Standard Model I: Hand-in exercises sheet 1

November 24, 2021

3 points

1. Define the left and right chiral projectors

$$P_{\mathrm{L,R}} = \frac{1}{2}(\mathbb{1} \mp \gamma^5) \tag{1}$$

expressed in the Dirac representation. Discuss under which circumstances right and left chiralities correspond to $+$ and $-$ helicities respectively.
Suggestion: align the 3-momentum as $\boldsymbol{p} = p_z \boldsymbol{e_z}$.

3 points

2. Consider the Dirac equation $(i\gamma^\mu \partial_\mu - m)\Psi = 0$ with the gamma matrices γ^μ expressed in the Weyl (or chiral) representation defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \qquad \sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma})\,. \tag{2}$$

Decompose Ψ in terms of left and right Weyl spinors $\Psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}$ and show that in the chiral limit, *i.e.* $m \to 0$, the Weyl equations are obtained

$$\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} &= 0 \\ i\sigma^\mu \partial_\mu \psi_{\mathrm{R}} &= 0\,. \end{aligned} \tag{3}$$

Discuss the result.

2 points

3. For a classical fermion in its rest frame determine the stationary states $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ and $u^{(4)}$ of the Dirac equation expressed in the Weyl representation and use the eigenvalue equation $Hu^{(i)} = E_i u^{(i)}$ to identify the particle and antiparticle solutions.

4 points

4. Determine the generic particle and antiparticle solutions of the Dirac equation in the Weyl representation . Show that your result agrees with the previous problem.
Suggestion: use the Feynman-Stuckelberg convention to define the antiparticle spinors $v^{(1,2)}(p)$.

4 points

5. Determine the helicity of the Dirac spinors $u^{(1,2)}$ and $v^{(1,2)}$ and their chiral components in the Weyl representation. Compare with the result obtained in the Dirac representation. **Suggestion:** align the 3-momentum as $\boldsymbol{p} = p_z \boldsymbol{e_z}$.

4 points

6. Let us define the left-handed Weyl spinors ψ and ξ such that

$$\begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix} \equiv \begin{pmatrix} \psi_\mathrm{L} \\ (\xi^\star)_\mathrm{R} \end{pmatrix} \equiv \begin{pmatrix} \psi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\xi}^{\dot{\beta}} \end{pmatrix} \tag{4}$$

where we have introduced a new notation with a bar to be equivalent to the complex conjugate. In this notation, the labels L and R refer, respectively, to a left handed or right handed 2-spinor and the greek indices run over $\alpha = 1, 2$ and $\dot{\beta} = 1, 2$. Here we note that if ψ is left handed (L) and $\bar{\xi}$ is right handed (R), their complex conjugates will have opposite chirality, i.e. $\bar{\psi}$ will be R and ξ will be L. It is in this sense that we are expanding Ψ in terms of the left handed spinors ψ and ξ.

a) Show that $(\sigma^i)^\star = -\sigma^2 \sigma^i \sigma^2$ where σ^2 denotes the second Pauli matrix.

b) Use the above identity to show that the coupled Weyl equations, *i.e.*

$$\begin{cases} i\sigma^\mu \partial_\mu \psi_\mathrm{R} = m\psi_\mathrm{L} \\ i\bar{\sigma}^\mu \partial_\mu \psi_\mathrm{L} = m\psi_\mathrm{R} \end{cases} \tag{5}$$

can be recast solely in terms of left-handed Weyl 2-spinors as

$$\begin{cases} i\bar{\sigma}^\mu \partial_\mu \chi = -mi\sigma^2 \bar{\psi} \\ i\bar{\sigma}^\mu \partial_\mu \psi = -mi\sigma^2 \bar{\chi} \end{cases} , \tag{6}$$

where we define the charge conjugate of $\bar{\xi}$ as $\chi = i\sigma^2 \xi$.