STANDARD MODEL I

LECTURE - 2

Maxwell equations

1st pair

$$\vec{\nabla} \cdot \vec{B} = 0 \longrightarrow \text{Gauss Law for } \vec{B}$$

$$\vec{\nabla} \wedge \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \longrightarrow \text{Faraday's Law}$$

charge density

2nd pair

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \longrightarrow \text{Gauss law}$$

$$\vec{\nabla} \wedge \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{j} \longrightarrow \text{Ampere's Law}$$

current density

Introduce the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad , \quad A^\mu \longrightarrow \text{electromagnetic field}$$

$$A^\mu = (A^0, \vec{A})$$

$$\vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \equiv (\vec{E}, \vec{B})$$

Dual of $F^{\mu\nu}$

$$F^{*\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} 1 & \text{if } \mu=0,\ \nu=1,\ \alpha=2,\ \beta=3 \\ -1 & \text{if odd permutations} \\ 0 & \text{if repeated indices} \end{cases}$$

$$F^{*\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\partial_\mu F^{*\mu 0} = \partial_i F^{*i0} = \vec{\nabla}\cdot\vec{B}$$

$$\partial_\mu F^{*\mu 1} = \partial_0 F^{*01} + \partial_2 F^{*21} + \partial_3 F^{*31}$$

$$= -\partial_t B_x - \partial_y E_z + \partial_z E_y$$

$$= -\left(\partial_t \vec{B} + \vec{\nabla} \wedge \vec{E}\right)_x$$

$$\partial_\mu F^{*\mu 2} = -\left(\partial_t \vec{B} + \vec{\nabla} \wedge \vec{E}\right)_y$$

$$\partial_\mu F^{*\mu 3} = -\left(\partial_t \vec{B} + \vec{\nabla} \wedge \vec{E}\right)_z$$

$$\partial_\mu F^{*\mu\nu} = 0$$

1st Pair of Maxwell equations

Faraday-Gauss Law

$$\boxed{\partial_\mu F^{\mu\nu} = 4\pi j^\mu}$$

2^{nd} pair of Maxwell equations

$$j^\mu = (\rho, \vec{j})$$

Ampere-Gauss Law

Covariance of Maxwell's equations

- $F^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{\alpha\beta}$

$$\Lambda^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}$$

- $F^{*\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F'^{*\alpha\beta}$
- $j^\nu = \Lambda^\nu{}_\beta j'^\beta$
- $\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\mu} = \partial'_\alpha [\Lambda^{-1}]^\alpha{}_\mu$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial'_\alpha \underbrace{[\Lambda^{-1}]^\alpha{}_\mu \Lambda^\mu{}_{\tilde\alpha}}_{\delta^\alpha{}_{\tilde\alpha}} \Lambda^\nu{}_\beta F'^{\tilde\alpha\beta} = \Lambda^\nu{}_\beta \, j'^\beta$$

$$\partial'_\alpha \Lambda^\nu{}_\beta F'^{\alpha\beta} = \Lambda^\nu{}_\beta \, j'^\beta \quad \Longrightarrow$$

$$\boxed{\partial'_\alpha F'^{\alpha\beta} = j'^\beta} \quad //$$

The Ampere-Gauss law is <u>invariant</u> under L.T.

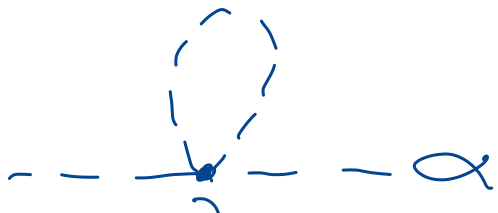
Gauge Invariance

- Gauge bosons exist
- photons and gluons are massless because of gauge invariance
- $W^\pm$ and Z bosons have small masses (when compared e.g. to the GUT or Planck scales) because of gauge protection.

$$m_H^2 = \underbrace{m_H^{0\,2}}_{\sim m_{Planck}^2 + \varepsilon} + \overbrace{\Delta m_H^2}^{m_{Planck}^2}$$

$\Delta m_H^2 \to$ 1-loop

$\sim m_{Planck}^2 + \varepsilon \to \sim M_{EW}^2$



λ

$$\propto \frac{\lambda \mu^2}{16\pi^2}$$

$$m_H^2 = -m_{Planck}^2 + \varepsilon + m_{Planck}^2 = \varepsilon \sim m_{EW}^2$$

A^μ is not unique!

Redefine it as $A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu \alpha(x)$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu + \partial^\mu \partial^\nu \alpha(x) - \partial^\nu A^\mu - \partial^\nu \partial^\mu \alpha(x)$$

$$= \partial^\mu A^\nu - \partial^\nu A^\mu + \cancel{\partial^\mu \partial^\nu \alpha(x)} - \cancel{\partial^\mu \partial^\nu \alpha(x)} \Longrightarrow$$

$$\boxed{F'^{\mu\nu} = F^{\mu\nu}} \longrightarrow F^{\mu\nu} \text{ is gauge invariant}$$

and so are $\partial_\mu F^{\mu\nu} = 4\pi j^\mu$, $\partial_\mu F^{*\mu\nu} = 0$

Relativistic Quantum Mechanics

① Klein-Gordon Equation

$$E = \frac{\vec{p}^{\,2}}{2m} \qquad \begin{cases} \hat{E} \longrightarrow i\frac{\partial}{\partial t} \\ \hat{\vec{p}} \longrightarrow -i\vec{\nabla} \end{cases}$$

$$\hat{E}\psi = \frac{\hat{\vec{p}}^{\,2}}{2m}\psi \Longrightarrow \boxed{i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\vec{\nabla}^2\psi}$$

S. E.

What about relativistic energies?

$$E = \sqrt{m^2 + \vec{p}^{\,2}} \Longleftrightarrow \hat{E}^2 - \hat{\vec{p}}^{\,2} = m^2$$

$$\left[\left(i\frac{\partial}{\partial t}\right)^2-\left(-i\vec{\nabla}\right)^2\right]\phi = m^2\phi \implies$$

$$-\underbrace{\left[\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2\right]}_{\Box}\phi = m^2\phi \implies$$

$$\boxed{\left(\Box + m^2\right)\phi = 0}$$

K. G. equation

1926

$$\Box = g^{\mu\nu}\partial_\mu\partial_\nu = g^{\mu\nu}\left[\Lambda^{-1}\right]^\alpha{}_\mu\left[\Lambda^{-1}\right]^\beta{}_\nu\,\partial'_\alpha\,\partial'_\beta$$

$$\phi(x) = S\,\phi'(x') \quad , \quad S^{-1}S = \mathbb{1}$$

$$\left(g^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\phi = 0$$

$$\left(g^{\mu\nu}[\Lambda^{-1}]^{\alpha}{}_{\mu}[\Lambda^{-1}]^{\beta}{}_{\nu}\,\partial'_\alpha\partial'_\beta + m^2\right)S\phi' = 0$$

$$\left(g'^{\alpha\beta}\partial'_\alpha\partial'_\beta + m^2\right)S\phi' = 0$$

$$\left(\Box' + m^2\right)S\phi' = 0$$

The K.G. equation becomes manifestly invariant under L.T if $S\phi' = \phi' \Longrightarrow S = S^{-1} = \mathbb{1}$

$\boxed{\phi \text{ is a scalar}}$ $\longrightarrow$ Spin 0

The K.-G. Equation cannot describe, e.g., relativistic electrons since ϕ must transform as a scalar.

Solutions

Ansatz: $\phi(x) = e^{-i p_\mu x^\mu} = e^{-iEt + i\vec{p}\cdot\vec{x}}$

$$g^{\mu\nu}\partial_\mu \partial_\nu e^{-i p_\alpha x^\alpha} + m^2 e^{-i p_\mu x^\mu} = 0$$

$$g^{\mu\nu}(-i p_\mu)(-i p_\nu) e^{-i p_\alpha x^\alpha} + m^2 e^{-i p_\mu x^\mu} = 0$$

$$\left(-p_\mu p^\mu + m^2\right)\phi = 0 \implies$$

$$(-E^2 + \vec{p}^{\,2} + m^2)\phi = 0$$

$$\Longrightarrow \quad E^2 = \vec{p}^{\,2} + m^2 \quad \Longrightarrow \quad E = \pm\sqrt{\vec{p}^{\,2} + m^2}$$

$$\{\phi_+(x), \phi_-(x)\}$$

$$\Phi(x) = c_1 \phi_+(x) + c_2 \phi_-(x)$$

To understand the meaning of the negative energy solutions, let us couple the scalar field to an electromagnetic field.

$$A^\mu = (A^0, \vec{A})$$

$$\partial_\mu \longrightarrow \partial_\mu + i Q A_\mu \quad , \quad Q \rightarrow \text{electric charge of } \Phi$$

(A) $$\left[g^{\mu\nu}\left(\partial_\mu + iQA_\mu\right)\left(\partial_\nu + iQA_\nu\right) + m^2\right]\phi_+ = 0$$

$$\phi_- = \phi_+^*$$

upon complex conjugation:

(B) $$\left[g^{\mu\nu}\left(\partial_\mu - iQA_\mu\right)\left(\partial_\nu - iQA_\nu\right) + m^2\right]\phi_- = 0$$

ϕ_- is the change conjugation of ϕ_+

with $\quad Q(\phi_-) = -Q(\phi_+)$

A charge conjugation operation transforms particles into their antiparticles.

Dirac Equation (1928)

$$i \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle \quad \text{S.E.}$$

$$\hat{H} = \vec{\alpha} \cdot \vec{p} + \beta m$$

$\vec{\alpha}$ and β are hermitian operators.

$$(\vec{\alpha}\cdot\vec{p}+\beta m)^2|\psi\rangle = E^2|\psi\rangle$$

This is a relativistic theory!

$$(\vec{\alpha}\cdot\vec{p}+\beta m)^2 = \vec{p}^{\,2}+m^2 \iff$$

$$(\vec{\alpha}\cdot\vec{p})^2 + m(\vec{\alpha}\beta+\beta\vec{\alpha}) + \beta^2 m^2 = \vec{p}^{\,2}+m^2$$

$$\begin{cases} (\vec{\alpha}\cdot\vec{p})^2 = \vec{p}^{\,2} \\ \vec{\alpha}\beta+\beta\vec{\alpha} = 0 \\ \beta^2 = 1 \end{cases}$$

$$(*) = \alpha_1^2 p_1^2 + \alpha_2^2 p_2^2 + \alpha_3^2 p_3^2 + \alpha_1\alpha_2 p_1 p_2 + \ldots$$

$$(*) \Longrightarrow \begin{cases} \alpha_i^2 = 1 = \beta^2 \\ \alpha_i \beta + \beta \alpha_i = 0 \\ \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \end{cases} \quad \text{for } i \neq j, \; i,j = 1,2,3$$

$$i\partial_t \psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi \iff$$

$$i\partial_t \psi = (-i\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi$$

$$i\beta\partial_t \psi = (-i\beta\vec{\alpha} \cdot \vec{\nabla} + m)\psi$$

Define $\beta = \gamma^0$, $\beta\vec{\alpha} = \vec{\gamma}$

Then

$$i\gamma^0 \partial_0 \psi = (-i\gamma^i \partial_i + m)\psi$$

$$\Rightarrow \boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

$$(i\not{\partial} - m)\psi = 0$$

$$\gamma^\mu = (\gamma^0, \vec{\gamma})$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$\begin{pmatrix} 4 \\ k \end{pmatrix} = 1$ if $k = 0$ Clifford Algebra (see Giunti)

$(\gamma^0)^2 = (\gamma^1)^2 = \mathbb{1}$ one component

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ one component

$\gamma^\mu \longrightarrow$ 4 components

$\gamma^\mu\gamma^5 \longrightarrow$ 4 components

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \longrightarrow 6 \text{ components}$$

$$\frac{n(n-1)}{2} = 6 \quad (n=4)$$

dimension 16 basis

$$\{ \mathbb{1}, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu} \}$$

$$M_{4\times 4} = m_1 \mathbb{1} + m_2 \gamma^5 + m_3 \gamma^1 + \cdots + m_{16} \sigma^{23}$$

The γ-matrices can be expressed as:

Pauli-Dirac representation

Pauli Matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Chiral representation

$$\gamma_c^5 = \gamma_D^0 \quad , \quad \gamma_c^i = \gamma_D^i \quad , \quad \gamma_c^0 = -\gamma_D^5$$

$$\gamma_c^0 = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma_c^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma_c^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \quad \gamma_c^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ -\sigma^\mu & 0 \end{pmatrix}$$

$$\bar{\sigma}^\mu = (-\mathbb{1}, \sigma^i) \quad , \quad \sigma^\mu = (\mathbb{1}, \sigma^i)$$

For the time being we will use the Pauli-Dirac representation.

- $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\boxed{\psi^\alpha = u^\alpha(p)\, e^{-ip_\mu x^\mu}}$$

$$\partial_\mu \psi^\alpha = -ip_\mu \psi^\alpha$$

$$(\gamma^\mu p_\mu - m)\, u^\alpha(p) = 0$$

multiply on the left by $(\gamma^\nu p_\nu - m)$

$$(\underbrace{\gamma^\nu \gamma^\mu p_\nu p_\mu}_{?} - m^2)\, u(p) = 0$$

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\, p_\nu\, p_\mu = g^{\mu\nu} p_\nu p_\mu$$

$$\frac{1}{2}\left(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right)p_\nu p_\mu = p_\nu p^\nu \iff$$

$$\frac{1}{2}\left(\gamma^\mu\gamma^\nu p_\nu p_\mu + \gamma^\nu\gamma^\mu p_\nu p_\mu\right) = E^2-\vec{p}^{\,2}$$

$$\frac{1}{2}\left(\gamma^\mu\gamma^\nu p_\mu p_\nu + \gamma^\nu\gamma^\mu p_\nu p_\mu\right) = E^2-\vec{p}^{\,2}$$

$$\gamma^\mu\gamma^\nu p_\mu p_\nu = E^2-\vec{p}^{\,2} \Longrightarrow$$

$$\left(E^2-\vec{p}^{\,2}-m^2\right)u(p) = 0 \Longrightarrow$$

$$\left(E^2-\vec{p}^{\,2}\right)u(p) = m^2 u(p)$$

If the particle is at rest

$$E^2 \mathbb{1}\, u^\alpha(p) = m^2 \mathbb{1}\, u^\alpha(p) \Longrightarrow E = \pm m$$

$$H u^\alpha = \gamma^0 m\, u^\alpha$$

$$\begin{pmatrix} E\mathbb{1} & 0 \\ 0 & E\mathbb{1} \end{pmatrix} u^\alpha = \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix} u^\alpha$$

$$\underbrace{u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{E = m}, \quad \underbrace{u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{E = -m}$$

$$\vec{\gamma} \wedge \vec{\gamma} = -2i \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \vec{e}_x - 2i \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \vec{e}_y + 2i \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \vec{e}_z$$

$$\vec{S} = \frac{i}{4} \vec{\gamma} \wedge \vec{\gamma} = S_x \vec{e}_x + S_y \vec{e}_y + S_z e_z$$

$$[\vec{S}, H] = 0$$

Thus $\{u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}\}$

is a basis that diagonalizes H and $\vec{S}$ simultaneously!

$$S_z u^{(1,3)} = \frac{1}{2} u^{(1,3)}$$

$$S_z u^{(2,4)} = -\frac{1}{2} u^{(2,3)}$$

$u^{(1)} \longrightarrow$ particle solution $E = m$, $S_z = \frac{1}{2}$

$u^{(2)} \longrightarrow$ " " " , $S_z = -\frac{1}{2}$

$u^{(3)} \longrightarrow$ antiparticle " " , $S_z = \frac{1}{2}$

$u^{(4)} \longrightarrow$ " " " , $S_z = -\frac{1}{2}$