Standard Model I: Hand-in exercises sheet 2

December 8, 2021

6 points

1. Given a Dirac spinor ψ, and defining $\bar{\psi} = \psi^\dagger \gamma^0$, show that $\bar{\psi}\psi \equiv \bar{\psi}^\alpha \psi_\alpha$ is Lorentz invariant while $\psi^\dagger \psi$ is not. Show also that $\bar{\psi}\gamma^\mu \psi$ is Lorentz invariant.

6 points

2. Consider the Dirac spinor Ψ expressed in terms of 2-component left-handed (LH) Weyl spinors ψ and ξ as:

$$\Psi \equiv \begin{pmatrix} \psi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\xi}^{\dot{\beta}} \end{pmatrix} . \tag{1}$$

Show that, up to a surface term, the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi} \left(i\gamma^\mu \partial_\mu - m \right) \Psi$$

can be written in terms of LH Weyl spinors as:

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi - m \left(\chi\psi + c.c. \right)$$

where we define the product

$$\chi\psi \equiv \chi^T \left(i\sigma^2 \right) \psi \equiv \chi_\alpha \epsilon^{\alpha\beta} \psi_\beta = \chi_2 \psi_1 - \chi_1 \psi_2 \quad ,$$

and where we have $\chi = i\sigma^2 \xi$.

Note: Use the Weyl representation of the gamma matrices.

8 points

3. Consider a theory invariant under a $\mathrm{U}(1)_\mathrm{A} \times \mathrm{U}(1)_\mathrm{B}$ symmetry where $\mathrm{U}(1)_\mathrm{A}$ is **local (or gauged)** while $\mathrm{U}(1)_\mathrm{B}$ is **global**. The matter content of this theory is composed by two fermions ψ and χ as well as a complex scalar Φ. The $\mathrm{U}(1)_\mathrm{A} \times \mathrm{U}(1)_\mathrm{B}$ charges are given in Tab. 1. The ψ and χ denote 4-component chiral fermions defined in terms of the Dirac spinor

$$\Psi_\mathrm{D} = \begin{pmatrix} \psi_{\mathrm{L}\,\alpha} \\ \bar{\chi}_\mathrm{L}^{\dot{\beta}} \end{pmatrix}$$

as

$$\psi \equiv P_\mathrm{L} \Psi_\mathrm{D} = \begin{pmatrix} \psi_{\mathrm{L}\,\alpha} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \chi \equiv P_\mathrm{R} \Psi_\mathrm{D} = \begin{pmatrix} 0 \\ \bar{\chi}_\mathrm{L}^{\dot{\beta}} \end{pmatrix}$$

Field	$\mathrm{U}(1)_\mathrm{A}$	$\mathrm{U}(1)_\mathrm{B}$
Φ	q_A	q_B
ψ	q_A	0
χ	0	$-q_\mathrm{B}$

Table 1: *Matter content local A-charges and global B-charges.*

with $P_{\mathrm{L,R}}$ the chiral projectors expressed in the Weyl representation.
Note: ψ and χ in this exercise are 4-component chiral fermions and must not be confused with those in the previous exercise which were 2-component spinors.

a) Taking into account the gauge and global charges as well as the allowed contractions of Lorentz indices α, $\dot{\beta}$, justify why ψ and χ are indeed purely chiral?

b) Write down all allowed terms in the Lagrangian invariant under both the $\mathrm{U}(1)_\mathrm{A} \times \mathrm{U}(1)_\mathrm{B}$ and Lorentz symmetries.

c) Determine the Noëther charge densities of the theory.