STANDARD MODEL I

LECTURE 3

$\vec{p} = 0$: $[H, \vec{\sigma}] = 0$

$\vec{\gamma} = \gamma^0 \vec{\alpha}$

$\gamma^0 \gamma^0 = \mathbb{1}$

- $\vec{p} \neq 0$, $H = \vec{\alpha} \cdot \vec{p} + \gamma^0 m \Leftrightarrow \gamma^0 H = \vec{\gamma} \cdot \vec{p} + m\mathbb{1}$

$$\gamma^0 H\, u^\alpha(p) = (\vec{\gamma} \cdot \vec{p} + m\mathbb{1})\, u^\alpha(p) \quad (*)$$

WE ARE USING THE PAULI-DIRAC REPRESENTATION

$$(*) \quad \begin{pmatrix} E\mathbb{1} & 0 \\ 0 & -E\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} m\mathbb{1} & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & m\mathbb{1} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

u_A, u_B are 2 - spinors

$$\begin{cases} (E\mathbb{1} - m\mathbb{1})\,u_A = \vec{\sigma}\cdot\vec{p}\,u_B \\ (E\mathbb{1} + m\mathbb{1})\,u_B = \vec{\sigma}\cdot\vec{p}\,u_A \end{cases} \Longrightarrow u_A = \frac{(\vec{\sigma}\cdot\vec{p})^2}{E^2 - m^2}\,u_A$$

$$(\vec{\sigma}\cdot\vec{p})^2 = \begin{pmatrix} \sum_i p_i^2 & 0 \\ 0 & \sum_i p_i^2 \end{pmatrix} = \vec{p}^{\,2}\,\mathbb{1}$$

$$\boxed{u_A = \frac{\vec{p}^{\,2}}{E^2 - m^2}\,u_A} \Longrightarrow \vec{p}^{\,2} = E^2 - m^2 \Longrightarrow$$

$$\Longrightarrow \boxed{E = \pm\sqrt{\vec{p}^{\,2} + m^2}}$$

We can then write:

$$\underbrace{u^{(+)}(p) = N \begin{pmatrix} u_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A \end{pmatrix}}_{E>0}\ , \qquad \underbrace{u^{(-)}(p) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\, u_B \\ u_B \end{pmatrix}}_{E<0}$$

Lets give to $u_{A,B}$ the eigenvectors of the σ^3 Pauli matrix

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$E>0$ $$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}, \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$E < 0$

$$u^{(3)} = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + i p_y}{E-m} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)} = N \begin{pmatrix} \frac{p_x - i p_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Identify negative energy particle solutions with positive energy antiparticle solutions using the Feynman-Stuckelberg convention:

$$\begin{cases} v^{(1)}(p) = u^{(4)}(-p) \\ v^{(2)}(p) = u^{(3)}(-p) \end{cases}$$

$-E \longrightarrow E$

$\vec{p} \longrightarrow -\vec{p}$

$E > 0$

$$v^{(2)}(p) = N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad v^{(1)}(p) = N \begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$

antiparticle $\frac{1}{2}$ antiparticle $-\frac{1}{2}$

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}, \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

particle $\frac{1}{2}$ particle $-\frac{1}{2}$

Particles

$$\psi = u^{(1,2)}(p)\, e^{-i p \cdot x}$$

antiparticles

$$\psi = v^{(1,2)}\, e^{i p \cdot x}$$

For the normalization factor N

use $u^{(i)\dagger} u^{(j)} = v^{(i)\dagger} v^{(j)} = 2E\delta^{ij}$

$$N = \sqrt{E+m}$$

Helicity

$$\left[\vec{S}, H\right]\Big|_{\vec{p}\neq 0} \neq 0$$

$$SH = \begin{pmatrix} m\vec{\sigma} & \boxed{\vec{\sigma}(\vec{\sigma}\cdot\vec{p})} \\ \vec{\sigma}(\vec{\sigma}\cdot\vec{p}) & m\vec{\sigma} \end{pmatrix} , \quad HS = \begin{pmatrix} m\vec{\sigma} & \boxed{(\vec{\sigma}\cdot\vec{p})\vec{\sigma}} \\ (\vec{\sigma}\cdot\vec{p})\vec{\sigma} & m\vec{\sigma} \end{pmatrix}$$

The Pauli matrices don't commute therefore

$$[S, H]\Big|_{\vec{p}\neq 0} \neq 0$$

$$\sum_{i,j} \sigma_i \vec{e}_i (\sigma_j \cdot p_j)$$

$$\sum_{i,j} (\sigma_j \cdot p_j)\, \sigma_i \cdot \vec{e}_i = \sum_{i,j} \overbrace{\sigma_j \sigma_i}^{\neq \sigma_i \sigma_j} p_j \vec{e}_i$$

$$[\sigma_i, \sigma_j] = i \sum_k \varepsilon_{ijk} \sigma_k$$

A new operator compatible with the Hamiltonian that reads as:

$$\hat{h} = 2 \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|}$$

Helicity operator

↳ represents the projection of the spin along the momentum direction.

$$\hat{h} = \begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix} \quad , \quad H = \begin{pmatrix} m\mathbb{1} & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m\mathbb{1} \end{pmatrix}$$

$$\hat{h}H = \begin{pmatrix} m\vec{\sigma}\cdot\vec{p} & \vec{p}^{\,2} \\ \vec{p}^{\,2} & m\vec{\sigma}\cdot\vec{p} \end{pmatrix} = H\hat{h} \implies \boxed{[\hat{h}, H] = 0}$$

compatible operators

$$\hat{h}^2 = \frac{1}{|p|^2} \begin{pmatrix} (\vec{\sigma}\cdot\vec{p})^2 & 0 \\ 0 & (\vec{\sigma}\cdot\vec{p})^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues of $\hat{h}$ are the $\boxed{h_\pm = \pm 1}$

$$\hat{h}\, u^{\alpha}(p) = \pm\, u^{\alpha}(p)$$

$$\vec{p} = p_z\, \vec{e}_z$$

$$\hat{h}\, u^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix}$$

Particle

$h = +$

$E > 0$

$$\hat{h}\, u^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p_z}{E+m} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \frac{p_z}{E+m} \end{pmatrix}$$

Particle

$h = -$

$E > 0$

$$\hat{h}\, v^{(1)} = -\, v^{(1)}$$

Antiparticle

$h = -$

$E > 0$

$\hat{h}\, v^{(a)} = v^{(a)}$

Antiparticle

$h = +$

$E > 0$

What about Chirality ($\neq$ Helicity)

This is your exercise to hand-in until Friday 3rd December.

$$\underbrace{\psi_D = \psi_L + \psi_R}_{\text{Dirac spinors}}$$

$$\begin{cases} \psi_L = P_L\, \psi_D \\ \psi_R = P_R\, \psi_D \end{cases}$$

$$P_{L/R} = \frac{1}{2}(1 \pm \gamma^5) \quad , \quad \underbrace{\gamma_c^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}}_{\text{Weyl Representation}}$$

$$\Psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \qquad \alpha = 1, 2$$

$$\Psi_R = \begin{pmatrix} 0 \\ \psi_{\dot\beta} \end{pmatrix} \qquad \dot\beta = 1, 2$$

dotted ($\dot\alpha$) and undotted (α) indices indicate the the two types of spinors transform differently under the Lorentz group.

$\varepsilon_{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$ raise and lower

Lorentz indices

$$\chi_\alpha = \varepsilon_{\alpha\beta}\,\chi^\beta , \qquad \bar\psi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\,\bar\psi_{\dot\beta}$$

Typically one uses ONLY left-handed Weyl L-spinors

ψ and χ

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix} \quad \text{Dirac Spinor}$$

Majorana spinors

$$\psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\beta} \end{pmatrix}$$

covariance of Dirac's equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow \left(i\gamma^\mu [\Lambda^{-1}]^\nu{}_\mu \partial'_\nu - m\right) S\psi' = 0$$

$$\Rightarrow \left(i\gamma^\mu S [\Lambda^{-1}]^\nu{}_\mu \partial'_\nu - mS\right)\psi' = 0$$

$$\Rightarrow S^{-1}\left(i\gamma^\mu S [\Lambda^{-1}]^\nu{}_\mu \partial'_\nu - mS\right)\psi' = 0$$

$$\Rightarrow \left(i\, S^{-1}\gamma^\mu S\, [\Lambda^{-1}]^\nu{}_\mu \partial'_\nu - m\right)\psi' = 0$$

$$\partial_\mu = [\Lambda^{-1}]^\nu{}_\mu \partial'_\nu$$

$$S^\alpha{}_\beta \psi^\beta = \psi'^\alpha, \quad S^{-1}S = 1\!\!1$$

S acts on spinor space

Λ acts on coordinates

so $[S, \Lambda] = 0$

$$\Rightarrow \left(i \gamma'^{\nu} \partial'_{\nu} - m \right) \psi' = 0$$ "Forcing" covariance

$$\gamma'^{\nu} = S^{-1} \gamma^{\mu} S \left[\Lambda^{-1} \right]^{\nu}{}_{\mu}$$ This has to be realized.

$$\left[\gamma'^{\nu} \right]_{\alpha\beta} = \left[S^{-1} \right]_{\alpha}{}^{\rho} \left[\gamma^{\mu} \right]_{\rho\eta} \left[S \right]^{\eta}{}_{\beta} \left[\Lambda^{-1} \right]^{\nu}{}_{\mu}$$

Lagrangian Formulation in Classical Field Theory

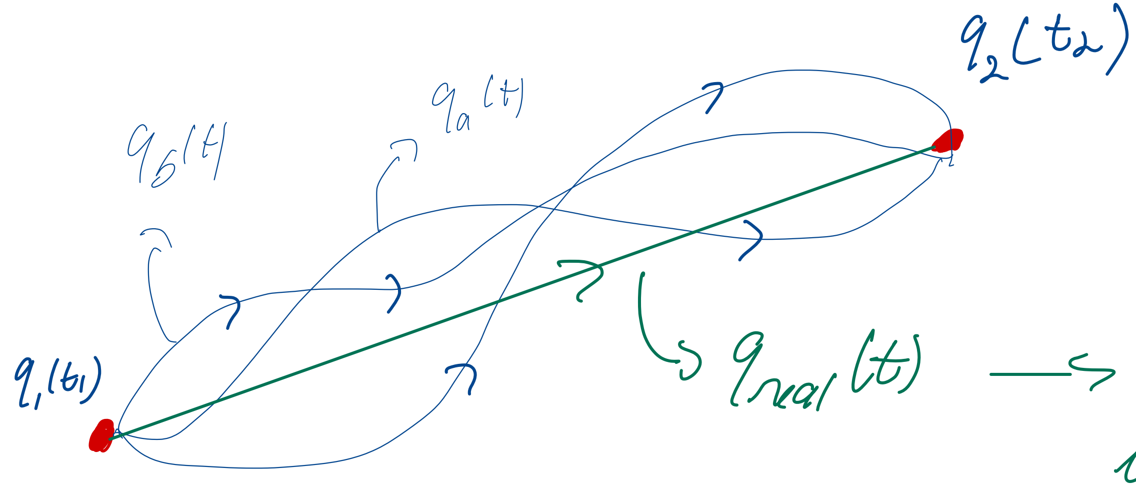
Recall the action $S[q]$ from Classical Mechanics

$f(x) : \mathbb{R} \longmapsto \mathbb{R}$, $S[f] : \mathcal{F} \longmapsto \mathbb{R}$

$$S[q] = \int dt \, L(q, \dot{q}, t)$$

$$L(q, \dot{q}, t) = T(\dot{q}) - V(q, t)$$

$q \longrightarrow$ generalized coordinates , $\dot{q} = \frac{dq}{dt}$



$\longrightarrow$ has, in average, the least difference between T and V

Inf. Shift $\longrightarrow$ $q_i'(t) = q_i(t) + \delta q_i(t)$

$$\delta S = \int_{t_1}^{t_2} dt\, L(q_i', \dot{q}_i', t) - \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t)$$

$$= \int_{t_1}^{t_2} dt \left[L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - L(q_i, \dot{q}_i, t) \right]$$

$$= \int_{t_1}^{t_2} dt \; \delta L(q_i, \dot{q}_i, t)$$

$$d f(x_i) = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i$$

$$= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q_i} \delta q_i + \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]$$

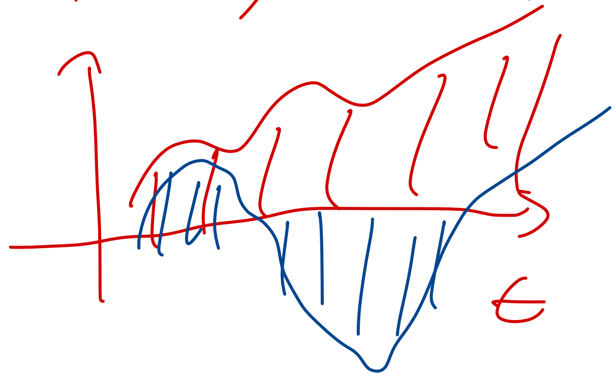
$$= \underbrace{\left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2}}_{0} + \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)$$

All trajectories converge at the endpoints q_1, q_2.

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

$$0 = \int_{t_1}^{t_2} dt\, f(q_i, \dot{q}_i)\, \delta q_i$$

Using the Principle of the stationary action



$$\Rightarrow \quad \delta S = 0$$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0}$$

Euler-Lagrange (EL) equations

$$\rightarrow \quad L = \frac{1}{2} m \dot{x}^2 - V(x) \longrightarrow F = ma$$

The EL in Field Theory

classical fields result from a generalization to a continuum of n-particles

$$\underbrace{\text{n-particles}}$$

$$L = \frac{1}{2} \sum_{k=1}^{n} m_k \dot{x}_k^2 - V(x_1, \dots, x_n)$$

Now to the continuum!

$$x_k(t) \longrightarrow \phi(t, \vec{x})$$

$$k \longrightarrow \vec{x}$$

$$\sum_k \longrightarrow \int d^3x$$

$$\dot{x}_k(t) \longrightarrow \partial_\mu \phi(t, \vec{x})$$

$\mathcal{L}(\phi, \partial_\mu \phi) \longrightarrow$ Lagrangian density

Lagrangian

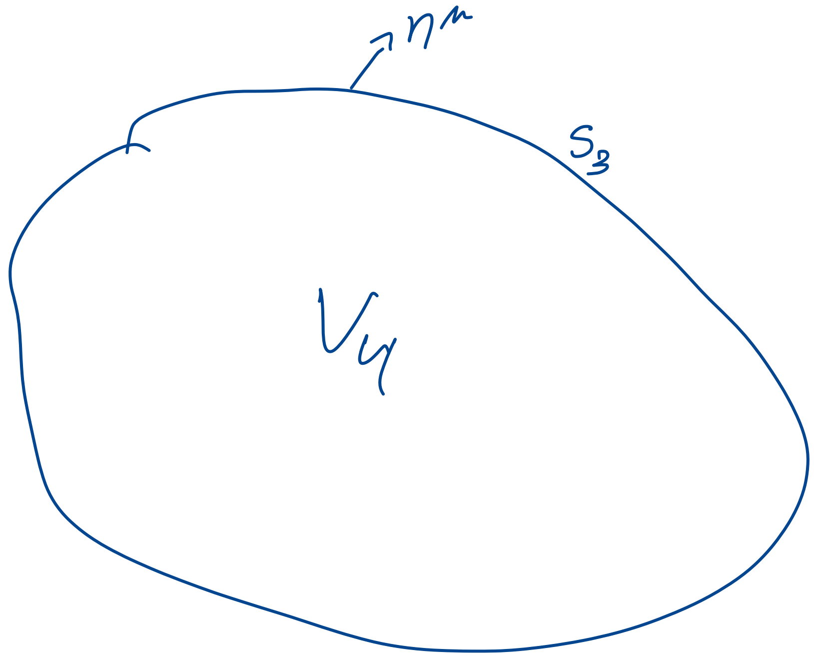
$$L = \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$E = \frac{hc}{\lambda} \qquad [E] = \text{length}^{-1} \equiv l^{-1}$$

$$[L] = \ell^3 [\mathcal{L}] \Leftrightarrow \ell^{-1} = \ell^3 [\mathcal{L}] \Rightarrow$$

$$\Rightarrow [\mathcal{L}] = \ell^{-4} = [E]^4 = \boxed{[m]^4}$$



S_3 is the generalization of t_1 and t_2

At the S_3 boundary $\phi(t,\vec{x})\big|_{S_3} = \phi'(t,\vec{x})\big|_{S_3} \Rightarrow$

$$\Rightarrow \delta\phi(t,\vec{x})\big|_{S_3} = 0$$

- $\phi'(t,\vec{x}) = \phi(t,\vec{x}) + \delta\phi(t,\vec{x})$

- $\partial_\mu \phi'(t,\vec{x}) = \partial_\mu \phi(t,\vec{x}) + \underbrace{\delta\left[\partial_\mu \phi(t,\vec{x})\right]}_{\partial_\mu \delta\phi(t,\vec{x})}$

$$\delta\left[\partial_\mu \phi\right] = \partial_\mu \phi' - \partial_\mu \phi = \partial_\mu(\phi' - \phi) = \partial_\mu \delta\phi$$

$$\delta S = \int_{V_4} d^4x \left[\underbrace{\mathcal{L}(\phi + \delta\phi, \partial_\mu + \delta(\partial_\mu\phi)) - \mathcal{L}(\phi, \partial_\mu\phi)}_{\delta\mathcal{L}(\phi, \partial_\mu\phi)} \right]$$

$$= \int_{V_4} d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta(\partial_\mu\phi)}_{\partial_\mu \delta\phi} \right]$$

$$= \int_{V_4} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi \right]$$

$$= \int_{V_4} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi + \boxed{\int_{V_4} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}$$

Using a generalization of Gauss-Ostrogradsky's Theorem

$$\int_V dV \, \vec{\nabla} \cdot \vec{F} = \int_{S_V} dS \, \vec{F} \cdot \vec{n}$$

$$\int_{V_4} d^4x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi = \int_{S_3} d^3x \, n_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi = 0$$

$$\delta S = \int_{V_4} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi = 0$$

$\Longrightarrow$ E.L. eqs.

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0}$$