\nu}$$

$$\circledast = \Psi(x') - \delta x^\nu \partial_\nu \Psi(x) + s\,\Psi(x) \simeq \Psi'(x')$$

$$\Rightarrow \underbrace{\Psi'(x') - \Psi(x')}_{\delta\Psi} \simeq -\delta x^\nu \partial_\nu \Psi(x) + s\,\Psi(x)$$

$$\Rightarrow \delta\Psi \simeq -\delta x^\nu \partial_\nu \Psi(x) + s\,\Psi(x)$$

To first order and making $\delta x^\nu = a^\nu$

$$\Longrightarrow \quad \boxed{\delta \Phi = -a^{\nu}\partial_{\nu}\Phi(x)}$$

$$\delta \mathcal{L} = \sum_i \left[\frac{\partial \mathcal{L}}{\partial \Phi^i}\,\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)}\,\delta(\partial_\mu \Phi^i) \right] = 0$$

$$\Longrightarrow \ \dots \quad \partial_\mu j^\mu = 0 \longrightarrow$$ It was done for internal symmetries, i.e., transformations in the field space.

For space-time transformations one has to consider the coordinates:

$$\delta \mathcal{L} = \sum_i \Bigg[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi^i}}_{\displaystyle \shortparallel \atop \displaystyle \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}} \delta \phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta(\partial_\mu \phi^i) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu \Bigg] = 0$$

(From E.L. Eqs.)

I will drop the $\sum_i$!!!

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta \phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \delta(\partial_\mu \phi^i) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu$$

$$+ \mathcal{L} \underbrace{\partial_\mu \delta x^\mu}_{=0} = 0$$

$=0$ because $\delta x^\mu \equiv a^\mu$ is a constant.

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \delta\varphi^i + \mathcal{L}\, \delta x^\mu \right) - \underbrace{\boxed{\mathcal{L}\, \partial_\mu \delta x^\mu}}_{=0} = 0$$

$$\Rightarrow \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \delta\varphi^i + \mathcal{L}\, \delta x^\mu \right) = 0}$$

- $\delta\varphi^i = -a^\nu \partial_\nu \varphi^i \,, \quad \delta x^\mu = a^\mu$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} (-a^\alpha \partial_\alpha \varphi^i) + \mathcal{L}\, a^\mu \right) = 0 \quad \Longrightarrow$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} a^\alpha \partial_\alpha \varphi^i - \mathcal{L} a^\alpha \delta^\mu_\alpha \right) = 0 \Longrightarrow$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \partial_\alpha \varphi^i - \mathcal{L} \delta^\mu_\alpha \right) a^\alpha = 0 \Longrightarrow$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \partial^\nu g_{\nu\alpha} \varphi^i - \mathcal{L} g^{\mu\nu} g_{\nu\alpha} \right) a^\alpha = 0 \Longrightarrow$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \partial^\nu \varphi^i - \mathcal{L} g^{\mu\nu} \right) \underbrace{a^\alpha g_{\nu\alpha}}_{a_\nu} = 0$$

Conservation Law :

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \partial^\nu \Psi^i - \mathcal{L} g^{\mu\nu} \right) = 0 \implies$$

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

STRESS ENERGY-MOMENTUM TENSOR !

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \partial^\nu \Psi^i - \mathcal{L} g^{\mu\nu}}$$

The conserved charges are the Energy and linear momentum:

$$\mathcal{H} = T^{00} \quad \text{Hamiltonian density}$$

$$H = \int d^3x \, T^{00}$$

$$\partial_0 T^{00} = 0$$

$$\frac{dH}{dt} = 0$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i} \dot{\varphi}^i - \mathcal{L}$$

$$\dot{\varphi}^i = \partial_0 \varphi^i$$

$$\mathcal{P}^{k} = T^{0k} \quad \text{Linear momentum density}$$

$$T^{0k} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^i)} \partial^k \varphi^i - \mathcal{L} \overbrace{g^{0k}}^{=0}$$

Pure Lorentz transformations

$$\Lambda = e^{\omega} = \mathbb{1} + \omega + O(\omega^2)$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \left[\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} + O(\omega^{\mu}{}_{\alpha}\omega^{\alpha}{}_{\nu})\right] x^{\nu}$$

(To 1st order) $\simeq x^{\mu} + \delta x^{\mu} \quad , \quad \delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu}$

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \implies$$

$$g_{\alpha\beta} = g_{\mu\nu} (\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta})$$

$$= (g_{\alpha\nu} + \omega_{\nu\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta})$$

$$= g_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha} + \cancel{O(\omega^2)}$$

$$g_{\alpha\beta} = g_{\alpha\beta} + \boxed{(\omega_{\alpha\beta} + \omega_{\beta\alpha})}$$

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} \Longrightarrow \omega \text{ is antisymmetric!}$$

Above we had that:

$$\delta\Psi^{r} \simeq -a^{\mu}\partial_{\mu}\Psi^{s}(x)\delta_{rs} + \tilde{s}_{rs}\Psi^{s}(x)$$

Now we recast it to:

$$\delta\Psi^{r} = -\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\Psi^{s}\delta_{rs} + s_{rs}\Psi^{s} \Longrightarrow$$

$$\delta\Psi^{r} = -\omega_{\alpha\nu}x^{[\nu}\partial^{\alpha]}\Psi^{s}\delta_{rs} + s_{rs}\Psi^{s}$$

$$A^{[ij]} = \frac{1}{2}(A^{ij} - A^{ji})$$

greek indices ⟶ space-time indices

latin indices ⟶ spinor indices

$$s_{rs} = -\frac{i}{2}\,\omega_{\alpha\beta}\,S^{\alpha\beta}_{rs}$$

$$\psi^r(x) \longrightarrow \psi'^r(x') = S_{rs}\,\psi^s(x) - \frac{i}{2}\,\omega_{\mu\nu}\,S^{\mu\nu}_{rs}\,\psi^s(x)$$

with $$S^{\mu\nu}_{rs} = \frac{1}{2}\,\sigma^{\mu\nu}_{rs} = \frac{i}{4}\left[\gamma^\mu,\gamma^\nu\right]_{rs}$$

- $$\boxed{\delta x^\mu} = \omega^\mu{}_\nu\, x^\nu = \omega_{\alpha\nu}\, x^{[\nu} g^{\mu\alpha]}$$
- $$\boxed{\delta \underline{\psi}^r} = -\omega_{\alpha\nu}\, x^{[\nu}\partial^{\alpha]}\underline{\psi}^s\,\delta_{rs} + s_{rs}\,\underline{\psi}^s$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \boxed{\delta \Psi^i} + \mathcal{L} \boxed{\delta x^\mu} =$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^r)} \Big(-\underbrace{\omega_{\alpha\nu}}_{=-\omega_{\nu\alpha}} x^{[\nu}\partial^{\alpha]} \Psi^s \delta_{rs} \Big) + \mathcal{L}\, \underbrace{\omega_{\alpha\nu}}_{=-\omega_{\nu\alpha}} x^{[\nu} g^{\mu\alpha]}$$

$$- \frac{i}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^r)} \omega_{\nu\alpha} S^{\nu\alpha}_{rs} \Psi^s \overset{\text{Using antisymmetry of } \omega_{\nu\alpha}}{=}$$

$$\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^r)} \left(x^{[\nu}\partial^{\alpha]} \Psi^s \delta_{rs} \right) - \mathcal{L}\, x^{[\nu} g^{\mu\alpha]} - \frac{i}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^r)} S^{\nu\alpha}_{rs} \Psi^s \right] \omega_{\nu\alpha}$$

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^i)} \partial^\nu \Psi^i - \mathcal{L}\, g^{\mu\nu}}$$

$$x^{[\nu}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^r)}\partial^{\alpha]}\psi^s\delta_{rs} - \mathcal{L}\, g^{\mu\alpha]}\right)}_{T^{\mu\alpha]}}\omega_{\nu\alpha}$$

$$-\frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^r)}S^{\nu\alpha}_{rs}\psi^s\omega_{\nu\alpha} =$$

$$x^{[\nu}T^{\mu\alpha]}\omega_{\nu\alpha} - \frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^r)}S^{\nu\alpha}_{rs}\psi^s\omega_{\nu\alpha} =$$

$$\left(x^\nu T^{\mu\alpha} - x^\alpha T^{\mu\nu} - i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^r)}S^{\nu\alpha}_{rs}\psi^s\right)\omega_{\nu\alpha}$$

Conserved Noether current :

$$\partial_\mu \left(x^\nu T^{\mu\alpha} - x^\alpha T^{\mu\nu} - i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^r)} S^{\nu\alpha}_{rs} \psi^s \right) = 0$$

$$\boxed{\partial_\mu M^{\mu\nu\alpha} = 0} \quad , \quad M^{\mu\nu\alpha} = l^{\mu\nu\alpha} + s^{\mu\nu\alpha}$$

- $l^{\mu\nu\alpha} = x^\nu T^{\mu\alpha} - x^\alpha T^{\mu\nu}$

orbital angular - momentum current density

$$x^\nu T^{00} - x^0 T^{0\nu}$$
$$= x^\nu \mathcal{H} - t p^\nu$$

$$\cdot\ s^{\mu\nu\alpha} = i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^r)} S^{\nu\alpha}_{rs} \psi^s$$

spin current density

$$j^0, \quad T^{00}, \quad P^k, \quad l^{0\nu\alpha}, \quad s^{0\nu\alpha}$$

see Eur. Phys. J. C (2018) 78:785

$$M^{0\alpha 0}$$

$$M^{012} = l^{012} + s^{012} = \underbrace{\overset{p^1}{T^{01}} x^2 - \overset{p^2}{T^{02}} x^1}_{L_z} + \underset{\downarrow\, S_z}{s^{012}}$$

SYMMETRY BREAKING!

Pure scalar theory

$$\mathcal{L} = [\partial^\mu \phi]^* [\partial_\mu \phi] - \boxed{V(\phi^*, \phi)}$$

$$V(\phi^*, \phi) = \frac{\lambda}{4}(\phi^* \phi)^2 + \mu^2 \phi^* \phi$$

U(1) global invariance

$$\mathcal{L}(\phi) = \mathcal{L}'(\phi e^{i\alpha})$$

$$\phi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right)$$

Let's study all possibilities for the signs of λ, μ^2

a) $\lambda > 0$ and $\mu^2 > 0$

b) $\lambda > 0$ and $\mu^2 < 0$

c) $\lambda < 0$ and $\mu^2 > 0$

d) $\lambda < 0$ and $\mu^2 < 0$

1. Stationarity conditions:

$$\phi^* \phi = \frac{1}{2}(\varphi_1^2 + \varphi_2^2)$$

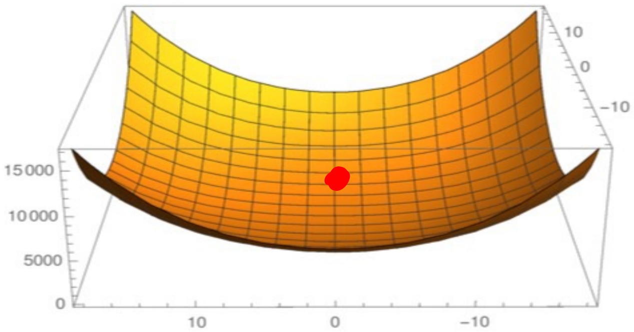
$$(\phi^* \phi)^2 = \frac{1}{4}(\varphi_1^4 + \varphi_2^4 + 2\varphi_1^2 \varphi_2^2)$$

$$\begin{cases} \dfrac{\partial V}{\partial \varphi_1} = \mu^2 \varphi_1 + \dfrac{1}{4}\lambda \varphi_1^3 + \dfrac{1}{4}\lambda \varphi_1 \varphi_2^2 = 0 \\ \dfrac{\partial V}{\partial \varphi_2} = \mu^2 \varphi_2 + \dfrac{1}{4}\lambda \varphi_2^3 + \dfrac{1}{4}\lambda \varphi_2 \varphi_1^2 = 0 \end{cases}$$

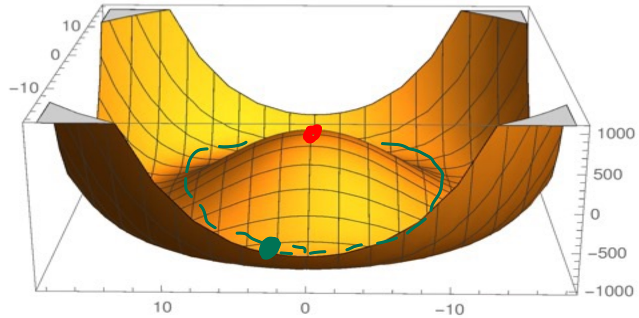
$$\varphi_1 = \varphi_2 = 0 \quad \text{or} \quad \mu^2 = -\frac{1}{4}\lambda(\varphi_1^2 + \varphi_2)^2$$

$$\begin{cases} \varphi_1 = v\cos\alpha \\ \varphi_2 = v\sin\alpha \end{cases} \longrightarrow \mu^2 = -\frac{1}{4}\lambda v^2$$

STABLE



(a) $\lambda > 0,\ \mu^2 > 0$

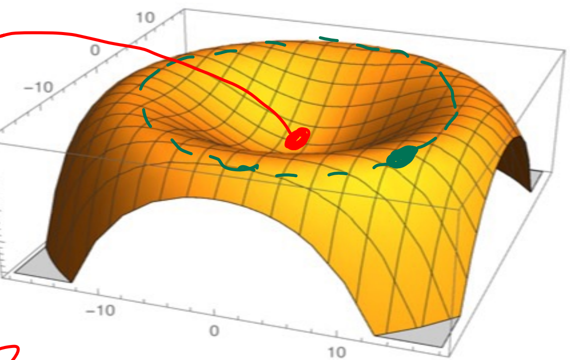


(b) $\lambda > 0,\ \mu^2 < 0$

Unstable

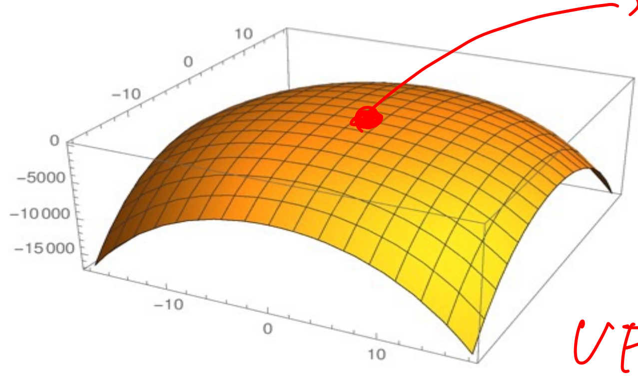
Unstable or metastable

C



UFB

(c) $\lambda < 0,\ \mu^2 > 0$



UFB

(d) $\lambda < 0,\ \mu^2 < 0$

Mass spectrum

$$\partial^2_{ij} V = \begin{pmatrix} \dfrac{\partial^2 V}{\partial \phi_1^2} & \dfrac{\partial^2 V}{\partial \phi_2 \partial \phi_1} \\ \dfrac{\partial^2 V}{\partial \phi_1 \partial \phi_2} & \dfrac{\partial^2 V}{\partial \phi_2^2} \end{pmatrix} \equiv \left[M^2 \right]_{ij}$$

$$M^2 = \begin{pmatrix} \mu^2 + \frac{3}{4}\lambda \phi_1^2 + \frac{1}{4}\lambda \phi_2^2 & \frac{1}{2}\lambda \phi_1 \phi_2 \\ \frac{1}{2}\lambda \phi_1 \phi_2 & \mu^2 + \frac{1}{4}\lambda \phi_1^2 + \frac{3}{4}\lambda \phi_2^2 \end{pmatrix}$$

$$\phi_i \longrightarrow \varphi_i$$

For case a) $\lambda > 0$, $\mu^2 > 0$

$$M^2\Big|_{\varphi_1=\varphi_2=0} = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^2 \end{pmatrix}$$

$$\Longrightarrow m_{\varphi_1}^2 = m_{\varphi_2}^2 = \mu^2 > 0 \quad \text{MINIMUM}$$

$$\Longrightarrow m_\phi^2 = \mu^2$$

For case a) $\varphi_1 = \varphi_2 = 0$, $\lambda < 0$, $\mu^2 < 0$

$$M^2\Big|_{\varphi_1=\varphi_2=0} = \begin{pmatrix} -|\mu^2| & 0 \\ 0 & -|\mu^2| \end{pmatrix}$$

$$\Longrightarrow \quad m_{\varphi_1}^2 = m_{\varphi_2}^2 = -|\mu^2| < 0$$

Case b) $\lambda > 0$, $\mu^2 < 0$

$$\varphi_1^2 + \varphi_2^2 = v^2 > 0, \quad \mu^2 = -\frac{1}{4}\lambda(\varphi_1^2 + \varphi_2^2)$$

$$M^2\Big|_{\mu^2=-\frac{1}{4}\lambda(\varphi_1^2+\varphi_2^2)} = \begin{pmatrix} \frac{1}{2}\lambda\varphi_1^2 & \frac{1}{2}\lambda\varphi_1\varphi_2 \\ \frac{1}{2}\lambda\varphi_1\varphi_2 & \frac{1}{2}\lambda\varphi_2^2 \end{pmatrix}$$

② To obtain the physical mass spectrum we need to diagonalize M^2

Gauge basis

mass basis

$$M^2 \longrightarrow m^2 = \begin{pmatrix} \frac{1}{2}\lambda(\varphi_1^2+\varphi_2^2) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$m_{\varphi_1}^2 = \frac{1}{2}\lambda v^2, \quad m_{\varphi_2}^2 = 0$$

φ_1 and φ_2 are eigenvectors of the Gauge basis

$\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are eigenvectors of the mass (or physical) basis.

Let's consider quantum fluctuations about the minimum v:

quantum fluctuations

$$\varphi_1(x) = v + h(x) \quad \text{radial}$$

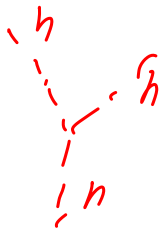
$$\varphi_2 = \sigma(x) \quad \text{angular}$$

$$\Rightarrow \phi(x) = \frac{1}{\sqrt{2}}\left(v + h(x) + \sigma(x)\right)$$

Using $\mu^2 = \frac{1}{4}\lambda v^2$ for $\mu^2 < 0$ and replacing $\phi(x)$ in $V(\phi^*, \phi)$:

$$V(h,\sigma) = \boxed{\frac{1}{16}\lambda\sigma^4} + \boxed{\frac{1}{8}\lambda\sigma^2 h^2} + \frac{1}{16}\lambda h^4 + \boxed{\frac{1}{4}\lambda v h \sigma^2}$$

$$+ \frac{1}{4}\lambda v h^3 + \underbrace{\boxed{\frac{1}{4}\lambda v^2 h^2}}_{\frac{1}{2}m_h^2} - \frac{1}{16}\lambda v^4$$



$\sigma \longrightarrow -\sigma \implies \mathbb{Z}_2$ symmetry

$U(1)_{global} \longrightarrow \mathbb{Z}_2$

$$\underbrace{m_h^2 = \frac{1}{2}\lambda v^2}_{\text{Higgs-like boson}} \quad , \quad \underbrace{m_\sigma^2 = 0}_{\text{Goldstone boson}}$$