Universidade de Aveiro
2020

Departamento de Matemática

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Aspetos de modelos Einstein-Maxwell-scalar: Solitões, Dualidade e Escalarisação

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática Aplicada, realizada sob a orientação científica do Doutor Carlos A. R. Herdeiro, Investigador Coordenador do Departamento de Matemática da Universidade de Aveiro e co-orientação do Doutor Filipe C. Mena, Professor Associado do Departamento de Matemática do Instituto Superior Técnico

Apoio financeiro da FCT e do FSE no âmbito do III Quadro Comunitário de Apoio

I dedicate this thesis to my parents and my love, Joana, for the much needed emotional support during my academic life.

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Acknowledgements

Various people made this work possible. Without their support, I would not be here and for that I am thankful.

First of all, I would like to thank my supervisor Prof. Carlos Herdeiro, for all the much needed scientific guidance given to me during my time as a PhD student. He also allowed me enough freedom and pushed me to develop my independence as a researcher which will definitely be crucial in my academic future.

Second, I want to thank my co-supervisor Prof. Filipe Mena, for all the help, motivation, and discussions of various ideas and possibilities. These discussions prevented me from ever feeling lost or without motivation and allowed me to stay confident towards the future.

Now I must thank Joana, for all the emotional help, trust, and care during the many difficult times of this thesis. Because of her, I grew to become a better person, in many aspects. After all this time I spent with her, and all the help she gave me, I just could not ever see myself without her.

I want to also thank my parents, for making this academic journey possible. Without their support and immense effort, I would never have the chance to get to where I am now. I am very thankful to have them.

For all the friendly discussions about science and various other topics, I must thank both Alexandre Pombo and Jorge Delgado. They were both good friends and great company whenever I was abroad. The friendly competition and feedback between us was a constant motivation to improve and allowed me to polish many of my skills to levels that I am now very proud of.

I must also thank Eugen Radu for various interesting and fruitful discussions about science.

And last but not least I want to thank Cristina Grosso, for being one of the nicest persons I know and for all the help and patience regarding documentation and plenty of bureaucracy.

This work received financial support from the Fundação para a Ciência e Tecnologia (FCT), with the PhD grant PD/BD/128184/2016.

Keywords: Scalar fields, Maxwell fields, Solitons, Black holes, Uniqueness, No go theorems, Axions, Duality, Scalarisation.

Abstract: In the recent years, the Einstein-Maxwell-scalar model has proven to be a very fruitful and interesting model. Due to the simple addition of a non-minimal coupling $f(\phi)$ between the scalar field ϕ and the electromagnetic field terms, we find various novel and unusual properties that have led to interesting results in the fields of general relativity (at least as a toy model) in the astrophysics of compact objects. This thesis will address some of these properties.

After a brief introduction of the topics to be covered, we present the various results that we obtained regarding the non-existence of solitons in certain conditions for this model. These results paved the way for us to obtain a new soliton solution for this kind of model, which is also presented here. Next, we discuss the duality symmetries of the Einstein-Maxwell-scalar model and take advantage of them to create a solution generating technique which we then apply to well-known solutions of the model. Lastly, we cover the much discussed concept of spontaneous scalarisation, but this time in higher dimensions for a generalised model. We end with some conclusions and remarks.

Palavras-chave: Campos escalares, Campos de Maxwell, Solitões, Buracos negros, Unicidade, Teoremas "no go", Axiões, Dualidade, Escalarisação.

Resumo: Nos anos recentes, o modelo de Einstein-Maxwell-scalar provou ser um modelo bastante rico e interessante. Devido à simples adição de um acoplamento não minimal $f(\phi)$ entre o campo escalar ϕ e o campo electromagnético, encontramos várias propriedades peculiares que levaram a resultados interessantes nos campos da relatividade geral e (pelo menos como um modelo teste) da astrofísica de objetos compactos. São algumas destas propriedades que vamos discutir nesta tese.

Após uma pequena introdução dos tópicos que vão ser abordados, apresentamos vários resultados que obtemos sobre a inexistência de solitões para certas condições neste modelo. Estes resultados abriram o caminho para obtermos uma nova solução solitónica para este tipo de modelo, que também é apresentada aqui. A seguir, discutimos as simetrias de dualidade do modelo Einstein-Maxwell-scalar e tiramos partido destas para obter uma técnica de geração de soluções que depois aplicamos a soluções conhecidas deste modelo. Por fim, abordamos o muito discutido conceito de escalarisação espontânea, mas desta vez em espaços de alta dimensão para um modelo generalizado. Finalizamos com algumas conclusões e comentários.

Contents

List of Figures

Chapter 1

Introduction

This chapter is dedicated to introducing the main topics that will be delved into in the following chapters. Every chapter starts with a brief introduction covering what will be discussed throughout that chapter and how it is structured. The structure of the thesis is underlined in section 1.6 of this introductory chapter.

Regarding notation, there are a few things that should be mentioned. We consider 4-dimensional spacetimes unless specified otherwise and use index notation, which implies that two repeated indices, one covariant (below) and one contravariant (above), constitute an implied sum

$$\sum_{\mu} a_\mu b^\mu := a_\mu b^\mu \ , \tag{1.0.1}$$

where a and b stand for 4-dimensional vectors and a^μ for each value of μ (from 0 to 3) represents each of the 4 components of the vector a. The same notation applies to tensor indices of any rank. Unless stated otherwise, we use the standard Minkowski (t, r, θ, φ) spherical coordinates to describe spacetime. Of note is that φ will also be used interchangeably with V to denote electric potential but only when specified and in a way that avoids confusion. In the same way, V will also be used to denote the norm of a stationary killing vector field. We consider both Newton's gravitational constant G and the speed of light c to be equal to unity $G = c = 1$.

1.1 The model

The model that will become the main topic of discussion in this thesis is the *Einstein-Maxwell-scalar* (EMS) model which has the following action

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \sqrt{-g} \left(\frac{R}{4} - \frac{f(\phi)}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right). \tag{1.1.2}$$

The elements of this action are the Ricci tensor R, the Maxwell field strength $F_{\mu\nu}$, the scalar field ϕ, which may or may not have a potential $U(\phi)$, and the non-minimal coupling function $f(\phi)$. We consider the theory in 4-dimensional spacetimes, therefore the integration in the action uses the covariant 4-dimensional volume element $\sqrt{-g}\, d^4x$, where g is the determinant of the spacetime metric tensor $g_{\mu\nu}$. The main novelty of this model is the coupling function $f(\phi)$, which directly couples the scalar and electromagnetic fields. Physically, this function can be interpreted as a varying electric permittivity, which is mediated by the scalar field equation of motion. For the sake of recovering the Einstein-Maxwell model when the scalar field vanishes, the following condition

$$f(0) = 1 \tag{1.1.3}$$

is imposed.

This kind of model can arise naturally in physics. Most notably are the well-known Kaluza-Klein models [1–3] where the electromagnetic and scalar terms result from the geometry of 5-dimensional spacetime. Other well-known contexts include supergravity and string theory [4]. In these theories, a very common non-minimal coupling function is the exponential function $f(\phi) \sim e^{-\alpha\phi}$, where α is a constant. This kind of coupling is usually referred to as *dilatonic* and the scalar field as the *dilaton*. A well-known example of a spherically symmetric black hole solution for this model with a dilatonic coupling was found in [5]. In the context of cosmology, more general classes of couplings were considered in [6,7] when considering the scalar field as the inflaton field and its coupling to gauge fields.

More recently studied in this kind of model is the phenomenon of *spontaneous scalarisation*. The study about scalarisation for this model was kickstarted in [8] where it was shown that EMS, under the condition $\frac{df}{d\phi}(0) = 0$, accommodates scalarisation of Reissner-Norström (RN) black holes. This means that there is a class of hairy black hole solutions which bifurcate from the RN solution. We will discuss this further in section 1.5 and in chapter 6.

Another topic of interest when studying this model is the search for *soliton* solutions. These are everywhere regular, asymptotically flat solutions that describe lumps of energy that are not dense enough to collapse and create an event horizon, but still manage to be in equilibrium. There has been a long extensive discussion of solitons in field theory starting with the Korteweg-de-Vries equation [9] and examples in general relativity include boson stars [10]. We will discuss solitons and theorems regarding their existence or non-existence more closely in section 1.2 and chapters 2 and 3.

In this thesis we will slowly build up on this model. After chapter 2 we will consider further generalisations of the action that allow us to recontextualise the model with other kinds of theories.

1.2 Uniqueness, no go theorems and solitons

In the context of general relativity, uniqueness theorems are theorems that, given a set of conditions, will specify the solution of our model (spacetime metric or field configuration) and rule out any other solution. This means that the solution is *unique* for that set of conditions.

1.2.1 Vacuum general relativity

Take for example vacuum general relativity which is described by the Einstein-Hilbert action

$$\mathcal{S}_{EH} = \frac{1}{16\pi}\int d^4x\sqrt{-g}R \; . \tag{1.2.4}$$

A plausible question for such a non-linear theory would be if there are solutions that are in equilibrium, are localised, and have finite energy. We know that one such class of such solutions exists: the *black hole* solutions.

1.2.1.1 Black Holes in GR

A theorem constructed by Carter and Robinson in [11, 12](see also [13–15]) states the following

Theorem 1 (Carter, Robinson). *The most general stationary (in equilibrium) and axisymmetric asymptotically flat vacuum black hole solution regular on and outside the event horizon, is provided by the two-parameter (mass and angular momentum) family of Kerr metrics [16].*

Note how the solution is already found and specified as the only possible solution for the conditions given at the start of the theorem.

This is a very general and strong result. In fact, this theorem tells us that any two black holes that have the same total mass and angular momentum are exactly the same, not unlike the properties of sub-atomic particles. Any other macroscopic object can have different density distributions of these same quantities that allow us to distinguish them, but not black holes! To describe this inability to distinguish black holes with those same quantities John Wheeler coined the expression that "*black holes have no hair*" [17], where "hair" stands as a metaphor for any other information that is not associated with a globally conserved quantity which has a Gauss law. But as we will see, for specific extended theories this expression's generality needs to be reconsidered as examples of hairy black holes have been found (see for example [8, 18, 19]).

If we impose stronger conditions on the theorem above, we should obtain a family of solutions that is a subset of the Kerr family. Another very important uniqueness theorem is the one proven by Israel in [20]:

Theorem 2 (Israel)**.** *The most general static and asymptotically-flat vacuum black hole solution regular on and outside the event horizon, is provided by the one-parameter (mass) family of Schwarzschild metrics [21].*

The one-parameter Schwarzschild metric has spherical symmetry and is a subset of the Kerr family of metrics. However, note that while staticity is a stronger condition than stationarity[1], we also relaxed the axisymmetry condition. This is the remarkable conclusion of Israel's theorem: spherical symmetry is obtained as a consequence of staticity and asymptotic flatness. This means that, while adding the axisymmetry condition would give us the same result, it would have been superfluous as a stronger result (spherical symmetry) is implied by the other conditions.

1.2.1.2 Solitons in GR

The main problem with black hole solutions is the presence of curvature singularities inside the event horizon. Moreover, fully regular black holes (*i.e.* regular also inside the event horizon) are impossible in vacuum gravity by the uniqueness theorems stated above. So if we once again consider equilibrium solutions that are localised and have finite energy, an interesting thought would be to search for solutions that are also everywhere non-singular. This kind of solution would be like a lump of energy, a particle-like solution also known as a self-gravitating *soliton*. A possible physical interpretation is to imagine such speculative solutions as a bundle of gravitational waves tied up under their own "weight", without enough density to collapse.

The problem with this kind of solutions is that they are ruled out by classical results of general relativity. These are the so-called *no go theorems* which instead of specifying a solution, they rule out possible solutions given a set of conditions. Stationary solitonic solutions with non-zero mass are ruled out by such theorems [22, 23] (see also [24]) even for non-trivial topologies. Zero mass solutions are then ruled out by the positive mass theorem [25, 26] which states that the only zero mass solution is Minkowski spacetime.

1.2.2 Einstein-Maxwell: Electro-vacuum general relativity

If our search for solitons in vacuum proves futile, a simple step is to add matter to the equation. The most natural generalisation is the Einstein-Maxwell theory, obtained by adding the Maxwell field $F_{\mu\nu}$ (without sources) to the action, also referred to as *electro-vacuum*. This model is described by the action

$$\mathcal{S}_{EM} = \mathcal{S}_{EH} - \frac{1}{16\pi}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \ . \tag{1.2.5}$$

[1] Staticity implies not only that the solution is stationary, but also invariant under time reversal $t \to -t$. More formal definitions are presented in chapter 2.

1.2.2.1 Black Holes in Einstein-Maxwell

In this theory, the black hole uniqueness theorems [11–15,27,28] stated for vacuum general relativity now generalise the Kerr and Schwarzschild families of solutions to the Kerr-Newman [29] and Reissner-Nordström [30–33] families respectively. A new parameter is added, the *charge* of the black hole.

There also exists a new class of solutions in electro-vacuum however, the Majumdar-Papapetrou family [34, 35] of multi-black hole solutions, which is also non-singular on and outside the (disconnected) event horizon [36] and unique [37, 38].

However, the BH uniqueness theorems once again rule out fully regular black holes and the Majumdar-Papapetrou family of solutions is still singular inside each of the black holes.

1.2.2.2 Solitons in Einstein-Maxwell

We then resort to finding everywhere regular solitons once again. This kind of particle-like solutions in Einstein-Maxwell were dubbed as *geons* (gravitational electromagnetic entities) by Wheeler in 1955 [39]. However, solitons are ruled out once again by no go theorems as static [24] or even strictly stationary [40] configurations. Thus, we must resort to theories beyond electro-vacuum to find self-gravitating solitons.

1.2.3 Solitons - Beyond electro-vacuum

Historically, two developments may be highlighted. Firstly, Kaup [41] (see also [42]) found a concrete realisation of "geons" but in Einstein-(complex, massive)-Klein-Gordon or Einstein-complex-scalar theory:

$$\mathcal{S}_{ECKG} = \mathcal{S}_{EH} - \frac{1}{4\pi}\int d^4x\sqrt{-g}\left[\frac{g^{\mu\nu}}{2}\left(\partial_\mu\Phi\partial_\nu\Phi^* + \partial_\mu\Phi^*\partial_\nu\Phi\right) + U(|\Phi|)\right], \tag{1.2.6}$$

which has a complex scalar field Φ, with Φ^* denoting its complex conjugate, subject to a potential $U(|\phi|)$. These solitons are now known as boson stars [10]. Secondly, Bartnik and McKinnon showed solitons also exist in the Einstein-Yang-Mills theory [43] - see also the discussions in [44, 45]. In both these cases, the existence of solitons is accompanied by the existence of "hairy" black holes, *i.e.* black holes that have macroscopic degrees of freedom not associated to a Gauss law - see [19, 46] for recent reviews. In the case of the Einstein-Yang-Mills theory these are known as coloured black holes [47]; in the case of the Einstein-(complex, massive)-Klein-Gordon model these are called black holes with synchronised hair [18]. As a rule of thumb, one observes that in models in which both solitons and the standard Schwarzschild/Kerr black holes exist, so does a (non-linear) bound state of both, which is a possible interpretation of the corresponding hairy black holes. But subtleties exist. For instance, in the Klein-Gordon case, the hairy black holes require rotation and do not exist in spherical symmetry [48]. Turning around the rule of thumb, one may wonder if in a model where both "bald" and hairy black holes exist, solitons should equally be found.

And so we are led to the aforementioned Einstein-Maxwell-scalar (EMS) model (1.1.2). Hairy black holes were found in this theory in the form of *scalarised* black holes which will be discussed further in section 1.5, so it is an interesting question to see if they also accompany the existence of solitons in this theory. In chapter 2 we will discuss how some of the no go theorems apply to the Einstein-Maxwell case. We first show for the Einstein-scalar case (a valid truncation of this model with no Maxwell field) a no go theorem regarding the existence of self-gravitating solitons in stationary and axisymmetric spacetimes. We use a scaling argument inspired by the one introduced by Derrick [49]. This result generalises previous results for the static [13, 50] and strictly stationary [40] cases, showing that rotation is not enough to support self-gravitating scalar solitons in this model, at least with real scalar fields[2]. We then show how previous no go theorem results [24, 40] can be generalised for static and strictly stationary solitons in the full EMS model.

1.2.4 Solitons in Einstein-Maxwell-scalar

The no go theorems tell us when we can not find a solution, this means that we also get more information about how we might find a soliton solution. In fact, this is what is discussed in chapter 4.

By dropping one of the assumptions of these no go theorems, we are able to find soliton solutions for the Einstein-Maxwell-scalar model which are in fact possible even for flat Maxwell-scalar spacetime. It is by then considering the gravitational backreaction that we are able to perturbatively and numerically obtain the self-gravitating solitons in the full Einstein-Maxwell-scalar model.

1.3 Axions and duality in generalised electrodynamics

1.3.1 Axions

Recently, a widely discussed type of scalar field is the *axion* field. The idea of the axion field originated in Quantum Chromodynamics (QCD). The QCD action admits a term that violates the combined CP (Charge conjugation and Parity) discrete symmetries. This kind of violation, however, is not observed experimentally when considering any kind of experimental process which is controlled only by the strong interaction. This suggests that

[2]One should note that there exist, however, quasi-stationary (indeed quasi-static) self-gravitating solitons in real scalar models with a mass term or more complicated positive potentials, named *oscillatons* [51]. Albeit, strictly speaking, non-static, these can be very long-lived [52–54].

if this CP violating term exists, its magnitude must be very small, leading to a fine tuning problem of its coefficient.

A solution to this problem known as the "strong CP problem" was later proposed by Peccei and Quinn [55, 56]. Their idea was that the coefficient of this term was actually a dynamical field which could dynamically be relaxed to zero. The way it was originally presented, this kind of mechanism would extend the standard model with a complex scalar field. This complex scalar field would then have a "Mexican hat" potential and possess a global $U(1)$ symmetry which could be broken below some high energy scale. This is when the field acquires a non-zero vacuum expectation value (vev) which would translate into the existence of a particle: the *axion* a, which parametrises the degeneracy of the potential vacuum manifold. As a particle emerging from a broken continuous symmetry, it is a Goldstone boson. If, moreover, at least one of the fermions in the model acquires its mass via a Yukawa coupling to the complex scalar, the axion acquires a potential under a chiral anomaly, driving it to a vev that precisely cancels the CP violating term and, moreover, endows the axion with a small mass [57, 58].

When later studied in a cosmological context, it was suggested that axions are also interesting dark matter candidates [59–61], see also [62]. Since then, the study of gravitational effects of axion-like particles has received considerable attention.

1.3.2 Axions in Einstein-Maxwell-scalar

In this work, we consider a model of axion electrodynamics with the following action [63–65]

$$\mathcal{S}_{Ax} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa a}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\nabla_\mu a\nabla^\mu a - U(a)\Big]\ , \quad (1.3.7)$$

where κ is simply a constant and $\tilde{F}_{\mu\nu}$ is the Hodge dual[3] of $F_{\mu\nu}$. We will use a for the scalar field when we refer specifically to the real axion field, mainly throughout chapter 3. The coupling between the axion field and the electromagnetic field is a linear non-minimal coupling which does not contribute to the Einstein equations but does contribute to the matter field equations of motion.

An idea developed in chapter 3 is to generalise the Einstein-Maxwell-scalar model with this axion coupling, obtaining then the Einstein-Maxwell-axion or generalised Einstein-Maxwell-scalar model:

$$\mathcal{S}_{A} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{f(a)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(a)}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\nabla_\mu a\nabla^\mu a - U(a)\Big]\ , \quad (1.3.9)$$

where we added back the non-minimal coupling $f(a)$ changed the linear coupling κa to a general $g(a)$ coupling. Some observations regarding symmetry are in order. The axion

[3]The hodge dual is defined as

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \quad (1.3.8)$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor.

field a is a *pseudoscalar* which means that it changes sign under a parity inversion. This property is necessary for the whole term $aF_{\mu\nu}\tilde{F}^{\mu\nu}$ to be invariant under a parity inversion as $F_{\mu\nu}\tilde{F}^{\mu\nu}$ also changes sign under parity inversion. If we want our Lagrangian to stay invariant under parity inversion, this consistency needs to be considered when we define the couplings $f(a)$ and $g(a)$. First of all, the scalar field in this theory must always be a pseudoscalar, or else the function $g(a)$ will not invert under a parity inversion unless it also depends on the coordinates. This means that even if we consider a to be a general scalar field, it must have this property of the axion field. Secondly, the function $f(a)$ must be expanded in even powers of the scalar field so that it stays a *scalar* while $g(a)$ must have a Taylor expansion in odd powers of the scalar field so that it stays a *pseudoscalar*.

Regarding our search of solitons in this theory, what we do is reconsider the no go theorems which were considered in chapter 2 for the basic EMS model in the context of this generalised model. This discussion is mostly done in chapter 3 for both the model (1.3.7) and then generalised to the model (1.3.9).

1.4 Duality

With the addition of electromagnetism to the model, an interesting question is how *electromagnetic duality* works when we consider the non-minimal interaction with the scalar field. But first, let's discuss duality in classical electromagnetism.

1.4.1 Classical electromagnetic duality

The parallelism between the laws that rule the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$, in the absence of sources, is transparent from Maxwell's equations. In vacuum, these equations are invariant under *electromagnetic duality*:

$$\mathbf{E} + i\mathbf{B} \longrightarrow e^{i\beta}(\mathbf{E} + i\mathbf{B}) \ , \tag{1.4.10}$$

which amounts to an $SO(2)$ rotation by an angle β. Two real β-independent quantities, quadratic in the electromagnetic fields, can be formed, namely:

$$\frac{1}{2}(\mathbf{E} + i\mathbf{B}) \cdot (\mathbf{E} + i\mathbf{B})^* = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \ , \qquad \frac{1}{2i}(\mathbf{E} + i\mathbf{B}) \times (\mathbf{E} + i\mathbf{B})^* = -\mathbf{E} \times \mathbf{B} \ . \tag{1.4.11}$$

This shows that, despite the change in the fields, electromagnetic duality preserves the electromagnetic energy and momentum densities.

Concrete formulations of electromagnetic duality appeared in the wake of Maxwell's equations. In 1893, Heaviside observed these equations are invariant under the discrete transformation $(\mathbf{E}, \mathbf{B}) \to (-\mathbf{B}, \mathbf{E})$ [66], which corresponds to (1.4.10) for $\beta = \pi/2$. This invariance was generalised to the continuous transformation (1.4.10) by Larmor [67]. It was studied in the context of general relativity by Rainich [68] and revisited by Misner and Wheeler in their attempt to understand classical physics as geometry, wherein the terminology *duality rotation* was introduced [69]. In its relativistic formulation, (1.4.10) can be expressed using differential forms as

$$\mathbf{F} \longrightarrow \cos\beta\mathbf{F} + \sin\beta\tilde{\mathbf{F}} \ , \tag{1.4.12}$$

where $\mathbf{F}$ is the Maxwell 2-form and $\tilde{\mathbf{F}}$ denotes its Hodge dual 2-form. This formulation makes clear that duality rotations remain a symmetry of Maxwell's equations in curved spacetime: the covariant theory remains *self-dual.*

Electromagnetic duality rotations are not ordinary rotations in 3-space. They define an equivalence class of electromagnetic fields; that is, there are different $(\mathbf{E}, \mathbf{B})$ solutions to Maxwell's equations which have the same energy and momentum density. However, they are only an invariance of Maxwell's equations in vacuum. For instance, applying (1.4.12) with $\beta = \pi/2$ to the electric field of a static, point electric charge Q, in standard spherical coordinates in flat spacetime (t, r, θ, φ), leads to

$$\mathbf{F} = \frac{Q}{r^2} dt \wedge dr \longrightarrow \tilde{\mathbf{F}} = -Q \sin\theta d\theta \wedge d\varphi \ , \tag{1.4.13}$$

which is the field of a static, magnetic monopole, with magnetic charge Q. Thus, preserving the duality in the presence of electric charges requires magnetic monopoles. From a different reasoning, Dirac noted that the existence of magnetic monopoles could explain electric charge quantisation [70]. Up to now, however, magnetic monopoles have no observational support, and thus electromagnetic duality is an unbroken symmetry in vacuum only. This example illustrates how the $\beta = \pi/2$ rotation, corresponding to the discrete symmetry observed by Heaviside, exchanges electric and magnetic fields.

1.4.2 Beyond classical electromagnetism

It is interesting to consider how duality rotations are affected if one generalises Maxwell's theory, modifying its equations of motion. Gibbons and Rasheed considered the case of relativistic non-linear electrodynamics [71]. They obtained the conditions under which a theory of non-linear electrodynamics, possibly coupled to gravity, has invariant equations of motion under duality rotations, and observed this is the case for Born-Infeld theory [72]. This is a rather exceptional theory, see *e.g.* [73], which naturally appears as the effective field theory describing open string excitations in string theory [74]. In this context, a low energy effective field theory is an Einstein-Maxwell-dilaton-axion model, where the dilaton is a scalar field and the axion a pseudo-scalar field. This is a further generalisation of the model in (1.3.9) by including another scalar field ϕ (dilaton) non-minimally coupled both to the electromagnetic field and to the other scalar field a (axion). The axion will only be coupled to the EM field through the coupling $g(a)$, which is the typical linear coupling κa, while the dilaton is coupled through the coupling $f(\phi)$, which takes the common exponential form $e^{-\alpha\phi}$. As $f(\phi)$ now depends on a (non-pseudo) scalar, it can be expanded in any power of ϕ and this preserves invariance under parity inversion. In [71] it was shown that this model is still self-dual under electromagnetic duality rotations as long as the axion and the dilaton transform in an appropriate way under this transformation. Thus, electromagnetic duality maps solutions of the Einstein-Maxwell-dilaton-axion equations to different solutions of the same model - see also [75–77].

There is, however, a broader notion of duality. Instead of considering self-dual models, which are left invariant (at least at the level of the equations of motion), by some transformation, we can consider *dual theories*: two different models related by a non-trivial duality map. Considering dual theories has been particularly rewarding when the mapping is a strong-weak coupling one. This allows us to relate a model in the weak coupling regime, wherein perturbative computations are possible, to a technically more challenging strongly coupled model, potentially extracting non-trivial information from the latter. Famous examples include the Sine-Gordon – Thirring duality [78], S-duality in string theory [79], and, of course, AdS-CFT [80]. The duality map, moreover, can be used at the level of specific solutions, as a means to obtain a solution of one of the models from a known solution of the dual model. In fact, it is often a non-trivial and useful solution generating technique.

1.4.3 Duality in generalised Einstein-Maxwell-scalar models

In chapter 5 we consider generalised Einstein-Maxwell-scalar models with the axion term, represented by the action (1.3.9) with no scalar potential (with a general scalar field denoted by ϕ):

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{f(\phi)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(\phi)}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) . \tag{1.4.14}$$

This is a family of models for which electromagnetic duality provides a simple realisation of "dual theories". Then, we shall use this mapping as a solution generating technique.

In this work we shall denote a solution of (1.4.14), for a specific choice of $f(\phi), g(\phi)$ as

$$[\mathbf{g}, \mathbf{A}, \phi; f(\phi), g(\phi)] \ , \tag{1.4.15}$$

where $\mathbf{g}, \mathbf{A}, \phi$ are our fields. We shall establish an electromagnetic duality transformation $\mathcal{D}_\beta$, defined by an angle β, that maps any solution (1.4.15) of a certain EMS model (1.4.14) to a different solution of a different (dual) model, within the same family,

$$[\mathbf{g}, \mathbf{A}, \phi; f(\phi), g(\phi)] \xrightarrow{\mathcal{D}_\beta} [\mathbf{g}, \mathbf{A}', \phi; f_\beta(\phi), g_\beta(\phi)] \ . \tag{1.4.16}$$

The rotation angle β parameterises orbits in the space of EMS models, that we shall call *duality orbits*. This space is spanned by the functions f, g. The orbits are closed and relate dual models. On the one hand, the electromagnetic variables and the couplings f, g are affected by the mapping, transforming from the original $\mathbf{A}$ and $f(\phi), g(\phi)$ to a new $\mathbf{A}'$ and $f_\beta(\phi), g_\beta(\phi)$, all of which depend on β. On the other hand, the metric and the scalar field shall remain invariant along the whole duality orbit. In particular, we shall consider the duality orbits passing through some of the EMS models recently studied, including black hole solutions [8, 81–84] and the solitonic solutions discussed in chapter 4. We will also discuss how we can generalise this duality transformation to more general variants of the Einstein-Maxwell-scalar model.

1.5 Scalarisation

In an era where we are now able to probe experimentally into the strong field regime of general relativity through gravitational waves, we are in a position to thoroughly test the black hole solutions that respect the uniqueness theorems stated in section 1.2. It is then important to find alternative models that can possibly explain any experimental deviation from these black hole solutions. One such alternative is the class of models that allow for *spontaneous scalarisation.*

1.5.1 Extended scalar-tensor-Gauss-Bonnet gravity

A class of models that have been considered in the context of scalarisation is the extended-scalar-tensor-Gauss-Bonnet (eSTGB) gravity class of models [85–87], which include a scalar field non-minimally coupled to the Gauss Bonnet gravity correction term. The Gauss-Bonnet term originates from Lovelock's theory of gravity [88]. Lovelock's theory is a higher dimensional generalisation of the Einstein-Hilbert action and comprises the most general metric theory of gravity which still has conserved second order equations of motion. The Lagrangian of Lovelock gravity in $D = 2d$ dimensions is given by the sum of Euler densities of order p with coupling constants α_p:

$$\mathcal{L}_L = \sqrt{-g}\sum_{p=0}^{d-1}\alpha_p\mathcal{L}_{(p)} \ , \tag{1.5.17}$$

which can be written as

$$\mathcal{L}_{(p)} \equiv \frac{(2p)!}{2^p}\delta^{\mu_1}_{[\rho_1}\cdots\delta^{\mu_{2p}}_{\rho_{2p}]}R_{\mu_1\mu_2}{}^{\rho_1\rho_2}\cdots R_{\mu_{2p-1}\mu_{2p}}{}^{\rho_{2p-1}\rho_{2p}} \ , \tag{1.5.18}$$

The Einstein-Hilbert action is the first term of this sum and the Gauss-Bonnet term is the second term of this sum

$$\mathcal{R}^2_{GB} = \mathcal{L}_{(2)} = \frac{4!}{4}\delta^{\mu_1}_{[\rho_1}\delta^{\mu_2}_{\rho_2}\delta^{\mu_3}_{\rho_3}\delta^{\mu_4}_{\rho_4]}R_{\mu_1\mu_2}{}^{\rho_1\rho_2}R_{\mu_3\mu_4}{}^{\rho_3\rho_4} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \ . \tag{1.5.19}$$

Gauss-Bonnet gravity is simply Lovelock gravity up to the second term of the expansion. However, this second term (and the following terms of the expansion) are all effectively trivial in 4 dimensions[4].

A simple way to make this term's contribution to the equations of motion nontrivial in four dimensions is to non-minimally couple it to a scalar field. And with this, we obtain extended-scalar-tensor-Gauss-Bonnet gravity which is described by the following action

$$\mathcal{S}_{eSTGB} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left[\frac{R}{4} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \alpha_{GB}f(\phi)\mathcal{R}^2_{GB}\right] . \tag{1.5.20}$$

This model can be considered a natural modification of general relativity. Any scalar free solution of this model will be a solution of the typical vacuum general relativity model. The question then is if there are any solutions with a non-trivial scalar field. A solution was indeed found in [85–87] which bifurcates from the Schwarzschild solution. At low mass (or high curvature) values, the Schwarzschild solution becomes unstable under scalar perturbations due to the Gauss-Bonnet term. This then causes the Schwarzschild solution to *scalarise*. One can imagine such a process as a phase transition where a small scalar field perturbation expands into a scalar cloud that surrounds the black hole. These scalarised solutions are not only stable under spherical perturbations [90] but also thermodynamically preferred over the vacuum solutions [85]. This phenomenon of spontaneous scalarisation was originally found for neutron stars in the similar context of scalar-tensor theories [91], the main difference being that the scalarisation was induced by matter instead of strong spacetime curvature. For further discussion, see [92, 93] for more on matter induced scalarisation and [94–98] for more on curvature induced scalarisation.

1.5.2 Charged black hole scalarisation

As we saw in the last section, the presence of the Gauss-Bonnet term created an instability in the original Schwarzschild solution of vacuum general relativity[5], causing the phenomenon of spontaneous scalarisation. To better understand this phenomenon we just need to consider the following action

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_\phi \ , \tag{1.5.21}$$

where $\mathcal{S}_\phi$ reads as (up to a constant):

$$\mathcal{S}_\phi = -\int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\nu\phi + \alpha\frac{f(\phi)}{4}\mathcal{I}\right) , \tag{1.5.22}$$

[4]In a somewhat recent paper [89], the authors have found that there is a non-trivial limit where the Gauss-Bonnet term does not vanish if we take the limit $D \to 4$ from above, as long as the term's coupling constant α_{GB} has a dimensional dependence of the form $\alpha_{GB} = \tilde{\alpha}/(D-4)$ where $\tilde{\alpha}$ is fixed.

[5]This instability is also relevant for Kerr spacetime as was shown in [99].

where $\mathcal{I}$ is an arbitrary function to be defined, and so the scalar field respects the following equation of motion

$$\Box^2\phi = \alpha\frac{df}{d\phi}\mathcal{I} \ . \tag{1.5.23}$$

In this kind of models, scalarisation is possible as long as the following three conditions are satisfied:

i) There exists a *scalar-free* solution with $\phi = \phi_0$. For $\mathcal{I} \neq 0$ eq. (1.5.23) implies the coupling function should satisfy the condition

$$\frac{df}{d\phi}\bigg|_{\phi=\phi_0} = 0 \ . \tag{1.5.24}$$

One may set $\phi_0 = 0$ via a field redefinition. Thus electrovacuum BHs plus a vanishing scalar field solve the model.

ii) The scalar-free solution suffers a tachyonic instability triggered by linear scalar perturbations. For a small scalar field $\phi = \delta\phi$, linearising (1.5.23) yields

$$(\Box - \mu_{\text{eff}}^2)\delta\phi = 0 \ , \qquad \text{where} \qquad \mu_{\text{eff}}^2 \equiv \alpha\frac{d^2f}{d\phi^2}\bigg|_{\phi=0}\mathcal{I} \ . \tag{1.5.25}$$

Let us assume that $d^2f/d\phi^2\Big|_{\phi=0}$ is strictly positive. Then the tachyonic condition $\mu_{\text{eff}}^2 < 0$ implies

$$\alpha\mathcal{I} < 0 \ , \tag{1.5.26}$$

which must hold *for some region* outside the horizon.

iii) A second set of solutions exists, with a nontrivial scalar field, the *scalarised BHs*. These solutions are continuously connected with the scalar-free set, approaching it as $\phi \to 0$. In this limit, the scalar field becomes a *scalar cloud* or *zero mode*. Although the quantitative properties of these solutions depend on the choice of the coupling function, qualitative properties are not so sensitive to this choice, as long as the condition (1.5.24) is satisfied.

We can see that if $\mathcal{I} = R^2_{GB}$ we recover the eSTGB model [85–87] and with $\mathcal{I} = R$ we obtain the original scalarisation mechanism [91]. But, of course, the model we are interested in this thesis is the EMS model (1.1.2) which is equivalent to having $\mathcal{I} = F_{\mu\nu}F^{\mu\nu}$.

Scalarisation in the EMS model was originally presented in [8]. As we know, the EMS model admits the scalar free solution of Reissner-Nordström (RN), which fits the criteria for (i), so it is intuitive to think that there might be a scalarisation process for this solution just as there is for the Schwarzschild solution. If, for example, we take a purely electric solution where $F^2 < 0$, we can find the Bekenstein type identity [8, 100] $\phi f_{,\phi} > 0$ which, along with the identities in condition (ii), allows us to narrow down the coupling $f(\phi)$. A simple coupling that is compatible with all these conditions is

$$f(\phi) = e^{-\alpha\phi^2} \ , \tag{1.5.27}$$

with $\alpha < 0$ being a dimensionless constant. In the RN background, we obtain $\mu_{\text{eff}}^2 = \alpha Q^2/r^4 < 0$, meaning it exhibits the tachyonic instability. This then leads to the scalarised charged black hole solutions obtained in [8] which are solutions that bifurcate away from the RN solution and are also thermodynamically preferred. This scalarisation process tends to occur for sufficiently high charge. See [81–83, 101–105] for further work on these scalarised EMS black holes.

In chapter 6, we discuss the scalarisation process of the RN solution in higher dimensions and explicitly construct these scalarised BHs for $d = 5$. We also observe how a conformal transformation between the Jordan and Einstein frames maps a model with a scalar field non-minimally coupled to the Ricci scalar to another model where a new scalar field is non-minimally coupled to the electromagnetic term (EMS model), relating two different scalarisation processes. Next, the spontaneous scalarisation of the Schwarzschild-Tangherlini BH in extended-scalar-tensor-Lovelock gravity in even dimensions is considered. These are models where the scalar field is non-minimally coupled to the $(d/2)^{th}$ Euler density, in d spacetime dimensions. Examples in $d = 6, 8$ are explicitly constructed, showing the properties of the four dimensional case are qualitatively generic, but with quantitative differences. These higher d scalarised BHs are then compared to the hairy BHs in shift-symmetric Horndeski theory, for the same d, which are also constructed.

1.6 Structure

Now we briefly summarise the structure of this thesis. Chapter 2 is dedicated to the existence or inexistence of soliton solutions in the EMS model. Uniqueness and no go theorems are more thoroughly discussed in this context and three no go theorems are proven. Chapter 3 is dedicated to the same topic of soliton solutions in an extended EMS model, where the axion is introduced, and the theorems of the preceding chapter are adapted to this more general model. In Chapter 4 we, taking into account the conditions imposed by the no go theorems, circumvent the restrictions to the existence of a soliton solution and find a concrete example of a soliton solution in the EMS model. The concepts covered in sections 1.2 and (1.3) of this introductory chapter introduce the context and necessary concepts to understand the work covered in these three chapters.

Then we proceed to Chapter 5 where duality of the EMS model, as discussed in section 1.4, is covered. We define the duality orbits of the EMS model after formulating a duality transformation that can be used as a map between solutions with different non-minimal couplings.

Chapter 6 will then discuss the topic of scalarisation, which was introduced in section 1.5, but in higher dimensions. A general model which includes the EMS model is presented and we do a qualitative and numerical analysis of various constructed scalarised solutions in higher dimensional spacetime.

Finally, we get to chapter 7 where comments and remarks are made about the work covered throughout the whole thesis.

Chapter 2

On the inexistence of solitons in Einstein-Maxwell-scalar models

In this chapter, we will introduce various concepts regarding the uniqueness of specific solutions in the Einstein-Maxwell-scalar model (1.1.2).

In section 2.1, a vanishing Maxwell field is considered and a search is then made for the most general asymptotically flat, stationary and axisymmetric line element in the Einstein-scalar theory. Following this we show, using a scaling argument, how this line element is incompatible with a solitonic solution, proving a no go theorem for solitons.

The Maxwell field is once again considered in section 2.2, where we first consider static scalar-electromagnetic solitons. A no go theorem for this kind of solitons is shown by adapting an argument by Heusler to the Einstein-Maxwell-scalar model which was originally considered for electro-vacuum [24].

At last, in section 2.3, we apply a modified Lichnerowicz-type argument to generalise the result of the no go theorem for strictly stationary spacetimes, which allows us to consider rotating metrics. Some remarks and conclusions are then presented in 2.4.

2.1 Absence of stationary solitons for vanishing Maxwell field

The first result to be established concerns the absence of asymptotically flat, stationary and axisymmetric self-gravitating scalar solitons. This means we have no electromagnetic field, thus we consider $F_{\mu\nu} = 0$. Observe this is a consistent truncation of the model (1.1.2). That is, taking a vanishing Maxwell tensor in the action is equivalent to taking a vanishing Maxwell tensor in the field equations. Thus, the model under consideration in this section is the following Einstein-scalar model[1]

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - U(\phi)\Big]. \tag{2.1.1}$$

[1] For a complex scalar field Φ, every $\nabla_\mu\phi\nabla_\nu\phi$ term is replaced by $\frac{1}{2}(\nabla_\mu\Phi^*\nabla_\nu\Phi + \nabla_\mu\Phi\nabla_\nu\Phi^*)$.

Our first task is to show the most general metric form for the configurations we seek to rule out is

$$ds^2 = -\frac{\rho^2}{X(\rho,z)}dt^2 + X(\rho,z)\left[d\varphi - w(\rho,z)dt\right]^2 + A(\rho,z)\left[d\rho^2 + B(\rho,z)dz^2\right] , \qquad (2.1.2)$$

which contains four unknown functions of the "cylindrical" coordinates ρ and z. The second task is to rule such non-trivial solitonic solutions by applying a scaling argument.

2.1.1 Most general line element

2.1.1.1 Isometries

Firstly, axisymmetry and stationarity implies the existence of two Killing vector fields m and k. Without loss of generality, these Killing vectors commute $[k,m] = 0$ [106], assuming the spacetime is asymptotically flat. Thus coordinates adapted simultaneously to *both* these vectors fields can be chosen. As k corresponds to the asymptotically timelike Killing vector field and m to the spacelike one, a temporal coordinate t and an angular coordinate φ are introduced along the orbits of the Killing vector fields as:

$$k = \frac{\partial}{\partial t} , \qquad m = \frac{\partial}{\partial \varphi} . \qquad (2.1.3)$$

Consequently, in coordinates (t,φ,x,y), the general line element can be cast in the form:

$$ds^2 = g_{\mu\nu}(x,y)dx^\mu dx^\nu . \qquad (2.1.4)$$

2.1.1.2 Circularity

We now want to prove that our metric is *circular*. That is, the surfaces orthogonal to the Killing fields k and m are integrable. By Frobenius' theorem (see [107] App. B.3), the surfaces orthogonal to the Killing fields are integrable if the following conditions hold:

$$dk \wedge k \wedge m = 0 = dm \wedge m \wedge k . \qquad (2.1.5)$$

Circularity means that the spacetime manifold $\mathcal{M}$ is locally a product of two 2-dimensional manifolds $\mathcal{M} = \mathcal{N}_1 \times \mathcal{N}_2$ and can be cast in the following form:

$$ds^2 = g_{\mu\nu}(x,y)dx^\mu dx^\nu = \sigma_{ab}(x,y)dx^a dx^b + \gamma_{ij}(x,y)dx^i dx^j , \qquad (2.1.6)$$

where $\boldsymbol{\sigma}$ corresponds to the metric in the (t,φ) manifold and $\boldsymbol{\gamma}$ corresponds to the metric in the (x,y) manifold. Establishing circularity requires using the Einstein equations and hence depends on the energy-momentum of the spacetime. Circularity can actually be established by first establishing *Ricci circularity* as we now discuss.

For the case under consideration, the energy-momentum tensor obtained from (2.1.1) is

$$4\pi T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + U(\phi)\right) . \qquad (2.1.7)$$

It follows that

$$T(k) \wedge k \wedge m = 0 = T(m) \wedge m \wedge k \ , \tag{2.1.8}$$

where the $T(k)$ and $T(m)$ 1-forms correspond to the contraction of the energy-momentum tensor with the Killing vectors. To establish this observe that, since the spacetime is stationary, the Einstein equations imply that $\pounds_k g_{\mu\nu} = 0 \Rightarrow \pounds_k T_{\mu\nu} = 0$ and this in turn implies, due to ϕ being real, that $\pounds_k \phi = k^\mu \nabla_\mu \phi = 0$. Thus,

$$4\pi T(k)_\mu = 4\pi T_{\mu\nu} k^\nu = -k_\mu \left(\frac{1}{2} \nabla_\alpha \phi \nabla^\alpha \phi + U(\phi) \right) \ , \tag{2.1.9}$$

meaning that $T(k)$ is proportional to k and, as such, $T(k) \wedge k = 0$. A similar argument shows that $T(m) \wedge m = 0$, proving the equality (2.1.8). Now, using Einstein's equations it follows that

$$R(k) \wedge k \wedge m = 0 = R(m) \wedge m \wedge k \ , \tag{2.1.10}$$

where the $R(k)$ and $R(m)$ 1-forms correspond to the contraction of the Ricci tensor with the Killing vectors. A spacetime which respects (2.1.10) is called *Ricci circular*. Thus, we have shown that an asymptotically flat, axisymmetric and stationary spacetime sourced by a real scalar field (with an arbitrary potential) is Ricci circular. But Ricci circularity and circularity are equivalent for asymptotically flat, stationary and axisymmetric spacetimes [13, 108, 109], concluding the proof of circularity.[2] It follows we can then write the line element as

$$ds^2 = -V dt^2 + 2W dt d\varphi + X d\varphi^2 + \gamma_{ij} dx^i dx^j \ , \tag{2.1.11}$$

where $V = -\langle k|k \rangle$, $X = \langle m|m \rangle$ and $W = \langle k|m \rangle$. We have now reduced the unknown metric functions from ten to six.

Two remarks are in order. Firstly, observe that circularity is equivalent to assuming the spacetime to be invariant under the simultaneous discrete symmetry transformations $(t, \varphi) \to (-t, -\varphi)$. However, the circularity argument shows that (for our matter content) this is not an assumption and no generality is lost. Secondly, for a *complex* scalar field Φ, circularity is actually lost. Indeed, we cannot guarantee the implication $\pounds_k T_{\mu\nu} = 0 \Rightarrow \pounds_k \Phi = 0$, as the field can have a harmonic dependence on both t and ϕ in the form of a phase. In spherical coordinates the field would be ($n \in \mathbb{Z}$ and $w \in \mathbb{R}$ are constants)

$$\Phi(t, r, \theta, \varphi) = \phi(r, \theta) e^{i(n\varphi - \omega t)} \ , \tag{2.1.12}$$

and this dependence does not manifest itself in the energy-momentum tensor, which is still preserved independently by the two Killing vector fields. Thus, the form (2.1.11) is no longer the most general metric form describing an asymptotically flat, axisymmetric,

[2] This proof can be extended for the full model (2.1.1) including the electromagnetic field. Such extension, however, is not relevant for our discussion so we will omit the details regarding the electromagnetic field part of the action which can be found in [13, 108].

stationary spacetime sourced by a complex scalar field.[3] Interestingly, this complex scalar field case allows circumventing no-scalar hair theorems for black holes, *e.g.* Bekenstein's theorem [100], and yields black holes with scalar hair that are asymptotically flat, stationary and axisymmetric [18, 110]. The known solutions have a geometry invariant under $(t, \varphi) \to (-t, -\varphi)$; the absence of circularity opens up the possibility that more general hairy black holes may exist in the complex scalar field model.

2.1.1.3 The orthogonal manifold

The simplification of the orthogonal (x, y) manifold, with metric γ_{ij}, can now be addressed. Due to the gauge freedom, *i.e.* the ability to redefine the (x, y) coordinates, one anticipates the possibility of reducing further the number of unknown functions from six to four. Indeed, one first introduces a scalar function ρ, defined as

$$\rho \equiv \sqrt{-\sigma} = \sqrt{VX + W^2} \ , \tag{2.1.13}$$

where σ corresponds to the determinant of the σ_{ab} metric. Assuming that both ρ and $\nabla_\mu \rho$ do not vanish (to be further discussed below), we can choose ρ as a coordinate on the orthogonal manifold. Introducing a second coordinate therein, z, chosen such that $\nabla_\mu z \nabla^\mu \rho = 0$ (by setting $z =$ constant along the integral curves of $\nabla^\mu \rho$), we can write the full metric in the form (2.1.2), where

$$w \equiv -\frac{W}{X} \ . \tag{2.1.14}$$

There are now four unknown metric components, X, w, A and B of two variables ρ and z. Considering the way ρ and z were defined, this coordinate system is valid for $\rho \in]0, \infty[$ and $z \in]-\infty, \infty[$. We have now reached the maximal possible simplicity for the line element under our assumptions and for the matter content we wish to consider. Nonetheless, it is instructive to consider a further a simplification, which, however, is non-generic for our model.

A further simplification can be made if ρ is harmonic on the 2-dimensional orthogonal manifold, $\Box^2_{(\gamma)} \rho = 0$. In this case, z will be the harmonic conjugate function of ρ and they will obey the Cauchy-Riemann equations, which give us the following expressions:

$$\gamma_{\rho z} = \langle \nabla_{(\gamma)} \rho | \nabla_{(\gamma)} z \rangle = 0 \ , \tag{2.1.15}$$

$$\gamma_{\rho\rho} = \langle \nabla_{(\gamma)} \rho | \nabla_{(\gamma)} \rho \rangle = \langle \nabla_{(\gamma)} z | \nabla_{(\gamma)} z \rangle = \gamma_{zz} \ . \tag{2.1.16}$$

We can then set $B = 1$ in (2.1.2) and, after redefining $A \equiv e^{2h}/X$ (for consistency with the literature), we obtain the Weyl-Lewis-Papapetrou (WLP) metric:

$$ds^2 = -\frac{\rho^2}{X(\rho, z)} dt^2 + X(\rho, z) \left[d\varphi - w(\rho, z) dt \right]^2 + \frac{e^{2h(\rho, z)}}{X(\rho, z)} \left[d\rho^2 + dz^2 \right] \ , \tag{2.1.17}$$

[3] We thank E. Ayón-Beato for this observation.

which has only three unknown functions.

Another important consequence of ρ being harmonic is that, if ρ is not a constant, it can be shown that it has *no critical points* in the orthogonal manifold (no points where the gradient vanishes) [111]. As this choice of coordinates is well behaved except for when $d\rho = 0$, it means the coordinate system is globally well behaved in the whole manifold, as long as there are no event horizons. So the metric (2.1.17) can be used to describe the whole stationary and axisymmetric spacetime.

The key question for this further simplification is: when can it be guaranteed that ρ is harmonic? It has been proven by Papapetrou and others [13, 112] that ρ is harmonic as long as the projection of the Ricci tensor $\boldsymbol{R}$ along the (t, φ) surfaces is trace free through the following equation [13]:

$$\frac{1}{\rho}\Box^2_{(\gamma)}\rho = -\frac{1}{X} tr_\sigma \boldsymbol{R} \ . \tag{2.1.18}$$

Using the metric (2.1.11) for simplicity, this translates into the following condition for ρ to be harmonic:

$$tr_\sigma \boldsymbol{R} = \sigma^{ab}R_{ab} = \frac{1}{\rho^2}\big[- XR(k,k) + 2WR(k,m) + VR(m,m)\big] = 0 \ . \tag{2.1.19}$$

So as long as (2.1.19) is respected, ρ is harmonic and the coordinates used in the metric (2.1.17) are globally well defined: the γ_{ab} metric is *globally conformally flat*.[4] Now all that is left is to check if our model respects condition (2.1.19). We can write the Ricci tensor in terms of the energy momentum tensor as follows:

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \ . \tag{2.1.20}$$

The Ricci tensor components $R(k,k)$, $R(k,m)$ and $R(m,m)$ for the real scalar field energy momentum tensor (2.1.7) read

$$R(k,k) = R_{\mu\nu}k^\mu k^\nu = -2U(\phi)V \ , \tag{2.1.21}$$

$$R(m,m) = R_{\mu\nu}m^\mu m^\nu = 2U(\phi)X \ , \tag{2.1.22}$$

$$R(k,m) = R_{\mu\nu}k^\mu m^\nu = 2U(\phi)W \ , \tag{2.1.23}$$

yielding

$$tr_\sigma \boldsymbol{R} = 4U(\phi) \ . \tag{2.1.24}$$

This can only be zero in the whole manifold if the potential $U(\phi)$ vanishes. So the metric (2.1.17) is the most general metric for a free real scalar field only; for a non-zero potential we must resort to the form (2.1.2), as we will do in the next subsection.

[4]Every two-dimensional metric is *locally conformally flat* due to the existence of isothermal coordinates (the uniformisation theorem [113,114]). Such choice of coordinates is, however, only *locally* and not *globally* conformally flat. Thus one cannot guarantee their validity throughout the whole orthogonal manifold. Invoking such conformal flatness would reduce the unknown metric functions from six to four, instead of the three obtained in the WLP metric, since ρ would not be used as a coordinate any longer.

2.1.2 Scaling argument

We have established that the most general line element that can describe the solitonic solutions we seek is given by eq. (2.1.2), where (t, ρ, φ, z) are "cylindrical" coordinates and A, B, w and X are unknown functions of the non-Killing coordinates (ρ, z) of which A, B and X are all positive. We can now proceed to show that there are no non-trivial solutions of this form for the model (2.1.1) by a scaling argument. It is useful to observe that the square root of minus the metric determinant is

$$\sqrt{-g} = \rho A \sqrt{B} \ . \tag{2.1.25}$$

We start with the Einstein-scalar action for the scalar field (2.1.1). The proof follows by contradiction. We assume that such a stationary, axisymmetric, asymptotically flat self-gravitating scalar soliton exists. Because the scalar field is real, it respects, for this hypothesised solution the same symmetries as the metric possesses, so it does not change under the action of the stationary and axisymmetric Killing vectors. The next step is to consider a scaling of the hypothesised solution by a scale factor λ. This rescales the coordinates (ρ, z) and defines a one-parameter family of configurations (not necessarily solutions) of the coupled geometry-scalar system:

$$A_\lambda(z,\rho) = A(\lambda z, \lambda\rho) \ , \quad B_\lambda(z,\rho) = B(\lambda z, \lambda\rho) \ , \quad w_\lambda(z,\rho) = w(\lambda z, \lambda\rho) \ , \tag{2.1.26}$$

$$X_\lambda(z,\rho) = X(\lambda z, \lambda\rho) \ , \quad \phi_\lambda(z,\rho) = \phi(\lambda z, \lambda\rho) \ . \tag{2.1.27}$$

Under this scaling transformation, the metric determinant transforms as, using (2.1.25),

$$\int d^4x\sqrt{-g} \to \int d^4x_\lambda \sqrt{-g_\lambda} = \lambda^3 \int d^4x \, \rho A_\lambda \sqrt{B_\lambda} \ , \tag{2.1.28}$$

and the kinetic scalar field term transforms as

$$-\nabla^\mu\phi\nabla_\mu\phi = -\frac{1}{A}\left[(\partial_\rho\phi)^2 + \frac{1}{B}(\partial_z\phi)^2\right] \to -\frac{1}{\lambda^2 A_\lambda}\left[(\partial_\rho\phi_\lambda)^2 + \frac{1}{B_\lambda}(\partial_z\phi_\lambda)^2\right] \ . \tag{2.1.29}$$

The action of the scaled solutions is $\mathcal{S}^\lambda = \mathcal{S}[\phi_\lambda, B_\lambda, A_\lambda, w_\lambda, X_\lambda]$. Since for $\lambda = 1$ we have the hypothesised solution, the variation of $\mathcal{S}^\lambda$ with respect to λ must have a stationary point at $\lambda = 1$. This condition yields the virial relation:[5]

$$\frac{1}{2}\int_0^{+\infty} d\rho \int_{-\infty}^{+\infty} dz \, \rho\left[\sqrt{B}(\partial_\rho\phi)^2 + \frac{1}{\sqrt{B}}(\partial_z\phi)^2\right] = -3\int_0^{+\infty} d\rho \int_{-\infty}^{+\infty} dz \, \rho A\sqrt{B} \ U(\phi) \ . \tag{2.1.30}$$

As the left side is always non-negative, and the right hand side is always non-positive, for positive $U(\phi)$ we get to a contradiction, which can only be settled if the hypothesised solution is trivial. Alternatively, a negative potential is mandatory, to have a non-trivial

[5]Only the matter field action enters the argument because the Einstein-Hilbert part of the action is invariant under a rescaling transformation as it corresponds to a *diffeomorphism*. In other words, we have that $\delta\mathcal{S}^\lambda_{EH}/\delta\lambda = 0$.

solitonic solution. Thus, the only asymptotically flat, everywhere regular, stationary and axisymmetric localised solution for the model (2.1.1) with $U(\phi) \geqslant 0$ is Minkowski spacetime. No self-gravitating scalar solitons exist in this model and under these assumptions.

The virial identity (2.1.30) can be written in a more compact form as

$$\int d^4x\sqrt{-g}\Big[\nabla^\mu\phi\nabla_\mu\phi + 6U(\phi)\Big] = 0 \ . \tag{2.1.31}$$

One observes the resemblance with the identity that Bekenstein deduced when attempting to rule out a black hole spacetime with scalar hair in the same model [100]:

$$\int d^4x\sqrt{-g}\left[\nabla^\mu\phi\nabla_\mu\phi + \phi\frac{dU(\phi)}{d\phi}\right] = 0 \ . \tag{2.1.32}$$

To obtain the latter identity for the model (2.1.1) one integrates the Klein-Gordon equation $\nabla^\mu\nabla_\mu\phi - U'(\phi) = 0$ multiplied by ϕ over the whole spacetime and then, upon integrating by parts the first term, a surface term at infinity emerges, which vanishes since $\phi\nabla_\mu\phi \to 0$ at infinity for an asymptotically flat spacetime. This procedure yields (2.1.32).[6]

A key difference between the Bekenstein identity (2.1.32) and the virial identity (2.1.31) is that the latter is independent from the equations of motion, while the former is a consequence of the scalar equation of motion. Moreover, they yield different (but complementary) conclusions. In particular, the Bekenstein identity with a positive potential is not enough to obtain the no go theorem we have just described. Rather it would rule out gravitating solitons under the assumption that $\phi U'(\phi) > 0$ everywhere (except for some discrete points where it can vanish), rather than the positivity of the potential.

Another remark concerns the case of a constant, but non-zero, potential $U(\phi) = \Lambda$, that can be interpreted as a cosmological constant. Does the virial identity (2.1.30) encode a no-go theorem for free spinning solitons in de Sitter spacetime ($\Lambda > 0$)? Actually no, since for de Sitter spacetime the metric (2.1.2) is not necessarily the most general metric. This is because the de Sitter spacetime is not asymptotically flat, a requirement to guarantee both that the commutativity of the Killing vectors [106] and that the circularity theorem [13, 108, 109] holds. Moreover, of course, de Sitter is not stationary, and the very definition of an equilibrium self-gravitating soliton has to be reconsidered. In the case of Anti-de-Sitter, no conclusions can be inferred either, but we remark that a no-go theorem for self-gravitating, purely gravitational solitons in Anti-de-Sitter was presented in [116].

2.2 Absence of static scalar-electromagnetic solitons

We now turn to the full model (1.1.2) to rule out *static*, asymptotically flat scalar-electromagnetic solitons. In this case no spatial symmetry assumption is made. The argument generalises

[6] Another slight variation consists on multiplying the Klein-Gordon equation by $dU/d\phi$ instead of ϕ [115]. In this case one gets an obstruction under the condition of the convexity of the potential, $d^2U/d\phi^2 > 0$ - see also [19].

an electro-vacuum argument by Heusler [24]. One condition we will set on the non-minimal coupling $f(\phi)$ of the EMS model is that it *must never diverge* or *vanish* in the spacetime. The requirement that f must not vanish is only necessary if we have a magnetic field. This is an important condition to ensure that our theorems are valid.

2.2.1 Heusler's argument for static spacetime

Our focus in this subsection is a static spacetime. We consider an asymptotically flat, everywhere regular and static spacetime with a strictly stationary Killing field k, obeying $V \equiv -k^\mu k_\mu > 0$, and shall prove that there are no solitons in the full model theory. The Einstein-Hilbert action $\mathcal{S}_{EH}$ will not play a role in this argument and neither will the metric as only the electromagnetic field equations of motion are used.

Define the electric and magnetic fields as:

$$E_\mu = -F_{\mu\nu}k^\nu \ , \tag{2.2.33}$$

$$B_\mu = -\frac{1}{2}\varepsilon_{\mu\alpha\beta\nu}F^{\alpha\beta}k^\nu \ , \tag{2.2.34}$$

where $\varepsilon_{\mu\alpha\beta\nu}$ is the Levi-Civita tensor. The Maxwell equations for the full model are

$$\nabla_{[\mu}E_{\nu]} = 0 \ , \tag{2.2.35}$$

$$\nabla_{[\mu}(fB_{\nu]}) = 0 \ , \tag{2.2.36}$$

$$\nabla_\mu\left(f\frac{E^\mu}{V}\right) = 0 \ , \tag{2.2.37}$$

$$\nabla_\mu\left(\frac{B^\mu}{V}\right) = 0 \ . \tag{2.2.38}$$

Due to the absence of currents, the first two Maxwell equations imply that the electric φ and magnetic-like ψ *scalar* potentials can be introduced, as $E_\mu = \partial_\mu\varphi$ and $fB_\mu = \partial_\mu\psi$. Note that the expression that defines the scalar ψ is not well defined when f is zero, which is why we restrict f to always be non-zero if we have a magnetic field[7]. A general mathematical identity states that for an arbitrary vector α that respects $\pounds_k\alpha = [k,\alpha] = 0$:

$$\int_{\partial\Sigma}\alpha^\mu k^\nu dS_{\mu\nu} = \frac{1}{2}\int_\Sigma \nabla_\mu\alpha^\mu k^\nu d\Sigma_\nu \ , \tag{2.2.39}$$

where Σ is an hypersurface, with volume element $d\Sigma_\nu$, while $\partial\Sigma$ corresponds to its boundary, with antisymmetric area element $dS_{\mu\nu}$. This is a version of Stokes' theorem in the presence of a Killing field. Applying this identity with $\alpha^\mu \to fE^\mu/V$ we obtain, using Maxwell's equations,

$$\int_{\partial\Sigma} f\frac{E^\mu k^\nu}{V}dS_{\mu\nu} = 0 \ . \tag{2.2.40}$$

[7] In the case where f can vanish, we will have non-zero contributions to surface integrals when applying the Stokes theorem. We will see in chapter 5 that soliton solutions with a magnetic field will always have a vanishing f.

If Σ is any Cauchy surface, we can take $\partial\Sigma$ to be the surface at infinity, in which case:

$$\int_{\partial\Sigma} f\frac{E^\mu k^\nu}{V} dS_{\mu\nu} = -4\pi Q_e f_\infty = -4\pi Q_e = 0 \ , \tag{2.2.41}$$

where Q_e corresponds to the electric charge. If we now replace α^μ by $\varphi f E^\mu/V$ and once again use the Maxwell equations, we obtain

$$\frac{1}{2}\int_\Sigma f\frac{E^\mu E_\mu}{V} k^\nu d\Sigma_\nu = \int_{\partial\Sigma} \varphi f\frac{E^\mu k^\nu}{V} dS_{\mu\nu} = -4\pi Q_e \varphi_\infty = 0 \ . \tag{2.2.42}$$

Since $k^\mu E_\mu = 0$ then E is never timelike and, assuming that the coupling f does not change sign, it follows that this expression only holds if the electric field vanishes. The same argument can be used for B by replacing φ by ψ, obtaining

$$\int_\Sigma f\frac{B^\mu B_\mu}{V} k^\nu d\Sigma_\nu = 0 \ , \tag{2.2.43}$$

from which we conclude that B must also vanish if f does not change sign, which is implied by requiring that f must always be non-zero if we have a magnetic field. Note that this is not required for a purely electric solution. Thus, for a constant sign coupling function, solitons with a non-trivial electromagnetic field are ruled out, regardless of the potential $U(\phi)$.

It remains the possibility that there could be self-gravitating solitons with a non-trivial scalar field. However, the scalar field must also vanish, as long as it obeys the dominant energy condition *and* violates the strong energy condition [13, 50]. The rationale is the following. If the scalar field violates the strong energy condition this implies its Komar mass M is negative; but if it respects the dominant energy condition, the positive mass theorem is applicable and its ADM (or Komar) mass is non-negative. This leads us to a contradiction. So it remains to see what these conditions mean for the model. Consider the energy-momentum tensor of the full action (1.1.2):

$$4\pi T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + U(\phi)\right) + f(\phi)\left(F_{\mu\alpha}F_\nu^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \ . \tag{2.2.44}$$

The strong energy condition requires $R_{\mu\nu}\tilde{k}^\mu\tilde{k}^\nu \geqslant 0$ for any timelike vector field $\tilde{k}^\mu$ (for instance one that obeys $\tilde{k}^\mu = k^\mu/\sqrt{V}$). For $F = 0$, and a static spacetime with a purely spatial scalar field distribution, this yields,

$$U \leqslant 0 \ . \tag{2.2.45}$$

Thus, a scalar field with a non-negative potential only obeys the strong energy condition if the potential is trivial. If, moreover, it obeys the dominant energy condition, then the aforementioned contradiction applies, except if $U = 0$. In that case, the Komar mass is zero, and by the positive energy theorem, the resulting solution is Minkowski spacetime.

Since the result in this section relies on the constancy of the sign of $f(\phi)$, one may ask what is the physical meaning of a change in the sign of the coupling function $f(\phi)$. To assess this, observe that from the energy-momentum tensor of the full action (2.2.44), the energy density is given as

$$\rho = 4\pi T_{\mu\nu}k^\mu k^\nu = V\left(\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + U(\phi)\right) + f(\phi)\left(E_\alpha E^\alpha + \frac{V}{4}F_{\alpha\beta}F^{\alpha\beta}\right) , \qquad (2.2.46)$$

where $V = -k^\mu k_\mu$ is always positive as the spacetime is strictly stationary. We see that the energy density contribution of the electromagnetic field will, generically, change sign along with $f(\phi)$. Another perspective is that the electric permittivity would also change sign. Both these observations make such sign change physically questionable, as it would make the electromagnetic contribution a sort of exotic matter. It is worth noting, however, that the weak energy condition $\rho > 0$ needs not be violated even if such sign change in the coupling occurs, as the scalar field contribution could compensate for the opposite sign contribution of the electromagnetic field.

To close this section, let us remark on the dominant energy condition. In the full model, if the coupling $f(\phi)$ changes sign to negative, then the full model may not respect the dominant energy condition even if, separately, the electromagnetic and scalar parts (excluding the coupling) abide it. This can be seen as follows: the dominant energy condition states that

$$T_{\mu\nu}X^\mu Y^\nu \geqslant 0 , \qquad (2.2.47)$$

for any two co-oriented causal vectors X^μ and Y^μ. Then, even assuming the scalar and electromagnetic EM tensors obey it

$$T^S_{\mu\nu}X^\mu Y^\nu \geqslant 0 , \qquad T^E_{\mu\nu}X^\mu Y^\nu \geqslant 0 , \qquad (2.2.48)$$

the full model EM tensor is $T_{\mu\nu} = T^S_{\mu\nu} + f(\phi)T^E_{\mu\nu}$, which needs not respect the dominant energy condition due to the sign of $f(\phi)$. While a negative $f(\phi)$ does not directly imply that the model does not respect the dominant energy condition, this possibility ties with the fact that a different sign for $f(\phi)$ implies that the electromagnetic field behaves like exotic matter. It seems, thus, that in the most reasonable physical scenarios, $f(\phi)$ should not change sign and the dominant energy condition will hold for the full model as long as it obeys, separately, for the electromagnetic and scalar sectors.

2.3 Absence of strictly stationary scalar-electromagnetic solitons

A further step beyond the last result in the direction of generality, is to rule out *strictly stationary*, but not necessarily static, asymptotically flat scalar-electromagnetic solitons in our full model (1.1.2). With this goal in mind, we consider a Lichnerowicz-type argument adapting the one presented in [40] where it was applied to Einstein-Maxwell-scalar models, but where the scalar field has no direct coupling to the electromagnetic field. This argument consists in finding a divergence identity from which we may restrict the ADM mass of the system to vanish. Thus, as long as the dominant energy condition holds, one can conclude, from the positive mass theorem, that the spacetime is Minkowski.

This argument generalises the one presented in last section in the sense it does not require staticity. Moreover, with respect to the argument in Section 2.1, it assumes an everywhere timelike Killing vector field (and hence an absence of ergo-regions) which is not a requirement in Section 2.1; in the latter, on the other hand, axial symmetry is assumed, unlike the argument here which has no spatial symmetry requirements.

2.3.1 Lichnerowicz argument for strictly stationary spacetimes

We consider an asymptotically flat, everywhere regular and strictly stationary spacetime with Killing field k. We can write the Einstein equations for our full model (1.1.2) as

$$\frac{R_{\mu\nu}}{2} = f(\phi)\left(F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F^2\right) + \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}U(\phi) \ . \tag{2.3.49}$$

We define the *twist vector* ω^μ using the timelike Killing vector

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}k_\nu\nabla_\alpha k_\beta \ , \tag{2.3.50}$$

which respects the identity

$$\nabla_\mu\left(\frac{\omega^\mu}{V^2}\right) = 0 \ , \tag{2.3.51}$$

where, as in Section 2.1, $V \equiv -k^\mu k_\mu$, and here it is assumed to be always positive, corresponding to a strictly stationary spacetime.

The electric and magnetic fields are defined as in (2.2.33) and (2.2.34). The Maxwell equations take the following form in a strictly stationary spacetime:

$$\nabla_{[\mu}E_{\nu]} = 0 \ , \tag{2.3.52}$$

$$\nabla_{[\mu}\big(fB_{\nu]}\big) = 0 \ , \tag{2.3.53}$$

$$\nabla_\mu\left(f\frac{E^\mu}{V}\right) = \frac{2}{V^2}f\omega_\mu B^\mu \ , \tag{2.3.54}$$

$$\nabla_\mu\left(\frac{B^\mu}{V}\right) = -\frac{2}{V^2}\omega_\mu E^\mu \ . \tag{2.3.55}$$

Observe how dropping the staticity assumption generalises the last two equations, as compared to their counterparts in the static case (2.2.37)-(2.2.38). On the other hand, since the two first equations remain the same, we can, as before, write the fields in terms of two potentials $E_\mu = \partial_\mu\varphi$ and $fB_\mu = \partial_\mu\psi$.

Using the relation

$$\nabla_{[\mu}\omega_{\nu]} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}k_{[\alpha}R_{\beta]\gamma}k^\gamma \ , \tag{2.3.56}$$

we can obtain through the Einstein equations (2.3.49) the following expression

$$\nabla_{[\mu}\omega_{\nu]} = 2fB_{[\mu}E_{\nu]} \ . \tag{2.3.57}$$

Roughly, the curl of ω is the Poynting vector. Then, from the existence of the potentials φ and ψ, we can obtain two more equations

$$\nabla_{[\mu}\left(\omega_{\nu]} - 2\psi E_{\nu]}\right) = 0 \ , \tag{2.3.58}$$

$$\nabla_{[\mu}\left(\omega_{\nu]} + 2f\varphi B_{\nu]}\right) = 0 \ , \tag{2.3.59}$$

which, in turn, imply the existence of two further scalar potentials

$$\nabla_\mu U_E = \omega_\mu - 2\psi E_\mu \ , \tag{2.3.60}$$

$$\nabla_\mu U_B = \omega_\mu + 2f\varphi B_\mu \ . \tag{2.3.61}$$

Using these equations, the Maxwell equations (2.3.52)-(2.3.55) and equation (2.3.51), we can obtain the following two identities

$$\nabla_\mu\left(U_E\frac{\omega^\mu}{V^2} - \frac{\psi}{V}B^\mu\right) = \frac{\omega_\mu\omega^\mu}{V^2} - f\frac{B_\mu B^\mu}{V} \ , \tag{2.3.62}$$

$$\nabla_\mu\left(U_B\frac{\omega^\mu}{V^2} - \frac{\varphi}{V}fE^\mu\right) = \frac{\omega_\mu\omega^\mu}{V^2} - f\frac{E_\mu E^\mu}{V} \ . \tag{2.3.63}$$

Another useful identity is the contraction of the Ricci tensor (2.3.49) with the stationary Killing field twice

$$R_{\mu\nu}k^\mu k^\nu = f\left(E_\mu E^\mu + B_\mu B^\mu\right) - 2VU(\phi) \ . \tag{2.3.64}$$

We now consider two different approaches to further the argument.

2.3.1.1 First approach

This approach follows the one in [40] but including now the non-minimal coupling function $f(\phi)$.

Using the following relation for the twist vector (see for example [13])

$$\frac{2}{V}R_{\mu\nu}k^\mu k^\nu = \nabla_\mu\left(\frac{\nabla^\mu V}{V}\right) + 4\frac{\omega_\mu\omega^\mu}{V^2} \ , \tag{2.3.65}$$

together with equation (2.3.64), we get

$$\nabla_\mu\left(\frac{\nabla^\mu V}{V}\right) + 4\frac{\omega_\mu\omega^\mu}{V^2} = 2\frac{f}{V}\left(E_\mu E^\mu + B_\mu B^\mu\right) - 4U(\phi) \ . \tag{2.3.66}$$

This relation together with (2.3.62) and (2.3.63) finally gives the divergence identity:

$$\nabla_\mu\left[\frac{\nabla^\mu V}{V} + 2(U_E + U_B)\frac{\omega^\mu}{V^2} - 2\frac{\psi B^\mu + f\varphi E^\mu}{V}\right] = -4U(\phi) \ . \tag{2.3.67}$$

Let us now analyse the consequences of this divergence identity, starting with the particular case of a free, massless, scalar field, so that $U(\phi) = 0$. The left hand side of (2.3.67) is the divergence of a vector v^μ which respects

$$k^\mu v_\mu = 0 \ , \tag{2.3.68}$$

which follows from the definitions (2.2.33), (2.2.34), (2.3.50) and $V \equiv -k^\mu k_\mu$. Now we integrate $\nabla_\mu v^\mu$ on a spacetime volume bounded by two (neighbouring) Cauchy hypersurfaces, Σ_1 and Σ_2, with exterior normals k^μ and $-k^\mu$, respectively, and a timelike hypersurface at spatial infinity, $\mathcal{T}$, whose spatial sections are round 2-spheres, and hence the normal is a unit radial vector n^μ. Then, applying the covariant divergence theorem:

$$0 = \int_{V^4} d^4x\sqrt{-g}\nabla_\mu v^\mu = \int_{\Sigma_1} d^3x\sqrt{g_\Sigma}k_\mu v^\mu - \int_{\Sigma_2} d^3x\sqrt{g_\Sigma}k_\mu v^\mu + \int_{\mathcal{T}} d^3x\sqrt{-g_{\mathcal{T}}}n_\mu v^\mu \ . \tag{2.3.69}$$

The first two integrals in the right hand side vanish by (2.3.68). To simplify the remaining term we note that, asymptotically, the leading behavior of the asymptotically flat metric is

$$ds^2 = -Vdt^2 + \frac{dr^2}{V} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + ... \tag{2.3.70}$$

where

$$V = 1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{2.3.71}$$

Then[8],

$$0 = \int_{\mathcal{T}} d^3x\sqrt{-g_{\mathcal{T}}}n_\mu v^\mu = \lim_{r\to\infty}\int_{t_2}^{t_1} dt\int_0^{2\pi} d\varphi\int_0^{\pi} d\theta r^2\sin\theta\sqrt{V}n_r\frac{\partial_r V}{V} = 8\pi M\Delta t \ . \tag{2.3.72}$$

where $\Delta t = t_1 - t_2$ and t_1, t_2 are the (arbitrary) time coordinates associated to the two Cauchy surfaces. This informs that the ADM mass M must vanish. Then, by the positive mass theorem, assuming the dominant energy condition holds for this model, the spacetime is Minkowski.

Concerning the scalar potential, unless $U(\phi)$ is written as a divergence, as to be included in the left side of the equation (2.3.67), the reasoning does not apply. This can be done for a constant negative potential.[9] But for the general case we shall follow a different approach.

2.3.1.2 Second approach

In order to accommodate a non-trivial potential in the no-go theorem, we now take advantage of an argument in [50], already used in Section 2.2, stating that a strictly stationary,

[8]The electromagnetic terms in the divergence (2.3.67) disappear at infinity because they all decay asymptotically faster than r^{-2}. Consequently, only the first term inside the divergence contributes to this integral.

[9]The process of integration in this case is exactly the same as in [40].

asymptotically flat spacetime coupled to a matter model satisfying the dominant energy condition will always be flat spacetime as long as it violates the strong energy condition for the Killing field k at every point, $R_{\mu\nu}k^\mu k^\nu \leqslant 0$.

We start from the following result in [50] for the Komar mass:

$$\begin{aligned} M &= -\frac{1}{4\pi}\int_\Sigma \left(\frac{R_{\mu\nu}k^\mu k^\nu}{V} - \frac{2\omega^\mu\omega_\mu}{V^2}\right) k^\alpha d\Sigma_\alpha \\ &= \frac{1}{4\pi}\int_\Sigma \left(R_{\mu\nu}k^\mu k^\nu - \frac{2\omega^\mu\omega_\mu}{V}\right) d\Sigma \ , \end{aligned} \tag{2.3.73}$$

where Σ is a spacelike Cauchy surface.[10] Since ω^μ is nowhere timelike, $\omega^\mu\omega_\mu > 0$ and, if $R_{\mu\nu}k^\mu k^\nu \leqslant 0$, both contributions to the integral will be negative and

$$M \leqslant 0 \ . \tag{2.3.74}$$

Assuming the dominant energy condition, on the other hand, the positive mass theorem $M \geqslant 0$ holds. Thus, $M = 0$ and the spacetime is flat.

For our full model, we cannot simply state that the strong energy condition is violated, due to the presence of the electromagnetic field terms in equation (2.3.64). Nonetheless, we can still show the mass is non-positive, for a non-negative potential. We then proceed as follows. Using equations (2.3.62), (2.3.63) and (2.3.64) we obtain

$$\int_\Sigma R_{\mu\nu}k^\mu k^\nu d\Sigma = \int_\Sigma \left(\frac{2\omega^\mu\omega_\mu}{V} - 2V\,U(\phi) - \frac{V}{2}\nabla_\mu W^\mu\right) d\Sigma \ , \tag{2.3.75}$$

where we defined the vector W^μ as

$$W^\mu \equiv 2(U_E + U_B)\frac{\omega^\mu}{V^2} - 2\frac{\psi B^\mu + f\varphi E^\mu}{V} \ . \tag{2.3.76}$$

Using the full Komar expression (2.3.73), equation (2.3.75) becomes

$$M = -\frac{1}{2\pi}\int_\Sigma V\,U(\phi) d\Sigma - \frac{1}{8\pi}\int_\Sigma V\,\nabla_\mu W^\mu d\Sigma \ . \tag{2.3.77}$$

For a positive potential, the first term will be clearly negative so we only have to deal with the second term which corresponds to the electromagnetic field contribution to the Komar mass. First, note that $\pounds_k W = 0$ so we can use the Stokes theorem identity (2.2.39) for this vector

$$\int_\Sigma V\,\nabla_\mu W^\mu d\Sigma = -\int_\Sigma \nabla_\mu W^\mu k^\nu d\Sigma_\nu = -2\int_{\partial\Sigma} W^\mu k^\nu dS_{\mu\nu} \ . \tag{2.3.78}$$

The surface $\partial\Sigma$ is the 2-surface at infinity and all the terms in W^μ decay, asymptotically, faster than r^{-2}, so the integral vanishes (see Appendix A). This means that the electromagnetic contribution to the Komar mass (2.3.77) given by the vector W^μ is zero and only the negative potential term is left, giving, again, $M \leqslant 0$. Thus, again, the positive mass theorem establishes $M = 0$.

[10] This means that $k^\alpha d\Sigma_\alpha = k^\alpha k_\alpha d\Sigma = -V d\Sigma$.

To conclude, in the full model (1.1.2), in an asymptotically flat and strictly stationary spacetime with positive scalar potential $U(\phi)$, there are no non-trivial solitonic solutions, even in the presence of any positive potential $U(\phi)$.[11] Observe that this generalises the argument of [40] even for the case of minimal coupling $f(\phi) = 1$, due to the inclusion of the potential.

2.4 Remarks

In flat spacetime, solitonic solutions occur in classical field theories that are, just like general relativity, nonlinear. As mentioned, examples date back as far as the Korteweg-de-Vries equation [9]. Coupling field theories to gravity opens up new possibilities: (i) solitonic solutions could arise even for linear field theories, with the required non-linearities being generated by the gravitational interaction; (ii) there are now also black hole solutions, which in vacuum are “bald". When both black hole solutions and solitonic solutions are possible, often so are “hairy" black hole solutions, which can be interpreted as a sort of non-linear bound state between both of these building blocks.

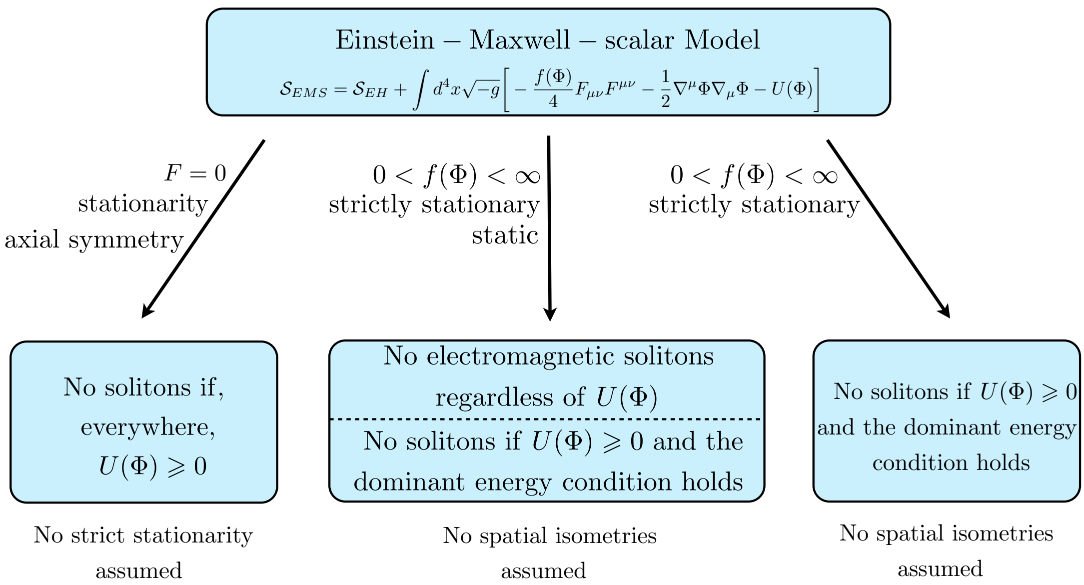


Figure 2.1: Schematic representation summarising the no-go theorems presented herein. In all cases asymptotically flat spacetimes are assumed. The units used in this image use $8\pi G = 1$.

In this chapter, motivated by the recently found "hairy" black hole solutions in Einstein-Maxwell-scalar models [8], which can arise dynamically from an instability of the RN black hole, we have addressed the existence of self-gravitating solitons in this family of models.

[11] We emphasise that throughout Sections 2.2 and 2.3 we have assumed, in the applications of Stokes' theorem, the absence of a boundary term at the origin, which, again, is only justified if the coupling is required to be finite therein. Singular solutions with divergent coupling have been reported, for instance, in [117, 118].

We obtained three results for asymptotically flat spacetimes. The first is for the absence of axisymmetric and stationary solitons with a vanishing Maxwell field, as long as the scalar potential is everywhere non-negative and another for the absence of static (without spatial symmetries assumed) scalar-electromagnetic solitons in the full model. In this case, no strict stationarity is assumed. The second result applies to strictly stationary and static spacetimes but without any assumptions on the spatial symmetries. If the coupling function $f(\phi)$ does not change sign, then no electromagnetic solitons exist, regardless of the scalar potential. Moreover, if one assumes the scalar potential to be non-negative and the dominant energy condition to hold, no scalar solitons exist either. Finally, the third result generalises the second by dropping the staticity assumption. A summary of these results is presented in a schematic way in Figure 2.1.

As a corollary of the results herein, the model studied in [8] illustrates that both bald and hairy black holes can exist without the existence of solitons. Therefore, even in models allowing both this sort of black holes, not all hairy black holes can be interpreted as a superposition of a soliton and a bald black hole.

It is now a natural question to ask if there might be soliton solutions in this model if either $f(\phi)$ changes sign (for a purely electric solution case) or if $f(\phi)$ either diverges or vanishes. While there are still no known solutions for the case where f changes sign (and it could present an interesting avenue for future work), in chapter 4 we discuss a solution which is found by circumventing the assumption that $f(\phi)$ must not diverge. Soliton solutions where $f(\phi)$ vanishes are also presented in chapter 5.

Regarding generalisations of this model, we discuss the effect of the addition of an axionic term to the model in the next chapter.

Chapter 3

On the inexistence of self-gravitating solitons in generalised axion electrodynamics

After establishing the no go theorems of chapter 2, two possible ways of finding soliton solutions are to either consider a more general model than the standard EMS model or to circumvent the conditions imposed by the no go theorems. The generalisation of the EMS model is what is considered in this chapter, where we discuss the consequences of adding an axionic term to the EMS model. An attempt to circumvent the conditions of the no go theorems is made in chapter 4. We recall that to distinguish the axion field from a more basic scalar field we use a to represent the axion field.

The model studied in this chapter is the basic axion electrodynamics model (1.3.7), where the only non-minimal coupling is the typical axion term $aF_{\mu\nu}\tilde{F}^{\mu\nu}$ ($f(a) = 1$). In section 3.1, we start by considering the possibility of static axionic solitons in this model. The same argument used in the last chapter (section 2.2), inspired by the one in [24], is adapted to this model. The process involves finding the new canonical form of the covariant Maxwell equations obtained from this model. It is shown that no solitons can exist under these conditions.

In section 3.2 we once again consider the possibility of static solitons but in the model (1.3.9), which is a generalisation of the typical EMS model. In this case, the non-minimal coupling $f(a)$ is once again considered and we promote the coupling κa to a general function $g(a)$. The result of section 3.1 persists even for this generalisation.

Finally, in section 3.3, the argument for strictly stationary spacetimes presented in the last chapter (section 2.3) is adapted to this generalised model, showing that there can be no soliton solutions with those conditions even in such a general model. A few remarks are discussed in section 3.4.

3.1 Absence of static axionic solitons

In this section we consider an asymptotically flat, static spacetime with no restrictions on the spatial symmetries. The gravitational part will play no role in the subsequent argument.

The equations of motion for the model (1.3.7) are

$$\nabla_\mu(F^{\mu\nu} - \kappa a \tilde{F}^{\mu\nu}) = 0 \ , \tag{3.1.1}$$

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \ , \tag{3.1.2}$$

$$\Box a = -\frac{\kappa}{4} F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{dU(a)}{da} \ , \tag{3.1.3}$$

where $\Box = \nabla_\mu \nabla^\mu$ is the covariant d'Alembertian. Since the spacetime is static and without horizons, we know that it admits an everywhere timelike Killing vector field k which can be used to define the electric and magnetic fields as in equations (2.2.33) and (2.2.34).

In Maxwell's theory, one can rewrite the covariant Maxwell equations in terms of E, B in a certain canonical form - see *e.g.* eqs.(2.2.35)-(2.2.37) from last chapter. In axion electrodynamics, a similar canonical form is obtained if we define two new fields E' and B' which are related to the original fields as

$$E'_\mu \equiv E_\mu - \kappa a B_\mu \ , \tag{3.1.4}$$

$$B'_\mu \equiv B_\mu + \kappa a E_\mu \ ; \tag{3.1.5}$$

now, the axion Maxwell equations (3.1.1)-(3.1.3) are written as

$$\nabla_{[\mu} E_{\nu]} = 0 \ , \tag{3.1.6}$$

$$\nabla_{[\mu} B'_{\nu]} = 0 \ , \tag{3.1.7}$$

$$\nabla_\mu \left(\frac{E'^\mu}{V} \right) = 0 \ , \tag{3.1.8}$$

$$\nabla_\mu \left(\frac{B^\mu}{V} \right) = 0 \ , \tag{3.1.9}$$

where $V \equiv -k^\mu k_\mu > 0$. Due to the absence of currents, the first two equations imply that we can once again introduce an electric φ and a magnetic ψ *scalar* potentials, as

$$E_\mu = \partial_\mu \varphi \ , \qquad B'_\mu = \partial_\mu \psi \ . \tag{3.1.10}$$

The remainder of the argument uses the method used in section 2.2 which was inspired by Heusler's argument described in [24]. We again make use of the identity (2.2.39) for an arbitrary vector α obeying $\pounds_k \alpha = [k, \alpha] = 0$. Specifying this identity for $\alpha^\mu = E'^\mu/V$ and using the axionic Maxwell equations yields

$$\int_{\partial\Sigma} \frac{E'^\mu k^\nu}{V} dS_{\mu\nu} = 0 \ , \tag{3.1.11}$$

where we took $\partial\Sigma$ to be the surface at spatial infinity (an $r = \infty$ 2-surface, which can be used near infinity due to asymptotic flatness).

Making a second use of the identity (2.2.39) but now with $\alpha^\mu = \varphi E'^\mu / V$ and once again using the axionic equations, we obtain

$$\frac{1}{2}\int_\Sigma \frac{E^\mu E'_\mu}{V} k^\nu d\Sigma_\nu = \int_{\partial\Sigma} \varphi \frac{E'^\mu k^\nu}{V} dS_{\mu\nu} = \varphi_\infty \int_{\partial\Sigma} \frac{E'^\mu k^\nu}{V} dS_{\mu\nu} = 0 \ , \tag{3.1.12}$$

where φ_∞ is the value of the electric potential at $r = \infty$ which is constant, and the last equality used (3.1.11).

The same argument can be used for B and B' by replacing φ by ψ, obtaining

$$\int_\Sigma \frac{B^\mu B'_\mu}{V} k^\nu d\Sigma_\nu = 0 \ . \tag{3.1.13}$$

We can now expand (E', B') in terms of (E, B), via (3.1.4)-(3.1.5) to obtain the identities:

$$\int_\Sigma \frac{E^\mu E_\mu}{V} k^\nu d\Sigma_\nu - \int_\Sigma \kappa a \frac{E^\mu B_\mu}{V} k^\nu d\Sigma_\nu = 0 \ , \tag{3.1.14}$$

$$\int_\Sigma \frac{B^\mu B_\mu}{V} k^\nu d\Sigma_\nu + \int_\Sigma \kappa a \frac{E^\mu B_\mu}{V} k^\nu d\Sigma_\nu = 0 \ . \tag{3.1.15}$$

Adding up the last two equations yields

$$\int_\Sigma \frac{E^\mu E_\mu + B^\mu B_\mu}{V} k^\nu d\Sigma_\nu = 0 \ . \tag{3.1.16}$$

From their definitions (2.2.33)-(2.2.34), $k^\mu E_\mu = 0 = k^\mu B_\mu$. Thus, these fields are never timelike. It follows that both $E^\mu E_\mu$ and $B^\mu B_\mu$ are always non-negative. Consequently, the only way for eq. (3.1.16) to be verified is if both fields vanish for every Cauchy surface Σ and, consequently, for the whole spacetime. This result is independent of the potential $U(a)$. With vanishing electromagnetic fields, all we have left is the possibility of self-gravitating axion (scalar) solitons. However, as was discussed in the last chapter and in [13, 50], there are no scalar field solitons as long as the dominant energy condition is obeyed and the strong energy condition is violated, which is the case for scalar fields with a positive potential. Therefore, the only possible solution for such potentials is Minkowski spacetime.

As a final remark in this section, the main difference between the result herein and the one for Einstein-Maxwell theory is that instead of establishing that the norms of both E and B vanish, we can only establish that the sum of these norms must vanish. Since both these norms are positive definite, however, the final conclusion is that each must vanish, recovering the result of Einstein-Maxwell theory.

3.2 Generalised axion electrodynamics

The result of section 3.1 can be straightforwardly extended to a model of generalised axion electrodynamics minimally coupled to Einstein's gravity (1.3.9) which introduces the arbitrary non-minimal coupling functions $f(a)$ and $g(a)$ of the axion field.

It is assumed, just like in last chapter, that both coupling functions *do not diverge* at any point in the spacetime and that $f(\phi)$ does not vanish for a non-zero magnetic field. The equations of motion are a simple generalisation of the previous ones (3.1.1)-(3.1.3) and read

$$\nabla_\mu (f F^{\mu\nu} - g\tilde{F}^{\mu\nu}) = 0 \ , \tag{3.2.17}$$

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \ , \tag{3.2.18}$$

$$\Box^2 a = -\frac{1}{4}\frac{dg}{da}F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{4}\frac{df}{da}F_{\mu\nu}F^{\mu\nu} - \frac{dU(a)}{da} \ . \tag{3.2.19}$$

Although the a equation can be considerably more difficult due to the arbitrary couplings, defining now the fields E' and B' as

$$E' = fE - gB \ , \tag{3.2.20}$$

$$B' = fB + gE \ , \tag{3.2.21}$$

it follows that these new fields respect the exact same equations as (3.1.6)-(3.1.9). Consequently, we follow the exact same procedure as in the last section to obtain the corresponding relations to (3.1.14)-(3.1.15), which now read

$$\int_\Sigma f\frac{E^\mu E_\mu}{V}k^\nu d\Sigma_\nu - \int_\Sigma g\frac{E^\mu B_\mu}{V}k^\nu d\Sigma_\nu = 0 \ , \tag{3.2.22}$$

$$\int_\Sigma f\frac{B^\mu B_\mu}{V}k^\nu d\Sigma_\nu + \int_\Sigma g\frac{E^\mu B_\mu}{V}k^\nu d\Sigma_\nu = 0 \ . \tag{3.2.23}$$

Adding these equations now yields

$$\int_\Sigma f\frac{E^\mu E_\mu + B^\mu B_\mu}{V}k^\nu d\Sigma_\nu = 0 \ . \tag{3.2.24}$$

As both $E^\mu E_\mu$ and $B^\mu B_\mu$ are non-negative, this identity implies a similar result to the one obtained in the previous chapter for the theory with no axions ($g = 0$): the fields must vanish and there are no solitonic solutions as long as the coupling $f(a)$ does not change sign. We can see that the main reason for this result to be similar to the one with $g = 0$ is because g, as complicated a function as it might be, does not contribute to the argument due to its contribution disappearing when we add equations (3.2.22) and (3.2.23).

3.3 Absence of strictly stationary axionic solitons

The method used above allowed us to rule out static solitons without requiring any spatial isometry. Now, after having considered static spacetimes, it is only natural to extend this treatment to strictly stationary spacetimes just like in the previous chapter. This accounts now for possibly rotating axionic solitons, as long as rotation does not create ergo-regions, since strict stationarity means that there exists an *everywhere* timelike Killing vector field.

Following a procedure similar to the one in section 2.3 where we use a Lichnerowicz type argument inspired by the one in [40], we shall also establish a no-go theorem for axionic solitons. In this case the Einstein equations play an important role in the argument.

The Einstein equations for this model are, just like in section 2.3

$$\frac{R_{\mu\nu}}{2} = f(a)\left(F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F^2\right) + \partial_\mu a\partial_\nu a + g_{\mu\nu}U(a) \ . \tag{3.3.25}$$

The axionic term is purely topological so it does not contribute to the Einstein equations. Using the timelike Killing vector field, we again define the twist vector ω^μ as (2.3.50) and obeying the identity (2.3.51). The Maxwell equations (3.1.6)-(3.1.9), with the primed fields defined by (3.2.20)-(3.2.21) are generalised for a strictly stationary spacetime as:

$$\nabla_{[\mu}E_{\nu]} = 0 \ , \tag{3.3.26}$$

$$\nabla_{[\mu}B'_{\nu]} = 0 \ , \tag{3.3.27}$$

$$\nabla_\mu\left(\frac{E'^\mu}{V}\right) = \frac{2}{V^2}\omega_\mu B'^\mu \ , \tag{3.3.28}$$

$$\nabla_\mu\left(\frac{B^\mu}{V}\right) = -\frac{2}{V^2}\omega_\mu E^\mu \ . \tag{3.3.29}$$

We can again consider the identity (2.3.56) so that using the Einstein equations (3.3.25) relates the curl of ω with the Poynting vector:

$$\nabla_{[\mu}\omega_{\nu]} = 2fB_{[\mu}E_{\nu]} \ . \tag{3.3.30}$$

One can freely add vanishing terms such as $-2gB_{[\mu}B_{\nu]}$ and $2gE_{[\mu}E_{\nu]}$ to rewrite the right hand side in two different ways

$$fB_{[\mu}E_{\nu]} = B'_{[\mu}E_{\nu]} = B_{[\mu}E'_{\nu]} \ . \tag{3.3.31}$$

We choose the expression with B' and E as these two fields are the ones which we can rewrite as potentials ψ and φ respectively, *cf.* (3.1.10). This means that equation (3.3.30) implies the following two identities

$$\nabla_{[\mu}\left(\omega_{\nu]} - 2\psi E_{\nu]}\right) = 0 \ , \tag{3.3.32}$$

$$\nabla_{[\mu}\left(\omega_{\nu]} + 2\varphi B'_{\nu]}\right) = 0 \ , \tag{3.3.33}$$

which in turn imply the existence of two new potentials $U_{B'}$ and U_E

$$\nabla_\mu U_E = \omega_\mu - 2\psi E_\mu \ , \tag{3.3.34}$$

$$\nabla_\mu U_{B'} = \omega_\mu + 2\varphi B'_\mu \ . \tag{3.3.35}$$

Using these potentials and the identity (2.3.51), the following divergence identity is obtained

$$\nabla_\mu W^\mu = \frac{4\omega^\mu\omega_\mu}{V^2} - 2\frac{E'_\mu E^\mu + B'_\mu B^\mu}{V} \ , \tag{3.3.36}$$

where

$$W^{\mu} = 2(U_E + U_{B'})\frac{\omega^{\mu}}{V^2} - 2\frac{\psi B^{\mu} + \varphi E'^{\mu}}{V} \ . \tag{3.3.37}$$

On the other hand, the contraction of the Einstein equations (3.3.25) with the Killing field yields

$$\frac{2}{V}R_{\mu\nu}k^{\mu}k^{\nu} = 2f\frac{E_{\mu}E^{\mu} + B_{\mu}B^{\mu}}{V} - 4U(a) \ . \tag{3.3.38}$$

The first term on the right hand side can be slightly reshaped by noting that $f(E_{\mu}E^{\mu} + B_{\mu}B^{\mu})$ may be written as

$$f(E_{\mu}E^{\mu} + B_{\mu}B^{\mu}) = (fE_{\mu} - gB_{\mu})E^{\mu} + (fB_{\mu} + gE_{\mu})B^{\mu} = E'_{\mu}E^{\mu} + B'_{\mu}B^{\mu} \ . \tag{3.3.39}$$

Then adding equations (3.3.36) and (3.3.38) yields

$$\frac{2}{V}R_{\mu\nu}k^{\mu}k^{\nu} - \frac{4\omega^{\mu}\omega_{\mu}}{V^2} = -\nabla_{\mu}W^{\mu} - 4U(a) \ . \tag{3.3.40}$$

The final step of the argument consists on taking the Komar mass integral on a Cauchy surface Σ [50]:

$$M = -\frac{1}{4\pi}\int_{\Sigma}\left(\frac{R_{\mu\nu}k^{\mu}k^{\nu}}{V} - \frac{2\omega^{\mu}\omega_{\mu}}{V^2}\right)k^{\alpha}d\Sigma_{\alpha} \ , \tag{3.3.41}$$

which, via (3.3.40), reads

$$M = \frac{1}{4\pi}\int_{\Sigma}\left(\frac{1}{2}\nabla_{\mu}W^{\mu} + 2U\right)k^{\alpha}d\Sigma_{\alpha} \ . \tag{3.3.42}$$

As $\pounds_k W = 0$, the identity (2.2.39) can be used to write the first term in the integral as

$$\int_{\Sigma}\nabla_{\mu}W^{\mu}k^{\alpha}d\Sigma_{\alpha} = 2\int_{\partial\Sigma}W^{\mu}k^{\alpha}dS_{\mu\alpha} \ . \tag{3.3.43}$$

The surface $\partial\Sigma$ is the 2-surface at infinity and all the terms in W^{μ} decay, asymptotically, faster than r^{-2}, so that (3.3.43) vanishes. Thus (3.3.42) becomes

$$M = \frac{1}{2\pi}\int_{\Sigma}Uk^{\alpha}d\Sigma_{\alpha} = -\frac{1}{2\pi}\int_{\Sigma}UVd\Sigma \ , \tag{3.3.44}$$

as $d\Sigma_{\alpha} = k_{\alpha}d\Sigma$. Consequently, as long as the potential $U(a)$ is positive, the only contribution to the Komar mass M will be negative. Then, by the the positive mass theorem[1], $M = 0$ and the only solution is flat spacetime. Therefore, no axionic solitons are possible in strictly stationary spacetimes, again regardless of the spatial symmetries.

[1]The energy conditions are unchanged from the Einstein-Maxwell-scalar theory by the axionic term, so we can take the same conclusions as in section 2.3. The dominant energy condition stays valid and, as consequence, the positive energy theorem is also valid.

3.4 Remarks

In this chapter, we have assessed the possible existence of self-gravitating solitons in axion electrodynamics and generalisations thereof. We established that the presence of axions and their coupling to the electromagnetic field does not change the results of (in)existence of Einstein-Maxwell solitons in static or strictly stationary spacetime established in chapter 2. This holds even when considering a model with rather generic couplings between the axion field and the electromagnetic invariants and, in particular, allowing an arbitrary coefficient function $g(a)$ in the axion term $F \cdot \tilde{F}$.

A possible generalisation would be to consider a coupling between the electromagnetic field and a different scalar field (rather than the axion). However, without any kind of coupling between these two scalar fields, the result will likely remain unchanged. One interesting future work route would be then to generalise this model to allow for two different scalar fields, coupled to each other, and to the electromagnetic field through the couplings f and g. An example of a model that corresponds to this type of framework is the Einstein-Maxwell-Dilaton-Axion model[2] [119], where the coupling $f(\phi) = e^{-\alpha\phi}$ depends on the dilaton field ϕ (α is a constant) and $g(a) = \kappa a$ has the usual dependence on the axion field a. These two fields also include a coupling between them, possibly allowing for the existence of scalar solitons in the model.

[2]The EMDA model will be considered in the context of duality in chapter 5.

Chapter 4

A class of solitons in Maxwell-scalar and Einstein-Maxwell-scalar models

After exploring the various ways in which we can not have soliton solutions in the Einstein-Maxwell-scalar model, we are tempted to ask: then when can we have solutions? One possible answer is to find a configuration of the fields in the discussed model which represents a soliton solution that does not respect every condition stated by the no go theorems. Another would be to generalise the model further, adding terms that would make the proof of the no go theorems impossible, like we tried to do in the previous chapter.

In this chapter we discuss the first possibility. An important condition specified by the no go theorems presented in the last chapters is that the non-minimal coupling $f(\phi)$, which directly couples the electromagnetic and scalar fields, does not diverge at any point in the spacetime. So in this chapter we present a solution with a non-minimal coupling that diverges at a single point, the origin.

In section 4.1 we discuss the flat spacetime Maxwell-scalar models. We start by introducing some physical motivation to contextualise a diverging coupling $f(\phi)$. The idea is that this diverging coupling can serve as a way to de-singularise the electric field of a point electric charge. We then present an integrable model that admits everywhere regular solutions with finite energy which represent flat spacetime solitons. The rest of the section is dedicated to presenting a class of examples and the peculiar case of the dilatonic coupling.

Section 4.2 is where the self-gravitating soliton solutions are constructed. Two procedures for constructing solutions are shown, perturbatively and numerically. A discussion of these solutions is made in section 4.3.

4.1 Flat spacetime Maxwell-scalar models

4.1.1 A physical motivation

An awkward feature of classical electromagnetism is that the energy E of the Coulomb field of a point charge Q is divergent:

$$E \sim \int_0^{+\infty} \frac{Q^2}{r^2} dr = \infty \ . \tag{4.1.1}$$

Quantum considerations naturally introduce an ultraviolet cut-off to the validity of the classical Coulomb solution, regularising this integral. Quantum Electrodynamics (QED), however, is itself incomplete as a quantum field theory, due to the Landau pole [120]. But it yields the important lesson that the coupling constant g, which determines the strength of the electromagnetic interaction in the Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} \ , \tag{4.1.2}$$

runs with the energy scale.

Whatever fundamental theory turns out to complete QED, it may admit a covariant effective field theory description that captures the dynamics of the coupling. Then, g would emerge as a spacetime function with some dynamics. In a simple model, g would be a real scalar field with a standard kinetic term. Allowing more general dynamics, one takes g as being an arbitrary function of the scalar field, keeping the latter with a standard kinetic term. This suggests considering the *naive* covariant effective field theory

$$\mathcal{S} = \frac{1}{4\pi} \int d^4x \sqrt{-g} \left(-\frac{f(\phi)}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \ , \tag{4.1.3}$$

which corresponds to the flat spacetime limit of the Einstein-Maxwell-scalar model. We see that the non-minimal coupling function $f(\phi)$ specifies the dynamics of the gauge coupling. This model ignores higher order corrections in F, so it is certainly incomplete. Nonetheless one may take the aforementioned reasoning as a motivation to consider this class of simple models. Can the Coulomb field of a point charge be de-singularised in this context?

4.1.2 An integrable model

The naive model (4.1.3), which is the decoupling limit of the EMS model (1.1.2) wherein back reaction is neglected, is integrable in the spherical sector. Taking the following ansatz for the fields in spherical coordinates in Minkowski spacetime (t, r, θ, φ):

$$\phi = \phi(r) \ , \qquad A = V(r)dt \ , \tag{4.1.4}$$

the Maxwell equations yield a first integral:

$$V(r) = \int \frac{Q}{r^2 f(\phi)} dr \ , \tag{4.1.5}$$

where Q is interpreted as the *electric charge*. Using this first integral, the Klein-Gordon equation reads

$$r^2\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)-\frac{Q^2}{2}\frac{d}{d\phi}\left(\frac{1}{f(\phi)}\right)=0\ , \tag{4.1.6}$$

which, introducing the coordinate $x\equiv 1/r$, yields another first integral

$$\left(\frac{d\phi}{dx}\right)^2-\frac{Q^2}{f(\phi)}=\mathcal{E}\ . \tag{4.1.7}$$

It is a simple application of the virial theorem, or a Derrick-type scaling theorem [49], to show that solutions must have $\mathcal{E}=0$. For instance, this can be seen from the condition [44]

$$\int d^3x T_{ij}=0\ , \tag{4.1.8}$$

that holds for time-independent, finite energy field configuration in Minkowski spacetime, where i,j are spatial indices in Cartesian coordinates. Relation (4.1.8) is a simple consequence of energy-momentum conservation and can be interpreted as the balancing of the total stresses in an extended object. There are regions where matter is in tension and regions where it is in compression, for any static balanced soliton. Thus, the problem of finding solutions is reduced to solving, from (4.1.7),

$$x(\phi)=\frac{1}{Q}\int\sqrt{f(\phi)}d\phi\ , \tag{4.1.9}$$

and then inverting $x(\phi)\to\phi(x)\to\phi(r)$. Fixing the coupling function $f(\phi)$ one can obtain $\phi(r)$ and, from (4.1.5), the electrostatic potential, both as line integrals. Due to the two first integrals the system is fully integrable.

4.1.3 Everywhere regular solutions

To assess if the solutions have finite energy one must consider the energy-momentum of the model,

$$4\pi T_{\mu\nu}=f(\phi)\left(F_{\mu\alpha}F_{\nu}{}^{\alpha}-\frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)+\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\ . \tag{4.1.10}$$

This yields the energy density ρ, after using (4.1.7):

$$\rho=T_{00}=\frac{Q^2}{4\pi r^4 f(\phi)}\quad . \tag{4.1.11}$$

and the total energy, E, obtained by integrating the energy density on a spacelike slice Σ

$$E=\int_\Sigma \rho\, d^3x=\int_0^{+\infty}\frac{Q^2}{r^2 f(\phi)}dr\ . \tag{4.1.12}$$

In order to obtain regular solutions at the origin we assume the scalar field admits a power series expansion near the origin:

$$\phi=\phi_0+\sum_{p=N}\phi_p r^p\ , \tag{4.1.13}$$

We do not constrain the constant coefficient ϕ_0, which may or may not vanish. Apart from ϕ_0, let ϕ_N, where $N \in \mathbb{N} \geqslant 1$ be the first non-vanishing coefficient in this expansion. Then, from (4.1.7),

$$(-Nr^{N+1}\phi_N + \dots)^2 = \frac{Q^2}{f(\phi)} \ . \tag{4.1.14}$$

Thus, as $r \to 0$,

$$f(\phi) \sim \frac{Q^2}{N^2\phi_N^2}\frac{1}{r^{2N+2}} \ . \tag{4.1.15}$$

Regularity of the scalar field at the origin then *requires* the coupling to diverge as $\sim 1/r^{2N+2}$. From (4.1.11) this implies that the energy density is finite therein and from (4.1.5),

$$V(r) = V(0) + \frac{N^2\phi_N^2}{(2N+1)Q}\, r^{2N+1} + \dots \ , \tag{4.1.16}$$

close to the origin. Thus, all physical quantities are finite close to the origin, for this class of behaviours of the coupling.

4.1.4 A class of examples

There is still, of course, some freedom in choosing the coupling function, within the class with the correct divergent behaviour at the origin. Let us consider examples.

4.1.4.1 A simple coupling yielding regular solutions

As an explicit example, consider

$$f(\phi) = \frac{1}{(1-\alpha\phi)^4} \ , \tag{4.1.17}$$

where α is a non-zero constant. Then (4.1.9) immediately yields, taking the integration constant such that $\phi \to 0$ as $r \to \infty$:

$$\phi(r) = \frac{Q}{Q\alpha + r} \ . \tag{4.1.18}$$

One observes that $\phi(r)$ is regular and smooth as $r \to 0$, $\phi(r) \simeq 1/\alpha - r/(Q\alpha^2)$; thus we expect, from (4.1.15), the coupling to diverge as $1/r^4$. Asymptotically , on the other hand, $\phi(r) \simeq Q/r$. Thus the scalar “charge" coincides with the electric charge. Plugging (4.1.18) into (4.1.17) yields:

$$f(r) = \left(1 + \frac{\alpha Q}{r}\right)^4 \ . \tag{4.1.19}$$

The coupling diverges as $1/r^4$ at the origin, as anticipated. This divergence precisely cancels the divergence of the Maxwell field at the origin, *cf.* (4.1.11), making it finite and non-zero. In fact, the energy density, from (4.1.11), is

$$\rho = \frac{Q^2}{4\pi(Q\alpha + r)^4} \ . \tag{4.1.20}$$

It follows that the total energy (4.1.12) is

$$E = \frac{Q}{3\alpha} \ . \tag{4.1.21}$$

Now, using (4.1.5) we obtain for the electrostatic potential:

$$V(r) = -\frac{rQ}{(Q\alpha + r)^2} - \frac{\alpha^2 Q^3}{3(Q\alpha + r)^3} \ . \tag{4.1.22}$$

All the quantities (4.1.20), (4.1.21), (4.1.22) manifestly reduce to the usual Coulombic ones upon taking $\alpha \to 0$. In such case (4.1.18) reduces to the profile of a scalar charge Q at the origin. The expressions make manifest how α regularises the solution.

4.1.4.2 A family of couplings yielding regular solutions

As further examples, with slightly different features, we generalise the coupling (4.1.17) as

$$f(\phi) = \frac{1}{(1 - \alpha\phi)^n} \ , \tag{4.1.23}$$

where n is an integer. Using this coupling, equation (4.1.9) gives

$$\frac{1}{r} = \frac{1}{Q} \int (1 - \alpha\phi)^{-n/2} d\phi \ , \tag{4.1.24}$$

which has a different indefinite integral for $n \neq 2$ and $n = 2$.

For $n \neq 2$, imposing $\phi(r \to \infty) = 0$ to fix the integration constant, one obtains

$$\phi(r) = \frac{1}{\alpha} - \frac{1}{\alpha} \left[1 + \frac{\alpha Q(n-2)}{2r}\right]^{\frac{2}{2-n}} \ , \tag{4.1.25}$$

which reduces to (4.1.17) for $n = 4$. For regular solutions at the origin we require $\lim_{r\to 0} \phi$ to be finite. This implies $n > 2$, in which case

$$\lim_{r\to 0} \phi(r) = \frac{1}{\alpha} - \frac{1}{\alpha} \left(\frac{2r}{\alpha Q(n-2)}\right)^{\frac{2}{n-2}} \ , \tag{4.1.26}$$

which is finite, as required. For $n = 3$ we see that the second term goes as r^2; but for $n > 4$, the second term has a non-integer power. In the former case we anticipate, from (4.1.15), that the coupling diverges as $1/r^6$. In the latter case, ϕ is not analytic at the origin. It will, nonetheless yield a regular solution, when analysing the usual physical quantities.

The coupling $f(\phi)$ as a function of r then reads:

$$f(r) = \left[1 + \frac{\alpha Q}{2r}(n-2)\right]^{\frac{2n}{n-2}} \ , \tag{4.1.27}$$

which diverges as $\sim 1/r^{\frac{2n}{n-2}}$ at the origin, for $n > 2$, but respects the condition $\lim_{r\to\infty} f(r) = 1$. We confirm, in particular, the $1/r^6$ divergence, for $n = 3$ and a divergence with (generically) a non-integer inverse power for $n \geqslant 5$. The electric field $E_\mu = -\partial_\mu V(r)$ has only one non-zero component which reads, from (4.1.5)

$$E_r(r) = -\frac{Q}{r^2 f} = -\frac{Q}{r^2} \left[1 + \frac{\alpha Q}{2r}(n-2)\right]^{-\frac{2n}{n-2}} \ , \tag{4.1.28}$$

which behaves as $r^{\frac{4}{n-2}}$ near the origin, and it is thus regular for $n > 2$.

The total energy now reads

$$E = \frac{2Q}{(n+2)\alpha} \ . \tag{4.1.29}$$

Thus, the family of cases with $n > 2$ illustrate how regular solutions can be obtained, with a different analytic behaviour of the scalar field near the origin (the cases $n = 3$ and $n = 4$) and non-analytic behaviour ($n > 4$).

With $n = 2$, following a similar reasoning one obtains

$$\phi(r) = \frac{1}{\alpha}\left(1 - e^{-\alpha Q/r}\right) \ , \tag{4.1.30}$$

which is a regular solution at $r = 0$ with $\lim_{r\to 0}\phi(r) = 1/\alpha$. The coupling function $f(\phi)$ becomes

$$f(r) = e^{2\alpha Q/r} \ , \tag{4.1.31}$$

which, as before, also diverges at $r = 0$ but respects $\lim_{r\to\infty} f(r) = 1$. Observe, however, it does not diverge as an inverse power of r, which was the conclusion in Section 4.1.3. This is because, again, ϕ in this case does not admit a power series expansion near the origin. This illustrates yet a different example of divergent coupling that yields regular solutions.

The electric field is now

$$E_r(r) = -\frac{Q}{r^2 f} = -\frac{Q}{r^2} e^{-2\alpha Q/r} \ , \tag{4.1.32}$$

and the total energy is

$$E = \frac{Q}{2\alpha} \ . \tag{4.1.33}$$

In these considerations αQ was assumed to be positive. Otherwise the total energy (4.1.29)-(4.1.33) would be negative, which would violate the weak energy condition. Interestingly enough, despite the seemingly different solution for $n = 2$, the total energy E is a smooth function of the power n, as (4.1.33) coincides with setting $n = 2$ in (4.1.29).

4.1.5 Dilatonic coupling: a spherically symmetric solution

As mentioned in the introduction, a dilatonic coupling

$$f(\phi) = e^{-\alpha\phi} \ , \tag{4.1.34}$$

where α is a constant, emerges in relevant scenarios. Let us thus briefly mention the existence of a spherically symmetric, exact solution for this coupling.

Considering (4.1.34) in (4.1.9), and taking the integration constant so that the scalar field vanishes asymptotically we immediately get

$$\phi = -\frac{2}{\alpha}\log\left[1 + \frac{\alpha Q}{2r}\right] \ . \tag{4.1.35}$$

Thus, the coupling, as a function of r is

$$f(\phi) = e^{-\alpha\phi} = \left[1 + \frac{\alpha Q}{2r}\right]^2 . \tag{4.1.36}$$

Thus, the coupling diverges at the origin and, if $\alpha Q > 0$ it is regular elsewhere. Moreover, using now (4.1.5) we get

$$V(r) = -\frac{2Q}{\alpha Q + 2r} . \tag{4.1.37}$$

One finds the following small-r expansion of the solution

$$\phi(r) = \frac{2}{\alpha}\left(\log r - \log\frac{\alpha Q}{2}\right) + \mathcal{O}(r) , \qquad V(r) = -\frac{2}{\alpha} + \frac{4r}{\alpha^2 Q} + \mathcal{O}(r^2) ; \tag{4.1.38}$$

thus, the scalar field diverges at the origin. Asymptotically, on the other hand, both fields are well behaved

$$\phi(r) = -\frac{Q}{r} + \frac{\alpha}{4}\frac{Q^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) , \qquad V(r) = -\frac{Q}{r} + \frac{\alpha}{2}\frac{Q^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) . \tag{4.1.39}$$

The energy density of this solution diverges at the origin:

$$\rho = -T_t^t = \frac{Q^2}{\pi r^2(\alpha Q + 2r)^2} ; \tag{4.1.40}$$

the total mass, however, is finite

$$M = 4\pi \int_0^\infty dr r^2 \rho = \frac{2Q}{\alpha} . \tag{4.1.41}$$

This solution is interesting in that it shows a divergent coupling can source a finite mass configuration which, nonetheless, is not fully regular, as the scalar field and the energy density diverge at the origin.

4.2 The gravitating solitons

The above flat spacetime solutions can be made to self-gravitate by coupling (4.1.3) to Einstein's general relativity. For the case of the regular solutions described in the previous section, this yields, perhaps, the simplest models of charged soliton.

One now considers the EMS model (1.1.2). In addition to the ansatz (4.1.4) we consider the metric form

$$ds^2 = -e^{-2\delta(r)}N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) , \qquad \text{where} \quad N(r) \equiv 1 - \frac{2m(r)}{r} , \tag{4.2.42}$$

and $m(r)$ is the Misner-Sharp mass; r is thus the areal radius, a geometrically meaningful coordinate.

The ansatz (4.1.4) and (4.2.42) yield the following effective Lagrangian:

$$\mathcal{L}_{\text{eff}} = e^{-\delta}m' - \frac{r^2}{2}e^{-\delta}N\phi'^2 + \frac{r^2}{2}f(\phi)e^{\delta}V'^2 . \tag{4.2.43}$$

As in the flat spacetime case, the equation of the electric potential possesses a first integral, which generalises (4.1.5), and reads

$$V' = e^{-\delta}\frac{Q}{r^2 f(\phi)} \ , \tag{4.2.44}$$

where again the integration constant Q is the electric charge, which we shall assume to be strictly positive, without any loss of generality. Using this integral, the remaining equations of motion become[1]

$$m' = \frac{r^2}{2}N\phi'^2 + \frac{Q^2}{2r^2}S(\phi) \ , \tag{4.2.45}$$

$$\delta' + r\phi'^2 = 0 \ , \tag{4.2.46}$$

$$(e^{-\delta}r^2 N\phi')' - \frac{e^{-\delta}}{2r^2}\frac{dS(\phi)}{d\phi}Q^2 = 0 \ , \tag{4.2.47}$$

where we have defined

$$S(\phi) \equiv \frac{1}{f(\phi)} \ . \tag{4.2.48}$$

The smooth of a spacetime configurations can be assessed by analysing the Ricci scalar

$$R = \frac{N}{r}(3r\delta' - 4) + \frac{2}{r^2}\left[1 + N(r^2\delta'' - (1 - r\delta')^2)\right] - N'' \ , \tag{4.2.49}$$

and the Kretschmann scalar

$$K = \frac{4}{r^4}(1-N)^2 + \frac{2}{r^2}\left[N'^2 + (N' - 2N\delta')^2\right] + \left[N'' - 3\delta' N' + 2N(\delta'^2 - \delta'')\right]^2 . \tag{4.2.50}$$

4.2.1 Asymptotic expansions

4.2.1.1 Near the origin

A small r analysis of the field equations confirms the conclusion observed in the flat space analysis: for a scalar field admitting a power series expansion near the origin $\phi = \phi_0 + \phi_1 r + \dots$ and $\phi_1 \neq 0$, if the coupling diverges as $1/r^4$, finite energy, everywhere regular solutions are possible. To see this, we again start by assuming the existence of a power series expansion of solutions, with the scalar field approaching a finite nonzero value

$$\phi(r) \to \phi_0 \quad \text{as} \quad r \to 0 \ , \tag{4.2.51}$$

where ϕ_0 is arbitrary. Then, the equations of motion, together with the assumption of regularity, impose, for the n^{th} derivative of $S(\phi)$ computed at the origin, denoted $S^{(n)}(\phi_0)$,

$$S(\phi_0) = S^{(1)}(\phi_0) = S^{(2)}(\phi_0) = S^{(3)}(\phi_0) = 0 \ , \qquad \text{whereas} \qquad S^{(4)}(\phi_0) > 0 \ . \tag{4.2.52}$$

This implies the advertised behaviour: the coupling function $f(\phi)$ diverges as $1/r^4$ as $r \to 0$. This behaviour cancels the divergence associated with the presence of an electric charge, providing a smooth configuration as $r \to 0$.

[1] There is an extra equation, which is a constraint and can be derived from (4.2.45)-(4.2.47).

The small r expansion of the matter functions reads

$$\phi(r) = \phi_0 - \frac{2\sqrt{6}r}{Q\sqrt{S^{(4)}(\phi_0)}} + \phi_2 r^2 + \dots , \qquad V(r) = -\frac{8e^{-\delta_0}}{Q^3 S^{(4)}(\phi_0)} r^3 + \dots , \tag{4.2.53}$$

while for the metric functions we find

$$m(r) = \frac{8}{Q^2}\frac{1}{S^{(4)}(\phi_0)} r^3 - \frac{2\sqrt{6}\phi_2}{Q\sqrt{S^{(4)}(\phi_0)}} r^4 + \dots , \qquad \delta(r) = \delta_0 - \frac{12r^2}{Q^2 S^{(4)}(\phi_0)} + \dots , \tag{4.2.54}$$

where ϕ_2 and δ_0 are constants that are fixed by the numerics when integrating the field equations from the origin to infinity and requiring the correct asymptotic behaviour. With this expansion, both the Kretschmann curvature scalar and Ricci scalar are finite as $r \to 0$, taking the form

$$K \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{3840}{Q^4[S^{(4)}(\phi_0)]^2} - \frac{2560\sqrt{6}\phi_2}{Q^3[S^{(4)}(\phi_0)]^{3/2}} r + \dots , \tag{4.2.55}$$

and

$$R = \frac{48}{Q^2 S^{(4)}(\phi_0)} - \frac{16\sqrt{6}\phi_2}{Q(S^{(4)}(\phi_0))^{1/2}} r + \dots . \tag{4.2.56}$$

The small r expansion of $S(\phi)$ reads

$$S(\phi) = \frac{24}{Q^4 S^{(4)}(\phi_0)} r^4 - \frac{8\sqrt{6}\phi_2}{Q^3\sqrt{S^{(4)}(\phi_0)}} r^5 + \dots , \tag{4.2.57}$$

which implies the following *generic* approximate form of the coupling function

$$S(\phi) = \frac{1}{f(\phi)} \sim (\phi - \phi_0)^4 \quad \text{as} \quad r \to 0 . \tag{4.2.58}$$

Of course, we could have assumed that in the scalar field expansion $\phi_1 = 0$ and the power series starts at a higher order term. This would impact in the way the coupling diverges at the origin, similarly to the flat spacetime analysis of section 4.1.3. For concreteness, here we focus on the case with $\phi_1 \neq 0$. This case corresponds, in the non-back-reacting case, to having $N = 1$ in equation (4.1.13). Choosing $\phi_1 = 0$ in the latter would imply a different behaviour for the divergence of $f(\phi)$, implied by the equation (4.1.15). In the back-reacting case this would correspond to having $S^{(4)}(\phi_0) = 0$. Non-trivial solutions with such behaviour should exist, as well.

4.2.1.2 Near infinity

A large r analysis of the field equations, on the other hand, imposing

$$f(\phi) \to 1 \quad \text{as} \quad r \to \infty, \tag{4.2.59}$$

yields the following approximate solutions:

$$m(r) = M - \frac{Q^2 + Q_s^2}{2r} + \dots , \qquad \phi(r) = \frac{Q_s}{r} + \dots , \tag{4.2.60}$$

$$V(r) = V_\infty - \frac{Q}{r} + \dots\ , \qquad \delta(r) = \frac{Q_s^2}{2r^2} + \dots\ . \tag{4.2.61}$$

Here M is the ADM mass and Q is the electric charge; V_∞ is the electrostatic potential at infinity and Q_s is the scalar 'charge' which in general needs not equal the electric charge (it *did* in the flat spacetime illustration above). In fact, the equations of motion possess again two first integrals implying that the *gravitating* solutions satisfy the following relation

$$M^2 + Q_s^2 = Q^2. \tag{4.2.62}$$

This last equation, in particular, shows manifestly the curved background breaks the equality between Q and Q_s.

Interestingly, one can show that there is a Smarr relation in terms of these asymptotic quantities, which is not affected by the scalar field,

$$M = V_\infty Q\ . \tag{4.2.63}$$

Moreover, a first law of thermodynamics can be obtained in the form

$$dM = V_\infty dQ\ . \tag{4.2.64}$$

We emphasise the absence of a scalar field contribution in these relations.

4.2.2 The full solutions

The gravitating version of the exact solution in Minkowski spacetime described in subsection 4.1.4.1, with coupling (4.1.17), and whose asymptotic limits have been described in subsection 4.2.1, can be constructed numerically. The set of four ordinary differential equations obtained from the above setup was solved numerically by using a standard Runge-Kutta ordinary differential equation solver and appropriate boundary conditions. Fixing α, gravitating solitons exist for arbitrary large values of Q. The profile of a typical solution is shown in Fig. 1. As one can see, the profiles of the various functions, and in particular that of the Kretschmann scalar K, are smooth as $r \to 0$. In Fig. 4.2 we show the ADM mass *vs.* electric charge diagram for families of solutions with different values of α. One can see that for all families, the solutions trivialise as $\alpha \to 0$. Moreover, a smaller α implies that the same value of the electric charge can support a more massive soliton. Obviously, the solutions also trivialise as $Q \to 0$. The electric charge supports the soliton. This is also manifest from the following virial identity that can be derived for these solutions:

$$\int_0^\infty dr\ e^{-\delta}\phi'^2 = \int_0^\infty dr\ \frac{e^{-\delta}}{r^2}\frac{Q^2}{f(\phi)}\ . \tag{4.2.65}$$

For $Q = 0$ the right hand side vanishes, and so must the left hand side, which implies $\phi' = 0$ and hence no non-trivial scalar profile exists.

Self-gravitating solitons with the coupling (4.1.23) and $n = 3$ were also obtained. They follow the same pattern as the $n = 3$ case, which is therefore illustrative.

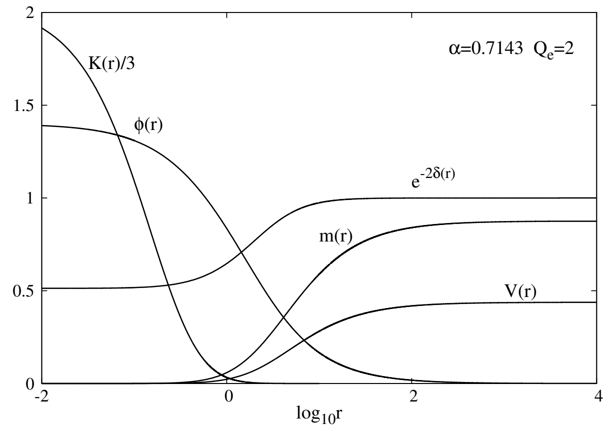


Figure 4.1: Profiles of an illustrative gravitating soliton with the coupling (4.1.17).

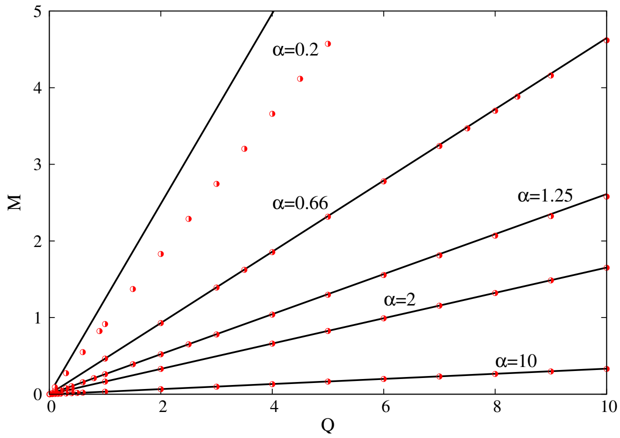


Figure 4.2: ADM *vs.* electric charge for families of gravitating solitons with different values of α. The straight lines are obtained from the perturbative solutions, whereas the dots represent the numerical solutions.

4.2.3 Perturbative solutions

The existence of a flat spacetime solution, whose total mass-energy is proportional to $1/\alpha$, suggests that the self-gravitating solitons may be expressed as a perturbative series in $1/\alpha$. Let us indeed show that the numerical solutions of the previous subsection can be approximated by such perturbative solutions. This approximation, as we will show and as one may anticipate, is accurate for sufficiently large α.

The perturbative solutions are obtained by performing a power series expansion for all relevant functions

$$m(r)=\sum_{k\geqslant 1}\left(\frac{1}{\alpha}\right)^k m_k(r)\ , \qquad \delta(r)=\sum_{k\geqslant 1}\left(\frac{1}{\alpha}\right)^k \delta_k(r)\ , \tag{4.2.66}$$

$$\phi(r)=\sum_{k\geqslant 1}\left(\frac{1}{\alpha}\right)^k \phi_k(r)\ , \qquad V(r)=\sum_{k\geqslant 1}\left(\frac{1}{\alpha}\right)^k V_k(r)\ . \tag{4.2.67}$$

As for the numerical solutions of the previous subsection, we focus on the quartic coupling function (4.1.17). Solving iteratively the field equations order by order in $1/\alpha$, we arrive at the following expressions[2]

$$\begin{aligned}
&m_1(r)=0=m_3(r)\ , \qquad m_2(r)=\frac{qr^3}{3(q+r)^3}\ , \qquad m_4(r)=-\frac{qr^4(10q^2+qr+r^2)}{90(q+r)^6}\ ,\\
&\delta_1(r)=0=\delta_3(r)\ , \qquad \delta_2(r)=\frac{q^2(q+3r)}{6(q+r)^3}\ ,\\
&\delta_4(r)=\frac{q^2(q^4+6q^3r+15q^2r^2+100qr^3-30r^4)}{540(q+r)^6}\ ,\\
&\phi_1(r)=\frac{q}{q+r}\ , \qquad \phi_2=0\ , \qquad \phi_3(r)=\frac{q(2q-r)r^2}{18(q+r)^4}\ ,\\
&V_1(r)=\frac{r^3}{3(q+r)^3}\ , \qquad V_2(r)=0\ , \qquad V_3(r)=-\frac{r^3(5q^3+25q^2r+6qr^2+r^3)}{90(q+r)^6}\ ,
\end{aligned} \tag{4.2.68}$$

where q is a free parameter, whose physical significance becomes transparent by computing the far field asymptotics of the electric potential. One finds it is related to the electric charge measured at infinity Q, as

$$Q=\frac{q}{\alpha}\ . \tag{4.2.69}$$

The perturbative solution yields the following ADM mass and scalar charge, valid to fourth order in perturbation theory:

$$M=\frac{Q}{3\alpha}\left(1-\frac{1}{30\alpha^2}+\frac{1}{1080\alpha^4}\right)\ , \qquad Q_s=\left(1-\frac{1}{18\alpha^2}+\frac{7}{3240\alpha^4}\right)Q\ . \tag{4.2.70}$$

Observe that the first terms in (4.2.70) reproduce the flat spacetime limit, eq. (4.1.21) and the fact that the electric and scalar charge coincide in that limit.

In Fig. 4.2 the perturbative solutions (4.2.70) are compared with the numerical solutions. One can observe that the former provide a good approximation for large values of α; for instance, for $\alpha=10$ the relative difference between the numerical result for $M(Q)$ and the theory one is around 10^{-4}. However, the differences start to increase for smaller α. This is illustrated by the results for $\alpha=0.2$ in Fig. 4.2.

Finally let us mention that a similar solution has been derived for the self-gravitating solitons with the coupling (4.1.23) and $n=3$. In this case one finds, *e.g.*

$$M=\frac{2Q}{5\alpha}\left(1-\frac{2}{45\alpha^2}+\frac{22}{14625\alpha^4}\right)+\dots\ . \tag{4.2.71}$$

[2] We have computed the solution up to eighth order and no obvious pattern could be found. Here we display only the first few terms for each function.

4.3 Discussion

A set of no go theorems were shown in chapters 2 and 3 establishing that the model (1.1.2) does not allow self-gravitating solitons. One of the observations therein is that if the coupling would diverge, the theorems could, potentially, be circumvented. The purpose of this chapter was to provide the mechanism of how this can happen by providing a simple construction of flat spacetime and gravitating solitons.

Preliminary analysis shows the solitons we have described herein are stable against spherical perturbations. If this is the case for generic perturbations, these solitons can be used for dynamical studies in many setups, as, for instance, boson stars [121]. Moreover, this construction reveals how to de-singularise the Coulomb field in a classical field theory, without resorting to non-linear electrodynamics, as in the Born-Infeld model [72], or invoking a manifestly extended object, such as in the Dirac model of the electron as a spherical membrane [122].

Finally, let us remark that there is a well-known similarity between the EMS model and the extended scalar-tensor-Gauss-Bonnet model, where black hole scalarisation was first pointed out in [85–87]. Very recently, a family of particle-like solutions in the latter model was discussed [123]. These particle-like solutions are also supported by a divergent coupling making them the counterparts of the solutions described herein. But in the cases reported in [123], the scalar field also diverges at the origin, in contrast with our fully regular solutions.

Chapter 5

Electromagnetic dual Einstein-Maxwell-scalar models

In this chapter we will discuss a specific solution generating technique, based on duality transformations. Here we construct the formalism of the duality orbits presented in the introduction (section 1.4) where we use a duality transformation as a map between solutions of similar Einstein-Maxwell-scalar models with different couplings $f(\phi)$ and $g(\phi)$. This contrasts with the usual self-dual models, like the pure Maxwell theory, where a duality transformation is simply a transformation between solutions of the exact same model.

In section 5.1 the formalism is presented for the electromagnetic duality in the EMS model, where we establish the map between different solutions in models with different coupling functions. These transformations preserve the metric and the scalar field and define the duality orbits.

Section 5.2 is where we apply this duality to various examples, which are explicitly known solutions of illustrative EMS models, in order to obtain duality orbits.

In section 5.3, we generalise this duality for further generalisations of Einstein-Maxwell-scalar, including possible multiple scalar and gauge fields. This allows us to consider this duality applied to, for example, the aforementioned Einstein-Maxwell-Dilaton-Axion model [71]. Some remarks are presented in section 5.4.

5.1 Electromagnetic duality in the EMS model

5.1.1 Fields and equations of motion

Consider the EMS family of models described by the action (1.4.14) which, for ease of reference, we present here again

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{f(\phi)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(\phi)}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) . \qquad (5.1.1)$$

We shall be interested in stationary asymptotically flat spacetimes, with associated asymptotically timelike Killing vector field k^μ. The scalar field will be regarded here as endowing spacetime with a medium, making the electric permittivity and the magnetic permeability spacetime dependent. Then, one uses the standard formalism for electrodynamics in a medium, definining the electric intensity, magnetic induction, electric induction and magnetic intensity 4-covectors as, respectively:[1]

$$E_\mu = k^\nu F_{\mu\nu} \ , \tag{5.1.2}$$
$$B_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\nu}F^{\alpha\beta}k^\nu = k^\nu \tilde{F}_{\mu\nu} \ , \tag{5.1.3}$$
$$D_\mu = k^\nu G_{\mu\nu}(\phi) \ , \tag{5.1.4}$$
$$H_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\nu}G^{\alpha\beta}(\phi)k^\nu = k^\nu \tilde{G}_{\mu\nu} \ , \tag{5.1.5}$$

where

$$G^{\mu\nu}(\phi) \equiv -2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = f(\phi)F^{\mu\nu} - g(\phi)\tilde{F}^{\mu\nu} \ , \tag{5.1.6}$$

and $\mathcal{L}$ is the Lagrangian density. The matter equations of motion obtained from (5.1.1) read:

$$\nabla_{[\mu}E_{\nu]} = 0, \tag{5.1.7}$$
$$\nabla_{[\mu}H_{\nu]} = 0, \tag{5.1.8}$$
$$\nabla_\mu\left(\frac{D^\mu}{V}\right) = 0, \tag{5.1.9}$$
$$\nabla_\mu\left(\frac{B^\mu}{V}\right) = 0, \tag{5.1.10}$$
$$\Box^2\phi = \frac{1}{4}\frac{df(\phi)}{d\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\frac{dg(\phi)}{d\phi}\tilde{F}_{\mu\nu}F^{\mu\nu}, \tag{5.1.11}$$

where $V = -k^\mu k_\mu$ is the norm of the Killing vector field.

5.1.2 Constitutive relations

For electrodynamics in a medium, the constitutive relations specify how the electric and magnetic inductions relate to the electric and magnetic intensities. From relations (5.1.4) and (5.1.5), the fields E_μ and H_μ are related to D_μ and B_μ fields through the following constitutive relations[2]:

$$\begin{pmatrix} E \\ H \end{pmatrix} = \frac{1}{f}\begin{pmatrix} 1 & g \\ g & f^2+g^2 \end{pmatrix}\begin{pmatrix} D \\ B \end{pmatrix} = M\begin{pmatrix} D \\ B \end{pmatrix} \ . \tag{5.1.12}$$

[1] The fields D and H here correspond to the fields E' and B' in chapter 3.

[2] These relations can be interpreted as a generalisation of the relations in the Einstein-Maxwell-Dilaton-Axion model of [71] as can be seen if we replace $f = e^{-\phi}$ and $g = a$. The sign differences are simply a consequence of the field definitions.

M shall be called the *constitutive matrix.* For $f = 1$ and $g = 0$, M becomes the identity matrix, and we recover standard vacuum electrodynamics, with $E = D$ and $H = B$ (recall we use units with $c = 1$). In general, however, E, H depend on both D, B. This is typically the case in non-linear materials and non-linear optics. Thus, one may envisage the non-minimally coupled scalar field as endowing spacetime with a non-linear material environment.

5.1.3 Duality map

We are interested in finding a duality transformation $\mathcal{D}_\beta$ that keeps equations (5.1.7)-(5.1.11) invariant in an appropriate sense. We consider duality $SO(2)$ rotations, by an angle β, acting on both the intensities and the inductions in the same way, namely [71][3]

$$\begin{pmatrix} E \\ H \end{pmatrix} \xrightarrow{\mathcal{D}_\beta} \begin{pmatrix} E' \\ H' \end{pmatrix} = S \begin{pmatrix} E \\ H \end{pmatrix} , \tag{5.1.13}$$

$$\begin{pmatrix} D \\ B \end{pmatrix} \xrightarrow{\mathcal{D}_\beta} \begin{pmatrix} D' \\ B' \end{pmatrix} = S \begin{pmatrix} D \\ B \end{pmatrix} , \tag{5.1.14}$$

where

$$S = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} , \tag{5.1.15}$$

or, equivalently,

$$F_{\mu\nu} \xrightarrow{\mathcal{D}_\beta} F'_{\mu\nu} = \cos\beta\, F_{\mu\nu} + \sin\beta\, \tilde{G}_{\mu\nu} , \tag{5.1.16}$$

$$G_{\mu\nu} \xrightarrow{\mathcal{D}_\beta} G'_{\mu\nu} = \cos\beta\, G_{\mu\nu} + \sin\beta\, \tilde{F}_{\mu\nu} . \tag{5.1.17}$$

Comparing (5.1.16) with (1.4.12) one observes this is the standard duality rotation of Maxwell's theory. From (5.1.12), it follows that the constitutive matrix becomes

$$M \xrightarrow{\mathcal{D}_\beta} M' = SMS^{-1} , \tag{5.1.18}$$

which reads, explicitly

$$M' = \frac{1}{f} \begin{pmatrix} f^2 \sin^2\beta + (g\sin\beta + \cos\beta)^2 & g\cos(2\beta) + (f^2 + g^2 - 1)\sin(2\beta)/2 \\ g\cos(2\beta) + (f^2 + g^2 - 1)\sin(2\beta)/2 & f^2\cos^2\beta + (g\cos\beta - \sin\beta)^2 \end{pmatrix} . \tag{5.1.19}$$

Thus, the duality rotation with an arbitrary angle β yields this new constitutive matrix. The duality orbit of models is defined as the continuous sequence of EMS models (5.1.1) where the coupling functions are

$$(f(\phi), g(\phi)) \xrightarrow{\mathcal{D}_\beta} (f_\beta(\phi), g_\beta(\phi)) , \tag{5.1.20}$$

[3] In a medium, the field E is dual to H while D is dual to B. This is because the dual fields share the same equation of motion, so a linear combination of them will still respect the same equation. For example, $\nabla_{[\mu}E'_{\nu]} = \cos\beta\nabla_{[\mu}E_{\nu]} + \sin\beta\nabla_{[\mu}H_{\nu]} = 0$.

such that

$$M' = \frac{1}{f_\beta}\begin{pmatrix} 1 & g_\beta \\ g_\beta & f_\beta^2 + g_\beta^2 \end{pmatrix} \ . \tag{5.1.21}$$

That is, the constitutive relations have the same functional form in terms of the coupling functions, along the whole duality orbit. Comparing (5.1.19) with (5.1.21) yields

$$f_\beta = \frac{f}{f^2 \sin^2\beta + (g\sin\beta + \cos\beta)^2} \ , \tag{5.1.22}$$

$$g_\beta = \frac{1}{2}\frac{2g\cos(2\beta) + (f^2 + g^2 - 1)\sin(2\beta)}{f^2\sin^2\beta + (g\sin\beta + \cos\beta)^2} \ . \tag{5.1.23}$$

The orbit of dual theories is therefore the 1-parameter family of actions

$$\mathcal{S}_\beta = \frac{1}{4\pi}\int d^4x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{4}f_\beta(\phi)F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{4}g_\beta(\phi)F'_{\mu\nu}\tilde{F}'^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right) \ , \tag{5.1.24}$$

where $\mathcal{S}_0$ equals the original action (5.1.1) and $(\mathbf{g}, \mathbf{A}', \phi)$, where $\mathbf{F}' = d\mathbf{A}'$, are taken as the independent fields in a variational principle. The tensor $G'_{\mu\nu}$ is found, as before, by the variation of the Lagrangian density in (5.1.24), $\mathcal{L}_\beta$, with respect to $\mathbf{F}'$:

$$G'^{\mu\nu}(\phi) \equiv -2\frac{\partial\mathcal{L}_\beta}{\partial F'_{\mu\nu}} = f_\beta(\phi)F'^{\mu\nu} - \frac{1}{4}g_\beta(\phi)\tilde{F}'^{\mu\nu} \ . \tag{5.1.25}$$

From the discussion above, it follows that if

$$[\mathbf{g}, \mathbf{A}, \phi; f(\phi), g(\phi)] \ , \tag{5.1.26}$$

is a solution of (5.1.1), then

$$[\mathbf{g}, \mathbf{A}', \phi; f_\beta(\phi), g_\beta(\phi)] \tag{5.1.27}$$

is a solution of the Maxwell equations obtained from (5.1.24). It remains to check the scalar and Einstein equations are also obeyed for the model (5.1.24).

The scalar equation of motion derived from (5.1.24) is

$$\Box^2\phi = \frac{1}{4}\frac{df_\beta}{d\phi}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{4}\frac{dg_\beta}{d\phi}\tilde{F}'_{\mu\nu}F'^{\mu\nu} \ . \tag{5.1.28}$$

Using the identities $F'_{\mu\nu}F'^{\mu\nu} = 2(\mathbf{B}'^2 - \mathbf{E}'^2)/V$ and $\tilde{F}'_{\mu\nu}F'^{\mu\nu} = -4\mathbf{E}'\cdot\mathbf{B}'/V$ and then reverting back to the original fields, it follows (5.1.28) reduces to the original equation of motion (5.1.11) for the scalar field, which is obeyed, since (5.1.26) is a solution of (5.1.1) by assumption.

It is also straightforward to check that the Einstein equations of (5.1.1) and (5.1.24) are the same. The energy-momentum tensor of the model (5.1.24), $T'_{\mu\nu}$ is obtained from the action by differentiating with respect to the metric, which is unchanged by the duality rotation. Then, the functional form of the energy-momentum tensor is the same as that of the model (5.1.1), $T_{\mu\nu}$, and they are mapped simply replacing $(\mathbf{A}, f, g) \to (\mathbf{A}', f_\beta, g_\beta)$. One can then show that

$$\begin{aligned} T'_{\mu\nu} &= f_\beta\left(F'_{\mu\alpha}F'^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F'_{\sigma\tau}F'^{\sigma\tau}\right) + \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi \\ &= f\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\right) + \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi \\ &+ \frac{f^2\sin\beta(g\sin\beta+\cos\beta)}{f^2\sin^2\beta+(g\sin\beta+\cos\beta)^2}\left(F_{\mu\alpha}\tilde{F}_{\nu}{}^{\alpha} + F_{\nu\alpha}\tilde{F}_{\mu}{}^{\alpha} - \frac{1}{2}g_{\mu\nu}F_{\sigma\tau}\tilde{F}^{\sigma\tau}\right) , \end{aligned} \qquad (5.1.29)$$

by a straightforward application of the transformations. The last term of this equation vanishes because for any 2-form in four dimensions we have that $F_{\mu\alpha}\tilde{F}_{\nu}{}^{\alpha} = \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}\tilde{F}^{\alpha\beta}$. Then, as expected,

$$T'_{\mu\nu} = f\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\right) + \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi = T_{\mu\nu} \ . \qquad (5.1.30)$$

We have thus established the duality orbit of solutions (1.4.16), under (5.1.16) and (5.1.20), the latter explicitly given by (5.1.22)-(5.1.23).

A represention of the duality orbits is obtained as follows. Consider a two dimensional space parameterised by $(x, y) = (f_\beta, g_\beta)$ as an illustration of the space of EMS models. It is simple to check that the duality orbits defined by (5.1.22)-(5.1.23) obey:

$$(f_\beta - A)^2 + g_\beta^2 = A^2 - 1 \ , \qquad \text{where} \quad A \equiv \frac{1+f^2+g^2}{2f} \ . \qquad (5.1.31)$$

Thus, they are circles, passing through the fiducial EMS model $(f_0, g_0) = (f, g)$. The radius of the circles vanishes at the self-dual model $(f_0, g_0) = (1, 0)$, that is, Maxwell's theory. This is illustrated in Fig. 5.1.

5.2 Examples of duality orbits

5.2.1 Closed form solution for a scalarised electric charge in flat spacetime

Our first example is a scalarised electric charge solution found in [8], for the model (5.1.1) in flat spacetime and with coupling functions

$$f(\phi) = \frac{1}{1-\phi^2} \ , \qquad g(\phi) = 0 \ . \qquad (5.2.32)$$

The scalar field and electric potential are radial functions

$$\phi(r) = \zeta\sin\left(\frac{Q}{r}\right) , \qquad V(r) = \frac{Q}{r} + \zeta^2\left[\frac{1}{4}\sin\left(\frac{2Q}{r}\right) - \frac{Q}{2r}\right] , \qquad (5.2.33)$$

from which the electric intensity and induction have only the radial component:

$$E_r = \frac{Q}{r^2}\left[1 - \zeta^2\sin^2\left(\frac{Q}{r}\right)\right] , \qquad D_r = f(\phi)E_r = \frac{Q}{r^2} \ , \qquad (5.2.34)$$

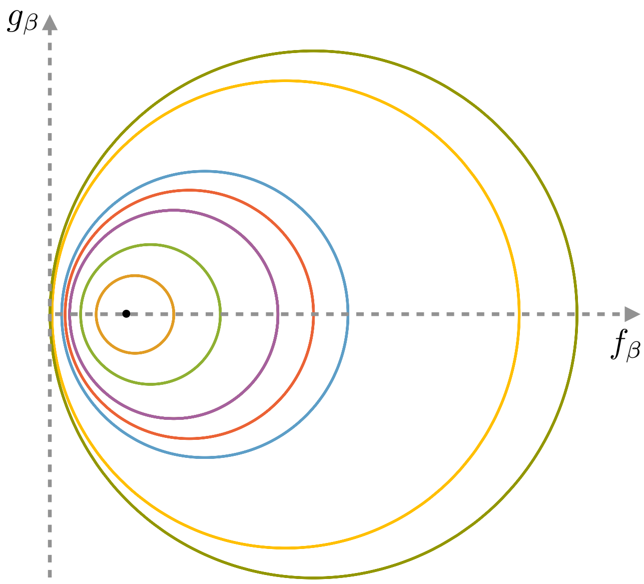


Figure 5.1: Duality orbits in the space of EMS models, for different values of the fiducial EMS model (f, g). The self-dual model, Maxwell's theory, is the black dot at $(1, 0)$.

whereas the magnetic induction and intensity vanish

$$\mathbf{B} = 0 = \mathbf{H} \; . \tag{5.2.35}$$

Observe that whereas the electric intensity is sensitive to the scalar field, the electric induction has the standard Coulombian form, and it is the same as when $\zeta = 0$.

The duality orbit that goes through the model (5.2.32) has:

$$f_\beta = \frac{1-\phi^2}{1-2\cos^2\beta\phi^2+\cos^2\beta\phi^4} \; , \qquad g_\beta = \frac{\phi^2(2-\phi^2)\sin\beta\cos\beta}{1-2\cos^2\beta\phi^2+\cos^2\beta\phi^4} \; . \tag{5.2.36}$$

Along this sequence of dual models, the seed (5.2.33)-(5.2.35) is mapped, generically, to dyonic solutions. For an arbitrary β, the fields along this orbit are:

$$E'_r = \frac{Q}{r^2}\left[1-\zeta^2\sin^2\left(\frac{Q}{r}\right)\right]\cos\beta \; , \qquad D'_r = \frac{Q}{r^2}\cos\beta \; , \tag{5.2.37}$$

$$B'_r = -\frac{Q}{r^2}\sin\beta \; , \qquad H'_r = -\frac{Q}{r^2}\left[1-\zeta^2\sin^2\left(\frac{Q}{r}\right)\right]\sin\beta \; . \tag{5.2.38}$$

Again, one observes the Coulombic form of the electric and magnetic induction fields, with electric and magnetic charges, respectively, $Q_\beta \equiv Q\cos\beta$ and $P_\beta \equiv Q\sin\beta$, such that

$$Q_\beta^2 + P_\beta^2 \equiv Q^2 = \text{constant} \; , \tag{5.2.39}$$

along the whole duality orbit.

Within this orbit there is, however, a pure magnetic solution at $\beta = \pi/2$, wherein the coupling functions are

$$f_{\pi/2}(\phi) = \frac{1}{f(\phi)} = 1-\phi^2 \; , \qquad g_{\pi/2}(\phi) = 0 \; , \tag{5.2.40}$$

the electric intensity and induction vanish

$$\mathbf{E}' = 0 = \mathbf{D}' \ , \tag{5.2.41}$$

and the magnetic induction and intensity are only radial functions:

$$B_r' = -\frac{Q}{r^2} \ , \qquad H_r' = f_{\pi/2}(\phi) B_r = -\frac{Q}{r^2}\left[1 - \zeta^2 \sin^2\left(\frac{Q}{r}\right)\right] \ . \tag{5.2.42}$$

We thus found a pure magnetic solution for the model with couplings (5.2.40). The original electric charge Q becomes the magnetic charge just as in the Maxwell theory example (1.4.13). For $\beta = \pi$ we would get the original purely electric solution but with opposite charge sign while for $\beta = 3\pi/2$ we get the pure magnetic solution once again with opposite charge sign. For any β value between these, we get a dyon whose magnetic and electric charges relative contributions depend on how close β is to the values mentioned above. There is a full orbit of solutions that can be obtained from the original solution.

Let us close this example with two observations. First, this formalism unveils the fact that although the original solution has a non-Coulombian electric intensity, the electric induction is Coulombian. The same holds along the whole duality orbit. Second, at $\beta = \pi/2$ the $f(\phi)$ coupling function is mapped into its inverse, whereas $g(\phi)$ remains zero. This is a generic feature starting with arbitrary $f(\phi)$ and vanishing $g(\phi)$, as can be appreciated from (5.1.22)-(5.1.23):

$$f_\beta \overset{\beta=\pi/2}{=} \frac{f}{f^2+g^2} \overset{g=0}{=} \frac{1}{f} \ , \tag{5.2.43}$$

$$g_\beta \overset{\beta=\pi/2}{=} -\frac{g}{f^2+g^2} \overset{g=0}{=} 0 \ . \tag{5.2.44}$$

Since $f(\phi)$ defines the coupling strength of the Maxwell field, this particular value of the map is an example of a strong $\leftrightarrow$ weak coupling duality, with an electric $\leftrightarrow$ magnetic mapping, reminscent of the Montonen-Olive duality [124].

5.2.2 Closed form Maxwell-scalar solitons in flat spacetime

Our second example uses the seed configuration found in chapter 4, section 4.1. It describes a purely electric, static, spherically symmetric soliton solution of (5.1.1) in flat spacetime, with

$$f(\phi) = \frac{1}{(1-\alpha\phi)^4} \ , \qquad g(\phi) = 0 \ . \tag{5.2.45}$$

The scalar field reads

$$\phi = \frac{Q}{\alpha Q + r} \ , \tag{5.2.46}$$

the electric intensity and induction have again only a radial component

$$E_r = \frac{Qr^2}{(r+\alpha Q)^4} \ , \qquad D_r = f(\phi) E_r = \frac{Q}{r^2} \ , \tag{5.2.47}$$

whereas the magnetic induction and intensity again vanish

$$\mathbf{B} = 0 = \mathbf{H} \ . \tag{5.2.48}$$

The $f(\phi)$ coupling (5.2.45) diverges at the origin $r = 0$; but all physical quantities are regular, such as the energy density and the electric intensity. Indeed, this solution was interpreted in the last chapter as a de-singularisation of the Coulomb solution of Maxwell's theory. Nonetheless, the electric induction $\mathbf{D}$ is again Coulombian and diverges at the origin.

The duality orbit that goes through the model (5.2.45), has:

$$f_\beta = \frac{(1-\alpha\phi)^4}{\sin^2\beta + \cos^2\beta(1-\alpha\phi)^8} \ , \qquad g_\beta = \frac{[1-(1-\alpha\phi)^8]\sin\beta\cos\beta}{\sin^2\beta + \cos^2\beta(1-\alpha\phi)^8} \ . \tag{5.2.49}$$

Once more, the duality map will generate dyonic solutions from the seed (5.2.46)-(5.2.48). The fields for this orbit are:

$$E'_r = \frac{Qr^2}{(r+\alpha Q)^4}\cos\beta \ , \qquad D'_r = \frac{Q}{r^2}\cos\beta \ , \tag{5.2.50}$$

$$B'_r = -\frac{Q}{r^2}\sin\beta \ , \qquad H'_r = -\frac{Qr^2}{(r+\alpha Q)^4}\sin\beta \ . \tag{5.2.51}$$

A pure magnetic solution is obtained $\beta = \pi/2$. The dual model at this β value has couplings

$$f_{\pi/2}(\phi) = \frac{1}{f(\phi)} = (1-\alpha\phi)^4 \ , \qquad g_{\pi/2}(\phi) = 0 \ , \tag{5.2.52}$$

and the dual configuration has vanishing electric intensity and induction

$$\mathbf{E}' = 0 = \mathbf{D}' \ , \tag{5.2.53}$$

and a spherical magnetic induction and intensity

$$B'_r = -\frac{Q}{r^2} \ , \qquad H'_r = f_{\pi/2}(\phi)B'_r = -\frac{Qr^2}{(r+\alpha Q)^4} \ . \tag{5.2.54}$$

The magnetic induction of the dual solution B'_r is Coulombic and diverges at $r = 0$ while $\mathbf{H}'$ is regular. All physical quantities are regular, including the energy density, making this a regular magnetic soliton.

Note how the solutions for the duality orbits (5.2.36) and (5.2.49) always have a vanishing f_β at the origin except for when $\beta = 0$ or $\beta = \pi$, in which case it diverges and we have purely electric solutions. This is because the presence of a magnetic field with a vanishing f is enough to circumvent the no go theorems of the previous chapters, which rely on the definition of a magnetic-like scalar potential ψ that respects $\partial_\mu\psi = fB$. And, as expected, when we have no magnetic field, the only way to circumvent the no go theorems is to have a diverging coupling f.

5.2.3 Closed form dilatonic solution in flat spacetime

As yet another example, consider the spherically symmetric solution discussed in chapter 4, section 4.1 for a dilatonic coupling, $f = e^{-\alpha\phi}$, in flat spacetime. The scalar field and electric potential read

$$\phi = -\frac{2}{\alpha}\ln\left(1+\frac{\alpha Q}{2r}\right) , \qquad V(r) = -\frac{2Q}{\alpha Q + 2r} . \tag{5.2.55}$$

whereas the electric intensity and induction fields are

$$E_r = \frac{4Q}{(\alpha Q + 2r)^2} , \qquad D_r = f(\phi)E_r = \frac{Q}{r^2} . \tag{5.2.56}$$

The magnetic induction and intensity are trivial

$$\mathbf{B} = 0 = \mathbf{H} . \tag{5.2.57}$$

In this case, the duality orbit that goes through this model, has:

$$f_\beta = \frac{1}{e^{\alpha\phi}\cos^2\beta + e^{-\alpha\phi}\sin^2\beta} , \qquad g_\beta = -\frac{\sin 2\beta \sinh \alpha\phi}{e^{\alpha\phi}\cos^2\beta + e^{-\alpha\phi}\sin^2\beta} . \tag{5.2.58}$$

The fields obtained from the seed solution are, along the duality orbit,

$$E'_r = \frac{4Q}{(\alpha Q + 2r)^2}\cos\beta , \qquad D'_r = \frac{Q}{r^2}\cos\beta , \tag{5.2.59}$$

$$B'_r = -\frac{Q}{r^2}\sin\beta , \qquad H'_r = -\frac{4Q}{(\alpha Q + 2r)^2}\sin\beta . \tag{5.2.60}$$

The reasoning is the same and we can see there is, once again, a magnetic solution for $\beta = \pi/2$ with trivial electric intensity and induction.

5.2.4 The GMGHS black hole

We now consider a curved spacetime generalisation of the example in the last subsection. This is the well known dilatonic electrically charged, spherically symmetric black hole (in four spacetime dimensions), obtained in the model (5.1.1) with

$$f(\phi) = e^{-2\phi} , \qquad g(\phi) = 0 , \tag{5.2.61}$$

It was first discussed by Gibbons and Maeda in [5] and later by Garfinkle, Horowitz and Strominger [125]. We shall call it the GMGHS black hole. The metric reads

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{r_-}{r}\right)(d\theta^2 + \sin^2\theta d\varphi^2) , \tag{5.2.62}$$

where M is the black hole mass, Q is the electric charge, $r_- = e^{2\phi_\infty}Q^2/M$ and ϕ_∞ is the asymptotic value of the scalar field. The scalar field and the gauge potential read

$$e^{2\phi} = e^{2\phi_\infty}\left(1-\frac{r_-}{r}\right) , \qquad \mathbf{A} = -\frac{Q}{r}e^{2\phi_\infty}dt , \tag{5.2.63}$$

whereas the electric intensity and induction are

$$E_r = \frac{Q}{r^2} e^{2\phi_\infty} \ , \qquad D_r = f(\phi) E_r = \frac{Q}{r(r - r_-)} \ , \tag{5.2.64}$$

and the magnetic induction and intensity vanish:

$$\mathbf{B} = 0 = \mathbf{H} \ . \tag{5.2.65}$$

We remark that $\mathbf{D}$ does not have a Coulombic form, unlike the above cases. This is because the radial coordinate in (5.2.62) is not the areal radius. Using the areal radius $r^* = r\sqrt{1 - r_-/r}$, the Coulombic form $D_r = Q/r^{*2}$ is recovered.

The duality orbit that goes through this model, has the form (5.2.58) with $\alpha = 2$. The fields obtained along the duality orbit, seeded by the GMGHS solution are

$$E'_r = \frac{Q}{r^2} e^{2\phi_\infty} \cos\beta \ , \qquad D'_r = \frac{Q}{r(r - r_-)} \cos\beta \ , \tag{5.2.66}$$

$$B'_r = -\frac{Q}{r(r - r_-)} \sin\beta \ , \qquad H'_r = -\frac{Q}{r^2} e^{2\phi_\infty} \sin\beta \ . \tag{5.2.67}$$

Once again for $\beta = \pi/2$ we obtain a purely magnetic configuration in the dual model with

$$f_{\pi/2}(\phi) = \frac{1}{f(\phi)} = e^{2\phi} \ , \qquad g(\phi) = 0 \ . \tag{5.2.68}$$

The magnetic induction and intensity are now non-trivial:

$$B'_r = -\frac{Q}{r(r - r_-)} \ , \qquad H'_r = f_{\pi/2}(\phi) B'_r = -\frac{Q}{r^2} e^{2\phi_\infty} \ , \tag{5.2.69}$$

whereas the electric intensity and induction are trivial

$$\mathbf{E}' = 0 = \mathbf{D}' \ . \tag{5.2.70}$$

This magnetic dilatonic black hole configuration was first obtained by Garfinkle, Horowitz and Strominger in [125], wherein the electric configuration was actually obtained by this duality rotation. The electromagnetic duality transformation of the EMS model reduces, for this specific choice of β, to this simple example of S-duality in low energy string theory, amounting to the change $\phi \to -\phi$, which in this context is the dilaton field.

We can just as easily find a dyon black hole for any other angle β, but in this case $g(\phi)$ becomes generically non-vanishing. As a concrete example take $\beta = \pi/4$. Then, the model along the duality orbit has $f_\beta = 1/\cosh 2\phi$, $g_\beta = -\tanh 2\phi$ and its action is, explictly:

$$\mathcal{S}_{\frac{\pi}{4}} = \frac{1}{4\pi} \int d^4x \sqrt{-g} \left(\frac{R}{4} - \frac{1}{4\cosh 2\phi} F'_{\mu\nu} F'^{\mu\nu} - \frac{\tanh 2\phi}{4} F'_{\mu\nu} \tilde{F}'^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \ . \tag{5.2.71}$$

This model admits a dyonic black hole solution with the GMGHS geometry and scalar field, (5.2.62) and (5.2.63), and the electromagnetic field (5.2.66)-(5.2.67) with $\beta = \pi/4$, which in covariant form reads:

$$\mathbf{F} = -\frac{Q'}{r^2} e^{2\phi_\infty} dt \wedge dr + Q' \sin\theta d\theta \wedge d\varphi \ \ \Leftrightarrow \ \ \mathbf{A} = -\frac{Q'}{r} e^{2\phi_\infty} dt - Q' \cos\theta d\varphi \ , \tag{5.2.72}$$

where $Q' = Q/\sqrt{2}$. Comparing with (1.4.13) one can confirm this describes an electric plus magnetic charge, a dyon. As far as we are aware, such closed form solution within model (5.2.71) has not been discussed previously in the literature. Moreover, one can compute other exact, closed form solutions of this model, *e.g.* rotating charged black holes.

5.2.5 Other models with scalarised and axionic black holes

Having understood the duality orbits, let us mention a set of other EMS models (5.1.1) wherein numerical black hole solutions have been constructed in the literature and whose duality orbits can be constructed. Examples include the following coupling functions:

1. Exponential coupling: $f_E^\alpha(\phi) = e^{-\alpha\phi^2}$, $g(\phi) = 0$;
2. Power-law coupling: $f_P^\alpha(\phi) = 1 - \alpha\phi^2$, $g(\phi) = 0$;
3. Fractional coupling $f_F^\alpha(\phi) = \frac{1}{1+\alpha\phi^2}$, $g(\phi) = 0$;
4. Higher power-law coupling $f_{HP}^\alpha(\phi) = 1 - \alpha\phi^4$, $g(\phi) = 0$;
5. Axionic coupling $f(\phi) = 1$, $g_A^\alpha = \alpha\phi$.
6. Axionic-type coupling $f(\phi) = 1$, $g_{AT}^\alpha = \alpha\phi^2$.

In all cases α is a coupling constant. Couplings 1-3 were discussed in [8, 81, 82] in the context of EMS models allowing spontaneous scalarisation of charged black holes (see also, *e.g.* [98, 101–105, 126–134]); all these coupling functions have the same behaviour for small values of $\alpha\phi^2$. Coupling 4 was discussed in [84]; it does not allow spontaneous scalarisations but it exhibits an interesting two-branch space of solutions with scalar hair, co-existing with the standard Reissner-Nordström black hole, in a trinity of non-uniqueness. Black holes with coupling 5 were first discussed in [135] and revisited recently in [83], wherein coupling 6 was also discussed, again in the context of spontaneous scalarisation of charged black holes. Various solutions for flat spacetime with coupling 5 were also found and discussed in [136] and [137]. See also, e.g., [138] for other forms of the scalar coupling $f(\phi)$, in the context of holography.

We shall not analyse the duality orbits for all these models in detail, but let us make some comments at the rotation point $\beta = \pi/2$. One can see the duality relates different couplings through the relations (5.2.43)-(5.2.44). For instance, we get the following identities:

$$f^\alpha_{E\pi/2} = f_E^{-\alpha} \ , \tag{5.2.73}$$

$$f^\alpha_{P\pi/2} = f_F^{-\alpha} \ , \tag{5.2.74}$$

$$f^\alpha_{F\pi/2} = f_P^{-\alpha} \ . \tag{5.2.75}$$

Thus, the exponential squared coupling enjoys a type of S-duality symmetry analogous to that of the dilatonic model of section 5.2.4, via (5.2.73), whereas the power law and fractional couplings are along the same duality orbit, and can be mapped into each other by also changing the sign of the coupling constant α. At $\beta = \pi/2$, moreover, the purely electric solutions of models 1-4, as before, become purely magnetic, with:

$$\mathbf{E}' = \mathbf{H} = 0 \ , \tag{5.2.76}$$

$$\mathbf{B}' = -\mathbf{D} = -f\mathbf{E} \tag{5.2.77}$$

These results are in agreement with the Bekenstein type identities found in [82], where both $f_{,\phi\phi}$ and $\phi f_{,\phi}$ must have the opposite sign of $F_{\mu\nu}F^{\mu\nu}$ for solutions with a scalar profile to exist. For purely electric ($F^2 < 0$) or magnetic ($F^2 > 0$) solutions, these conditions imply a different sign for the coupling constant α for the couplings mentioned above.

5.3 Duality in generalisations of Einstein-Maxwell-scalar

In here we consider various other possible generalisations of the Einstein-Maxwell-scalar model to which we can still apply some adaptation of the duality formalism presented in this chapter.

5.3.1 Dual scalar field generalisation

Consider once again an asymptotically flat stationary spacetime with an asymptotically timelike Killing field k_μ and following action

$$\begin{aligned}\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\bigg[&-\frac{f(\phi,a)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(\phi,a)}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{h(\phi)}{2}\partial_\mu a\partial^\mu a - V(\phi,a)\bigg] \ ,\end{aligned} \tag{5.3.78}$$

where ϕ and a are both scalar fields. The functions $f(\phi,a)$, $g(\phi,a)$ and $h(\phi)$ are arbitrary non-minimal couplings between the fields and $V(\phi,a)$ is a potential term depending on the scalar fields. We then have the following equations of motion:

$$\Box^2\phi = \frac{1}{4}\frac{\partial f}{\partial\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\frac{\partial g}{\partial\phi}\tilde{F}_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{\partial h}{\partial\phi}\partial_\mu a\partial^\mu a + \frac{\partial V}{\partial\phi} \ , \tag{5.3.79}$$

$$\Box^2 a = \frac{1}{4h(\phi)}\frac{\partial f}{\partial a}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4h(\phi)}\frac{\partial g}{\partial a}\tilde{F}_{\mu\nu}F^{\mu\nu} + \frac{\partial V}{\partial a} \tag{5.3.80}$$

$$\nabla_{[\mu}E_{\nu]} = 0, \tag{5.3.81}$$

$$\nabla_{[\mu}H_{\nu]} = 0, \tag{5.3.82}$$

$$\nabla_\mu\left(\frac{D^\mu}{V}\right) = 0, \tag{5.3.83}$$

$$\nabla_\mu\left(\frac{B^\mu}{V}\right) = 0, \tag{5.3.84}$$

where the electromagnetic fields are defined analogously to (5.1.2)-(5.1.5) as

$$E_\mu = k^\nu F_{\mu\nu} \ , \tag{5.3.85}$$

$$B_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\nu}F^{\alpha\beta}k^\nu = k^\nu \tilde{F}_{\mu\nu} \ , \tag{5.3.86}$$

$$D_\mu = k^\nu G_{\mu\nu}(\phi, a) \ , \tag{5.3.87}$$

$$H_\mu = \frac{1}{2}\varepsilon_{\mu\alpha\beta\nu}G^{\alpha\beta}(\phi, a)k^\nu = k^\nu \tilde{G}_{\mu\nu} \ , \tag{5.3.88}$$

with $V = -k_\mu k^\mu$ and

$$G_{\mu\nu}(\phi, a) = -2\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = f(\phi, a)F_{\mu\nu} - g(\phi, a)\tilde{F}_{\mu\nu}. \tag{5.3.89}$$

The constitutive matrix for this model is still the same as the one above (5.1.12).

It is worth noting that this model includes an Einstein-Maxwell-scalar model with a complex scalar field. To see this we can just define the complex scalar field as $\psi = \phi + ia$ and have $h = 1$. This results in the following action

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\left[-\frac{f(\psi,\psi^*)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(\psi,\psi^*)}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi^* - V(\psi,\psi^*)\right] . \tag{5.3.90}$$

5.3.2 General duality map

We now want to construct, in the same way as above, a general duality transformation $\mathcal{D}_\beta$ that also keeps the Maxwell equations (5.3.81) -(5.3.84) invariant. This duality transformation is parametrised by an angle β and can be represented as the duality in expressions (5.1.13) to (5.1.17) but with the new field definitions. The rest of the process is similar to the less general case.

The duality orbit of models is now defined as the continuous sequence of EMS models (5.3.78) where the coupling functions that couple the scalar and electromagnetic fields are

$$(f(\phi, a), g(\phi, a)) \xrightarrow{\mathcal{D}_\beta} (f_\beta(\phi, a), g_\beta(\phi, a)) \ , \tag{5.3.91}$$

and can be analogously obtained as in (5.1.22) and (5.1.23) but now dependent on two scalar fields. The orbit of dual theories is therefore the 1-parameter family of actions

$$\begin{aligned}\mathcal{S}_\beta = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\bigg(&- \frac{f_\beta(\phi, a)}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{g_\beta(\phi, a)}{4}F'_{\mu\nu}\tilde{F}'^{\mu\nu} \\ &- \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{h(\phi)}{2}\partial_\mu a\partial^\mu a - V(\phi, a)\bigg) \ ,\end{aligned} \tag{5.3.92}$$

where $\mathcal{S}_0$ this time equals the original action (5.3.78) and $(\mathbf{g}, \mathbf{A}', \phi, a)$, are taken as the independent fields in a variational principle. The tensor $G'_{\mu\nu}$ is now

$$G'^{\mu\nu}(\phi, a) \equiv -2\frac{\partial \mathcal{L}_\beta}{\partial F'^{\mu\nu}} = f_\beta(\phi, a)F'^{\mu\nu} - g_\beta(\phi, a)\tilde{F}'^{\mu\nu} \ . \tag{5.3.93}$$

In the same way as before, it follows that if

$$[\mathbf{g}, \mathbf{A}, \phi, a; f(\phi, a), g(\phi, a), h(\phi)] \ , \tag{5.3.94}$$

is a solution of (5.3.78), then

$$[\mathbf{g}, \mathbf{A}', \phi, a; f_\beta(\phi, a), g_\beta(\phi, a), h(\phi)] \tag{5.3.95}$$

is a solution of the Maxwell equations obtained from (5.3.92). It remains to check the scalar, axion and Einstein equations are also obeyed for the model (5.3.92). The Einstein equations are shown to be invariant by the same argument used above for the EMS case, where the invariance of the energy momentum tensor was shown under this duality rotation. The scalar equation was also shown to be invariant by showing that

$$\frac{\partial f_\beta}{\partial \phi} F'_{\mu\nu} F'^{\mu\nu} - \frac{\partial g_\beta}{\partial \phi} \tilde{F}'_{\mu\nu} F'^{\mu\nu} = \frac{\partial f}{\partial \phi} F_{\mu\nu} F^{\mu\nu} - \frac{\partial g}{\partial \phi} \tilde{F}_{\mu\nu} F^{\mu\nu} . \tag{5.3.96}$$

Now we need to show the same but also with the derivatives with respect to a. In fact this is easily verified because what really is invariant under these transformations is

$$df F_{\mu\nu} F^{\mu\nu} - dg \tilde{F}_{\mu\nu} F^{\mu\nu}, \tag{5.3.97}$$

meaning that the same is valid for any derivative we take of the functions f and g, be it with respect to ϕ or a. This means that, because the duality transformation does not act on ϕ and a, equations (5.3.79) and (5.3.80) are invariant under duality.

5.3.3 Examples of models

In here we specify examples of models where we can apply these duality transformations.

5.3.3.1 Einstein-Maxwell-Dilaton-Axion

Here we find the duality orbit of a model which is included in the Einstein-Maxwell-Dilaton-Axion class of models and whose duality properties were studied in [71] where a self-duality was formulated. The model, with $f(\phi, a) = e^{-\phi}$, $g(\phi, a) = a$, $h(\phi) = e^{2\phi}$ and $V(\phi, a) = 0$, has the following action in four dimensions

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi} \int d^4x \sqrt{-g} \left[-\frac{e^{-\phi}}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a}{4} \tilde{F}_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{e^{2\phi}}{2} \partial_\mu a \partial^\mu a \right] . \tag{5.3.98}$$

We can use relations (5.1.22) and (5.1.23) to find the duality orbits which are

$$f_\beta = \frac{1}{e^{-\phi} + e^{\phi}(a \sin\beta + \cos\beta)^2} \quad g_\beta = \frac{1}{2} \frac{2a\cos(2\beta) + (e^{-2\phi} + a^2 - 1)\sin(2\beta)}{e^{-2\phi} + (a\sin\beta + \cos\beta)^2} \tag{5.3.99}$$

A few solutions were found for this model on which this transformation can be applied. Examples are the wide range of stationary and axisymmetric solutions found in [139] and the rotating soliton solution in [140] (see also [141]).

5.3.3.2 Potentials in Einstein-Maxwell-scalar

The inclusion of a scalar potential in this generalisation allows us to apply this treatment to plenty of different models. The scalar potential is invariant under these transformations, meaning that the duality orbits will have different couplings with scalar fields that will have the exact same self interaction.

An example we can find is a fairly studied model which is the Einstein-Maxwell-Dilaton model with an exponential or Liouville potential [142]. This kind of potential arises as a cosmological constant in specific contexts of string theory or in non-trivial curved AdS spacetimes (see for example [143] and [144]). This kind of model has $f = e^{\alpha\phi}$, $g = 0$, $h = 0$ and $V = 2\Lambda e^{-\delta\phi}$, so the action is[4]

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{e^{\alpha\phi}}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 2\Lambda e^{-\delta\phi}\Big] . \tag{5.3.100}$$

We can see that for $\delta = 0$, we recover a basic Einstein-Maxwell-Dilaton model with a fixed cosmological constant Λ. The duality orbits of this kind of model were already calculated in section 5.2 for the dilatonic coupling without a potential and are represented by the couplings (5.2.58).

Still in the Einstein-Maxwell-Dilaton model, in [145] an exact solution was obtained for the following potential

$$V(\phi) = 2\gamma(2\phi + \phi\cosh\phi - 3\sinh\phi) , \tag{5.3.101}$$

with dilaton coupling constant $\alpha = 1$. The coupling function and scalar field for this solution can be obtained from

$$f(\phi) = e^{\phi} = 1 + \frac{Q_s^2}{2r^2} + \frac{Q_s^2}{2r^2}\sqrt{1 + \frac{4r^2}{Q_s^2}} , \tag{5.3.102}$$

where Q_s is a scalar charge. The metric of this solution has the following form

$$ds^2 = -N(r)\sigma^2(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \text{with} \quad N(r) = 1 - \frac{2m(r)}{r} \tag{5.3.103}$$

where

$$m(r) = \frac{r^2}{2\sigma^2(r)}\left[\frac{Q_e^2}{r^2}\left(\sqrt{1 - \frac{4r^2}{Q_s^2}} - 1\right) - \alpha\left(\frac{Q_s^2}{2}\sqrt{1 - \frac{4r^2}{Q_s^2}} - r^2\phi(r)\right)\right] - \frac{Q_s^2}{8r} , \tag{5.3.104}$$

$$\sigma(r) = \left(1 + \frac{Q_s^2}{4r^2}\right)^{-1/2} . \tag{5.3.105}$$

The electric intensity and induction are

$$E_r = \frac{Q_e}{f(\phi)\sigma(r)r^2} , \qquad D_r = \frac{Q_e}{\sigma(r)r^2} . \tag{5.3.106}$$

[4] As the a field does not contribute we can also consider $h = 1$ and $a = 0$, it makes no difference.

Applying the duality transformation we get

$$E'_r = \frac{Q_e}{f(\phi)\sigma(r)r^2}\cos\beta \ , \qquad D'_r = \frac{Q_e}{\sigma(r)r^2}\cos\beta \ , \tag{5.3.107}$$

$$B'_r = -\frac{Q_e}{\sigma(r)r^2}\sin\beta \ , \qquad H'_r = -\frac{Q_e}{f(\phi)\sigma(r)r^2}\sin\beta \ . \tag{5.3.108}$$

So any solution in this duality orbit is compatible with the scalar field and metric which are obtained from (5.3.102) and (5.3.103) respectively. Note that we get a purely magnetic solution at $\beta = \pi/2$. In the same paper a solution with

$$V(\phi) = \gamma\left[\sinh(\sqrt{3}\phi) + 9\sinh\left(\frac{\phi}{\sqrt{3}}\right) - 4\sqrt{3}\cosh\left(\frac{\phi}{\sqrt{3}}\right)\right] \ , \tag{5.3.109}$$

was also found for $\alpha = \sqrt{3}$, for which this same process is also applicable.

Another kind of potential that we can consider is a basic self-interaction one. A dyon solution was found in [146] for the Einstein-Maxwell-Axion model where the axion field has a self-interaction. This model has $f = 1$, $g = \alpha a$, $h = 1$ and $V = -\frac{1}{2}m^2a^2$, so its action is

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha a}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu a\partial^\mu a + \frac{1}{2}m^2a^2\right] \ . \tag{5.3.110}$$

The duality orbits are found in a straightforward fashion to be

$$f_\beta = \frac{1}{1+\alpha a\sin\beta(\alpha a\sin\beta + \cos\beta)} \qquad g_\beta = \frac{\alpha a\cos(2\beta) + (\alpha a)^2\sin(2\beta)/2}{1+\alpha a\sin\beta(\alpha a\sin\beta+\cos\beta)} \tag{5.3.111}$$

5.3.3.3 Complex axion field model

Axion boson stars were found in [147], where boson stars were found using a complex scalar field $\psi = |\psi|e^{i\omega t}$ that respected the following QCD axion potential

$$V(|\psi|) = 2k_0 - 2k_0\sqrt{1 - 4k_1\sin^2\Big(k_2|\psi|\Big)} \tag{5.3.112}$$

where k_0, k_1 and k_2 are constants. As mentioned earlier, our model can include complex scalar fields so we can consider a more general version of this complex axion field morel where we also include its coupling to the electromagnetic field. With $\psi = \phi + ia$, $f = 1$, $g = \alpha|\psi| = \alpha\sqrt{\phi^2 + a^2}$ and $h = 1$, we get the following model

$$\begin{aligned}\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Bigg[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha|\psi|}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}\\ &-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi^* - 2k_0 - 2k_0\sqrt{1-4k_1\sin^2\Big(k_2|\psi|\Big)}\ \Bigg] \ . \end{aligned} \tag{5.3.113}$$

which is equivalent to having two axion fields interacting in the same way with the electric field, but also interacting with each other through the scalar potential. The dual orbits of this model are also naturally represented by (5.3.111) by replacing a with $|\psi|$.

5.3.4 Further Generalisations

In here we discuss possible further generalisations of this model.

5.3.4.1 Multiple scalar fields

As the duality transformation does not affect the scalar fields directly we can easily generalise this procedure to a model with multiple scalar fields. Consider the following action with n scalar fields

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{f(\phi_1,...,\phi_n)}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g(\phi_1,...,\phi_n)}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} - K(\phi_1,...,\phi_n) - V(\phi_1,...,\phi_n)\Big] , \tag{5.3.114}$$

where

$$K(\phi_1,...,\phi_n) = \sum_{k=1}^{n}\frac{h_k(\phi_1,...,\phi_n)}{2}\partial_\mu\phi_k\partial^\mu\phi_k \ . \tag{5.3.115}$$

As you can see we can allow for any possible couplings h_k between the fields themselves. The only actual elements that will be altered by a duality transformation in this model are the electromagnetic fields and the couplings f and g, once again transformed according to the duality transformations (5.1.13), (5.1.14) and the relations (5.1.22) and (5.1.23).

5.3.4.2 Multiple Gauge fields

We can also consider various gauge fields, all with their own duality transformation. Consider the following action

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[-\frac{f_{(1)}(\phi,a)}{4}F_{(1)\mu\nu}F^{\mu\nu}_{(1)} + \frac{g_{(1)}(\phi,a)}{4}\tilde{F}_{(1)\mu\nu}F^{\mu\nu}_{(1)} + (...) - \frac{f_{(n)}(\phi,a)}{4}F_{(n)\mu\nu}F^{\mu\nu}_{(n)} + \frac{g_{(n)}(\phi,a)}{4}\tilde{F}_{(n)\mu\nu}F^{\mu\nu}_{(n)} + \mathcal{L}_s(\phi,a)\Big] , \tag{5.3.116}$$

where

$$\mathcal{L}_s(\phi,a) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{h(\phi)}{2}\partial_\mu a\partial^\mu a - V(\phi,a) \ . \tag{5.3.117}$$

We can consider dualities for each group of gauge fields coupled to the scalar fields. This duality transforms the fields as

$$\begin{pmatrix} E_{(n)} \\ H_{(n)} \end{pmatrix} \xrightarrow{\mathcal{D}^{(n)}_\beta} \begin{pmatrix} E'_{(n)} \\ H'_{(n)} \end{pmatrix} = S\begin{pmatrix} E_{(n)} \\ H_{(n)} \end{pmatrix} , \tag{5.3.118}$$

$$\begin{pmatrix} D_{(n)} \\ B_{(n)} \end{pmatrix} \xrightarrow{\mathcal{D}^{(n)}_\beta} \begin{pmatrix} D'_{(n)} \\ B'_{(n)} \end{pmatrix} = S\begin{pmatrix} D_{(n)} \\ B_{(n)} \end{pmatrix} , \tag{5.3.119}$$

where the fields are defined just as in equations (5.3.85)-(5.3.88) just by replacing the corresponding gauge tensor F. Each duality transformation $\mathcal{D}_\beta^{(n)}$ acts on the functions as follows.

$$(f_{(n)}(\phi,a),g_{(n)}(\phi,a)) \xrightarrow{\mathcal{D}_\beta^{(n)}} (f_{(n)\beta}(\phi,a),g_{(n)\beta}(\phi,a)) \ . \tag{5.3.120}$$

So each set of coupling functions $(f_{(n)},g_{(n)})$ will transform along with the gauge fields $F_{(n)}$ they are coupled to.

Now we ask if we can include terms like $F_{(1)\mu\nu}F_{(2)}^{\mu\nu}$ that couple two different gauge fields. Considering the following action

$$\begin{aligned}\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[&-\frac{f_{(1)}(\phi,a)}{4}F_{(1)\mu\nu}F_{(1)}^{\mu\nu} + \frac{g_{(1)}(\phi,a)}{4}\tilde{F}_{(1)\mu\nu}F_{(1)}^{\mu\nu} \\ -\frac{f_{(2)}(\phi,a)}{4}F_{(2)\mu\nu}F_{(2)}^{\mu\nu} + \frac{g_{(2)}(\phi,a)}{4}\tilde{F}_{(2)\mu\nu}F_{(2)}^{\mu\nu} &- \frac{c(\phi,a)}{2}F_{(1)\mu\nu}F_{(2)}^{\mu\nu} + \mathcal{L}_s(\phi,a)\Big] \ ,\end{aligned} \tag{5.3.121}$$

where we have two gauge fields and coupling term $c(\phi,a)F_{(1)\mu\nu}F_{(2)}^{\mu\nu}$ between the two which is also non-minimally coupled to the scalar fields. The Maxwell equations will be

$$\nabla_{[\mu}E_{(1)\nu]} = 0, \tag{5.3.122}$$

$$\nabla_{[\mu}(H_{(1)\nu]} + cB_{(2)\nu]}) = 0, \tag{5.3.123}$$

$$\nabla_\mu\left(\frac{D_{(1)}^\mu + cE_{(2)}^\mu}{V}\right) = 0, \tag{5.3.124}$$

$$\nabla_\mu\left(\frac{B_{(1)}^\mu}{V}\right) = 0, \tag{5.3.125}$$

$$\nabla_{[\mu}E_{(2)\nu]} = 0, \tag{5.3.126}$$

$$\nabla_{[\mu}(H_{(2)\nu]} + cB_{(1)\nu]}) = 0, \tag{5.3.127}$$

$$\nabla_\mu\left(\frac{D_{(2)}^\mu + cE_{(1)}^\mu}{V}\right) = 0, \tag{5.3.128}$$

$$\nabla_\mu\left(\frac{B_{(2)}^\mu}{V}\right) = 0. \tag{5.3.129}$$

We now observe that we can redefine the medium fields H and D as

$$\bar{D}_{(1)}^\mu = D_{(1)}^\mu + cE_{(2)}^\mu \ , \qquad \bar{H}_{(1)}^\mu = H_{(1)}^\mu + cB_{(2)}^\mu \ , \tag{5.3.130}$$

$$\bar{D}_{(2)}^\mu = D_{(2)}^\mu + cE_{(1)}^\mu \ , \qquad \bar{H}_{(2)}^\mu = H_{(2)}^\mu + cB_{(1)}^\mu \ , \tag{5.3.131}$$

to recover the original Maxwell equations. This means we can define a new tensor that we can use to define these fields:

$$\bar{G}_{(1)\mu\nu} = G_{(1)\mu\nu} + c(\phi,a)F_{(2)\mu\nu} \ , \tag{5.3.132}$$

$$\bar{G}_{(2)\mu\nu} = G_{(2)\mu\nu} + c(\phi,a)F_{(1)\mu\nu} \ . \tag{5.3.133}$$

Note that this allows us to write the action (5.3.121) as

$$\mathcal{S} = \mathcal{S}_{EH} + \frac{1}{4\pi}\int d^4x\sqrt{-g}\left[-\frac{1}{4}\bar{G}_{(1)\mu\nu}F^{\mu\nu}_{(1)} - \frac{1}{4}\bar{G}_{(2)\mu\nu}F^{\mu\nu}_{(2)} + \mathcal{L}_s(\phi, a)\right] . \tag{5.3.134}$$

We can do this process to include any term $c(\phi, a)F_{(n)\mu\nu}F^{\mu\nu}_{(m)}$. Then we can define the following duality transformations

$$\begin{pmatrix} E_{(n)} \\ \bar{H}_{(n)} \end{pmatrix} \xrightarrow{\mathcal{D}^{(n)}_{\beta}} \begin{pmatrix} E'_{(n)} \\ \bar{H}'_{(n)} \end{pmatrix} = S\begin{pmatrix} E_{(n)} \\ \bar{H}_{(n)} \end{pmatrix} , \tag{5.3.135}$$

$$\begin{pmatrix} \bar{D}_{(n)} \\ B_{(n)} \end{pmatrix} \xrightarrow{\mathcal{D}^{(n)}_{\beta}} \begin{pmatrix} \bar{D}'_{(n)} \\ B'_{(n)} \end{pmatrix} = S\begin{pmatrix} \bar{D}_{(n)} \\ B_{(n)} \end{pmatrix} , \tag{5.3.136}$$

where $\bar{D}_{(n)}$ and $\bar{H}_{(n)}$ will always have a new terms $cE_{(m)}$ and $cB_{(m)}$ respectively in their definitions for each new coupling $cF_{(n)\mu\nu}F^{\mu\nu}_{(m)}$ for any m. Note that the duality transformations for the n and m terms do not need to be done simultaneously as Maxwell equations are still consistent by considering only one of them.

If we instead add terms involving the hodge dual like $-\frac{c(\phi,a)}{2}F_{(n)\mu\nu}\tilde{F}^{\mu\nu}_{(m)}$, we just need to make use of the identity $F_{(n)\mu\nu}\tilde{F}^{\mu\nu}_{(m)} = \tilde{F}_{(n)\mu\nu}F^{\mu\nu}_{(m)}$ and the new fields are redefined as

$$\bar{D}^{\mu}_{(1)} = D^{\mu}_{(1)} + cB^{\mu}_{(2)} , \qquad \bar{H}^{\mu}_{(1)} = H^{\mu}_{(1)} - cE^{\mu}_{(2)} , \tag{5.3.137}$$

$$\bar{D}^{\mu}_{(2)} = D^{\mu}_{(2)} + cB^{\mu}_{(1)} , \qquad \bar{H}^{\mu}_{(2)} = H^{\mu}_{(2)} - cE^{\mu}_{(1)} , \tag{5.3.138}$$

where the additions of the fields E and B are simply replaced by B and $-E$. The duality transformations can then be applied with these fields.

5.3.4.3 Other considered generalisations

It might be intuitive to think that we could consider this model in dimensions higher than 4, however in that case we can not guarantee that the energy-momentum is the same. This is because to prove that $T'_{\mu\nu} = T_{\mu\nu}$ we require the formula $F_{\mu\alpha}\tilde{F}_{\nu}{}^{\alpha} = \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}\tilde{F}^{\alpha\beta}$ to be valid, something which is not guaranteed in $d > 4$ dimensions as $\tilde{F}$ will not have the same tensor rank as F.

For models with specific couplings there is also the possibility of finding an extra duality between the scalar fields, that then allows us to construct a self duality of the full model. An example of this duality is shown in [71] for the Einstein-Maxwell-Dilaton-Axion model where it was shown that the scalar fields can be transformed into other scalar fields that are combinations the original fields to accomodate the duality transformation, leaving the model invariant. However, the need of specifying the coupling functions makes it so it is not possible to generalise this process for any general coupling.

5.4 Remarks

Understanding the symmetries of any physical theory is always of great importance. Electromagnetic duality is a symmetry of the vacuum Maxwell equations which has led to important insights and generalisations in classical and quantum field theory, as well as in relativistic gravity. In this chapter we have considered EMS models described by the action (5.1.1) for which, in general, electromagnetic duality rotations are not a symmetry of a specific model, but define an orbit in the space of EMS models, which encompasses all possible choices for the coupling functions $f(\phi)$ and $g(\phi)$. This orbit is a one parameter closed orbit. There can be fixed points of the duality action, which are self-dual theories. In our analysis, the only such point corresponds to Maxwell's theory, which, in our setup has $f = 1$ and $g = 0$. For this self-dual theory, the orbit shrinks down to a point.

For self-dual theories, electromagnetic duality relates different solutions of the same theory. For non-self dual theories, electromagnetic duality relates different solutions of different theories. In either case, electromagnetic duality is a useful solution generating technique. In the case considered herein, the duality map generically relates models with different coupling functions and electromagnetic fields, leaving the scalar field and background geometry unchanged.

To illustrate how the duality orbits can be used as a solution generating technique we have considered some simple electrically charged solitonic and black hole solutions, obtaining the corresponding dyons along the duality orbit and, in particular, pure magnetic configurations that emerge at the particular rotation corresponding to $\beta = \pi/2$. In these examples, the models had a vanishing coupling $g(\phi)$; but since $\tilde{F}_{\mu\nu}F^{\mu\nu} = 0$ for these purely electric, spherically symmetric solutions, these are also solutions for any $g(\phi)$ coupling one may choose. A different orbit of solutions exists for each possible $g(\phi)$. We have also obtained a new dyonic black hole solution of the model (5.2.71), which illustrates the usefulness of this technique.

In the last section we discussed the possible avenues where this work can be applied to even more general models. The possibilities of having various fields (gauge vector fields and scalar fields) and couplings between these fields are discussed, creating a formalism that covers a wide range of models.

Chapter 6

Higher dimensional black hole scalarisation

We now come back to the topic of scalarisation that was presented in section 1.5. As we know, the EMS model admits scalarisation of black holes. What we consider here is the scalarisation of various models, including EMS and various closely related models, in higher dimensions.

Section 6.1 is where we introduce a new action and discuss how the various models which admit scalarisation, described in section 1.5, are included in this more general model.

In section 6.2 we address the scalarised electrovacuum BHs in scalar-tensor $d > 4$ models, constructing the zero modes for general d and the scalarised BHs for the simplest coupling function allowing scalarisation in $d = 5$, exhibiting some of their properties. We also address the mapping into the Einstein frame and the relation with Einstein-Maxwell-scalar models.

In section 6.3 we consider higher d extended scalar-tensor theories, where the scalar field non-minimally couples to the appropriate Lovelock density. Again we construct the scalarised BHs for $d = 6, 8$ (besides $d = 4$) and discuss some of their properties. We also compare them with the hairy BHs in shift-symmetric Horndeski models for the same d emphasizing some of the differences between the two models. We conclude with a summary and discussion in section 6.4.

6.1 Model and Recontextualisation

As discussed in chapter 1 (section 1.5), spontaneous scalarisation triggered by strong gravity effects emerges in some classes of scalar-tensor models. This phenomenon could provide a smoking gun for scalar-tensor theories and may be interpreted as a strong gravity phase transition. Here, for concreteness, we shall be considering models described by the generic

d-dimensional action

$$\mathcal{S} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}\Big\{[1-\alpha_{\rm st}f_{\rm st}(\phi)]R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ + \alpha_{\rm L}f_{\rm L}(\phi)\mathcal{L}_{(p)} - \alpha_{\rm emg}f_{\rm emg}(\phi)F_{\mu\nu}F^{\mu\nu} - \mathcal{L}_{\rm mat}\Big\} \ , \qquad (6.1.1)$$

where $\mathcal{L}_{\rm mat}$ is an unspecified matter Lagrangian and $\mathcal{L}_{(p)}$ is the $p^{\rm th}$ Euler density as defined in (1.5.18).

The functions $f_{\rm st}(\phi), f_{\rm L}(\phi)$ and $f_{\rm emg}(\phi)$ are three unspecified non-minimal coupling functions[1] that, when appropriately chosen, lead to spontaneous scalarisation, in each case triggered by a different source; the strength of each of the effects is controlled by the coupling *constants*, $\alpha_{\rm st}, \alpha_{\rm L}, \alpha_{\rm emg}$. In geometrised units, $\alpha_{\rm st}$ and $\alpha_{\rm emg}$ are dimensionless, whereas $\alpha_{\rm L}$ has dimensions of length squared.

Now we recontextualise the introduction of scalarisation presented in section 1.5 with regards to this model. The original scalarisation mechanism [91], which was proposed in $d = 4$ scalar-tensor theories, has

$$\alpha_{\rm st} \neq 0 \ , \qquad \alpha_{\rm L} = 0 = \alpha_{\rm emg} \ . \qquad (6.1.2)$$

Scalar-free objects that may become scalarised must have $R \neq 0$. This is the case, *e.g.* of neutron stars, but not of the electrovacuum black holes (BHs), which are immune to scalarisation in this framework.

More recently, scalarisation of vacuum BHs in $d = 4$ *extended* scalar-tensor theory was observed [85–87] with

$$\alpha_{\rm L} \neq 0 \ , \qquad \alpha_{\rm st} = 0 = \alpha_{\rm emg} \ . \qquad (6.1.3)$$

Scalarisation now requires a non-vanishing Gauss-Bonnet (GB) invariant, which holds for vacuum BHs, whose Kretschmann scalar is non-vanishing, despite being Ricci flat[2]. Scalarised solutions have been constructed in these models, but a dynamical study of the full scalarisation process, from the initial trigger around a vacuum BH until the settling into a scalarised BH is still lacking.

As we know, it was observed [8] that scalarisation of $d = 4$ electrovacuum BHs occurs for the EMS model with

$$\alpha_{\rm emg} \neq 0 \ , \qquad \alpha_{\rm st} = 0 = \alpha_{\rm L} \ . \qquad (6.1.4)$$

In this guise of spontaneous scalarisation, the trigger is a non-vanishing Maxwell invariant $F_{\mu\nu}F^{\mu\nu}$ and thus the phenomenon needs no gravity; moreover, in this case it was possible

[1] Note that $f_{\rm emg}(\phi)$ is the coupling function that was the main subject of study in chapters 2 through 5.

[2] Scalarisation should occur also in Einstein-Chern-Simons models with a suitable coupling between the scalar field and the Pontryagin density. No static scalarised solutions, however, have yet been studied, the only case investigated so far being the NUT generalisation of the Schwarzschild BH [148].

to establish dynamically that the scalarisation of electrically charged electrovacuum BHs indeed occurs, when these BHs have sufficiently high charge, and that the evolution settles into the scalarised solutions that can be constructed as stationary states of the field equations [8, 81].

These three guises of spontaneous scalarisation have been considered in $d = 4$. Considering $d \neq 4$ raises interesting questions, which also address the universality of the phenomenon. Firstly, for the $d > 4$ electrovacuum BHs, *i.e.* the higher dimensional generalisations of the Reissner-Nordström (RN) solution (see *e.g.* [149]), $R = 0$ ceases to hold, since classical electromagnetism is only conformally invariant in $d = 4$. Thus, higher dimensional charged BHs can be scalarised in the original scalar-tensor models (6.1.2), with $\mathcal{L}_{\rm mat} = F_{\mu\nu}F^{\mu\nu}$. Here, we shall show this indeed occurs and construct explicit scalarised RN BHs in these models.[3]

Secondly, one may inquire if there is anything special about the scalarisation in $d = 4$ extended scalar-tensor models (6.1.3), or if similar scalarised BHs occur in $d \neq 4$. We shall show that, indeed, the phenomenon is universal, and the properties of the higher dimensional scalarised BHs, using the appropriate Euler density, are similar to the ones of the four dimensional model with the GB term.

Finally, the simultaneous consideration of these three different guises of spontaneous scalarisation raises the following question: models (6.1.2) and (6.1.4) can be mapped into one another (for particular couplings) via a conformal transformation; how does this mapping allow relating scalarised solutions of both models in $d > 4$? Here, we shall provide the explicit mapping and exemplify how information can be extracted from it.

6.2 Scalarised electrovacuum BHs in $d > 4$ scalar-tensor models

6.2.1 The framework

For our first analysis we consider a scalar-tensor model, with the matter Lagrangian describing classical electromagnetism. Thus, we take (6.1.1) with (6.1.2) and $\mathcal{L}_{\rm mat} = F_{\mu\nu}F^{\mu\nu}$. Moreover, we take the simplest coupling function allowing for spontaneous scalarisation:

$$f_{\rm st}(\phi) = \phi^2 \ , \tag{6.2.5}$$

and for ease of notation we drop the subscript label in the coupling constant: $\alpha_{\rm st} \to \alpha$. As such, the action of the model reads

$$\mathcal{S} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}\left\{(1-\alpha\phi^2)R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - F_{\mu\nu}F^{\mu\nu}\right\} ; \tag{6.2.6}$$

[3]Taking into account quantum corrections, electrovacuum BHs can also become scalarised in the original scalar-tensor model [126].

observe that the scalar and electromagetic fields interact only indirectly, via the backreaction on the spacetime metric.

Restricting to spherically symmetric configurations, we consider a metric ansatz in Schwarzschild-like coordinates, together with a scalar field and electric potential which depend on the radial coordinate only,

$$ds^2 = -N(r)\sigma^2(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega_{d-2}^2 \ , \qquad \phi \equiv \phi(r) \ , \qquad A = V(r)dt \ . \quad (6.2.7)$$

The coordinates (r, t) possess the usual meaning and $d\Omega_{d-2}^2$ is the line element on the unit $(d-2)$-sphere. This ansatz results in the following equations (where the "prime" denotes radial derivatives):

$$(d-2)N' - (d-2)(d-3)\frac{(1-N)}{r} + \frac{1}{2}rN\phi'^2 + \frac{2rV'^2}{\sigma^2}$$
$$+ 4\alpha\left\{rN\phi\phi'' + rN\phi'^2 + \frac{1}{2}\phi\phi'\left[rN' + 2(d-2)N\right] + \frac{d-2}{4}\phi^2\left[N' - \frac{d-3}{r}(1-N)\right]\right\} = 0 \ , \quad (6.2.8)$$

$$\sigma' - \frac{r\sigma\phi'^2}{2(d-2)} - \frac{\alpha}{(d-2)}\left\{\phi\sigma'[(d-2)\phi + 2r\phi'] - 2r\sigma(\phi'^2 + \phi\phi'')\right\} = 0 \ , \quad (6.2.9)$$

$$(r^{d-2}N\sigma\phi')' + 2\alpha\phi\left\{r^{d-3}[\sigma(rN'' + (d-2)N') + \sigma'(3rN' + 2(d-2)N) + 2rN\sigma''] - \mathcal{L}_E\right\} = 0 \ , \quad (6.2.10)$$

$$V'' + \left[\frac{(d-2)}{r} - \frac{\sigma'}{\sigma}\right]V' = 0 \ . \quad (6.2.11)$$

These equations can also be derived from the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{E}} + \mathcal{L}_s + \mathcal{L}_{\text{M}} + \mathcal{L}_{\text{R}} \ , \quad (6.2.12)$$

where

$$\mathcal{L}_{\text{E}} = (d-2)r^{d-4}\sigma\left\{rN' + (d-3)(N+1) + 2Nr\frac{\sigma'}{\sigma}\right\} \ , \qquad \mathcal{L}_s = -\frac{1}{2}N\sigma r^{d-2}\phi'^2 \ , \quad (6.2.13)$$

and

$$\mathcal{L}_{\text{M}} = \frac{2r^{d-2}V'^2}{\sigma} \ , \qquad \mathcal{L}_{\text{R}} = -\alpha\phi\left[\phi\mathcal{L}_E + 2r^{d-2}\phi'(\sigma N' + 2N\sigma')\right] \ .$$

The equation for the electric potential possesses the first integral, which, for convenience we write as

$$V' = \frac{(d-3)Q_0\sigma}{r^{d-2}} \ , \quad (6.2.14)$$

where Q_0 is an integration constant fixing the electric charge.

We are interested in BH solutions, with an horizon at $r = r_h > 0$. Restricting to non-extremal configurations, the solutions possess a near horizon expansion with the first terms being

$$
\begin{aligned}
&N(r) = N_1(r - r_h) + \dots \ , \qquad \sigma(r) = \sigma_0 + \sigma_1(r - r_h) + \dots \ , \\
&\phi(r) = \phi_0 + \phi_1(r - r_h) + \dots \ , \qquad V(r) = v_1(r - r_h) + \dots \ ,
\end{aligned} \tag{6.2.15}
$$

which contains two essential parameters ϕ_0 and σ_0 (the remaining ones are determined in terms of these).

The approximate form of the solutions in the far field reads

$$
\begin{aligned}
N(r) = 1 - \frac{m}{r^{d-3}} + \dots \ , \ \phi(r) = \frac{Q_s}{r^{d-3}} + \dots \ , \\
V(r) = V_\infty - \frac{Q_0}{r^{d-3}} + \dots \ , \ \sigma(r) = 1 - \frac{[d - 3 - 4\alpha(2d - 5)]Q_s^2}{4(d - 2)r^{2(d-3)}} + \dots \ .
\end{aligned} \tag{6.2.16}
$$

Apart from m and Q_0, the essential parameters here are V_∞ (the electrostatic potential at infinity) and Q_s (the scalar 'charge'). Thus, the data at infinity is specified by the ADM mass M, electric charge Q, electrostatic potential V_∞ and scalar 'charge' Q_s, which are read off from the far field asymptotics (6.2.16), where the physical M, Q relate to the parameters m, Q_0 as

$$
M = \frac{(d - 2)V_{(d-2)}}{16\pi} m \ , \qquad Q = \sqrt{2(d - 2)(d - 3)} \frac{V_{(d-2)}}{8\pi} Q_0 \ , \tag{6.2.17}
$$

and $V_{(d-2)}$ is the area of a $(d - 2)$-sphere.

The horizon data, on the other hand, is the Hawking temperature T_H and the event horizon area A_H, which are given by

$$
T_H = \frac{(d - 3)}{4\pi r_h} \left(1 - \frac{[(d - 3)Q_0]^2}{r_h^{2(d-3)}} \right) \ , \qquad A_H = V_{(d-2)} r_h^{d-2} \ , \tag{6.2.18}
$$

together with the value at $r = r_h$ of the scalar field $\phi = \phi_0$.

We also define the reduced electric charge, horizon area and temperature as

$$
q \equiv \frac{Q}{M} \sqrt{\frac{d - 2}{2(d - 3)}} \ , \qquad a_H \equiv \frac{A_H}{M^{\frac{d-2}{d-3}}} c_a \ , \qquad t_H \equiv T_H M^{\frac{1}{d-3}} c_t \ , \tag{6.2.19}
$$

where

$$
c_a = \frac{V_{(d-2)}^{\frac{1}{d-3}} (d - 2)^{\frac{d-2}{d-3}}}{(16\pi)^{\frac{d-2}{d-3}}} \ , \qquad c_t = \frac{2^{\frac{2(d-1)}{d-3}} \pi^{\frac{d-2}{d-3}}}{(d - 3)(d - 2)^{\frac{1}{d-3}} V_{(d-2)}^{\frac{1}{d-3}}} \ . \tag{6.2.20}
$$

The scalar-free solution of the model[4] is the RN BH, which is specified by (6.2.7) with $\phi = 0$ and

$$
N(r) = 1 - \frac{m}{r^{d-3}} + \frac{2(d - 3)}{(d - 2)} \frac{Q_0^2}{r^{2(d-3)}} \ , \qquad \sigma(r) = 1 \ , \qquad V(r) = V_\infty - \frac{Q_0}{r^{d-3}} \ . \tag{6.2.21}
$$

[4] Here we follow the conventions used for this solution in [149].

The RN BH possess an (outer) horizon at $r = r_h$, where r_h is the largest (positive) solution of the equation $N(r_h) = 0$. Working in a gauge with $V(r_h) = 0$, the constant V_∞ corresponds to the electrostatic potential at infinity, $V_\infty = \frac{Q_0}{r_h^{d-3}}$. We also remark that for $d > 4$, the RN BH possesses a nonvanishing Ricci scalar:

$$R = \frac{d-4}{d-2}F^2 = -\frac{2(d-3)^2(d-4)Q_0^2}{(d-2)r^{2(d-2)}} \ . \tag{6.2.22}$$

6.2.2 The zero mode for general d

Let us start by treating the scalar field as a small perturbation around the d-dimensional RN background. This will allow us to compute the *zero modes*: linear scalar field bound states that are supported by a discrete set of RN backgrounds. Zero modes define the onset of the scalarisation instability and the bifurcation towards the new family of scalarised BHs.

Restricting to a spherically symmetric scalar field, the equation for ϕ reads

$$(r^{d-2}N\phi')' + \frac{4\alpha(d-3)^2(d-4)}{d-2}\frac{Q_0^2}{r^{d-2}}\phi = 0 \ . \tag{6.2.23}$$

One can see that ϕ-coefficient in the above equation acts as an r-dependent *effective* mass for the perturbations, with the condition $\alpha > 0$ being necessary for a tachyonic mass.

We are interested in solutions of the above equation which are regular for $r \geqslant r_h$ and vanish at infinity. Remarkably, one finds the following exact solution

$$\phi(r) = P_u\left[1 - \frac{2}{1 - \frac{r_h^{2(d-3)}}{\frac{2(d-3)Q_0^2}{d-2}}}\left\{1 - \left(\frac{r_h}{r}\right)^{d-3}\right\}\right] \ , \quad \text{with} \ \ u \equiv \frac{1}{2}\left(-1 + \sqrt{1 - \frac{8\alpha(d-4)}{(d-3)}}\right) \ , \tag{6.2.24}$$

P_u being the Legendre function. One can show that, in general, the function $\phi(r)$ approaches a constant *non-zero* value as $r \to \infty$,

$$\phi(r) \to {}_2F_1\left[\frac{1}{2}\left(1 - \sqrt{1 - \frac{8\alpha(d-4)}{(d-3)}}\right), \frac{1}{2}\left(1 + \sqrt{1 - \frac{8\alpha(d-4)}{(d-3)}}\right), 1; \frac{1}{1 - \frac{r_h^{2(d-3)}}{\frac{2(d-3)Q_0^2}{(d-2)}}}\right] + \mathcal{O}\left(\frac{1}{r}\right) \ . \tag{6.2.25}$$

Thus finding the spherically symmetric zero mode of the RN BH within the model (6.2.6), corresponding to a scalar field that vanishes asymptotically, reduces to a study of the zeros of the hypergeometric function ${}_2F_1$.

The *existence line*, *i.e.* the RN backgrounds that support scalar clouds, correspond to the set of values of $q \sim Q/M$, as a function of α, for which $\phi(r) \to 0$ asymptotically. Such existence lines are illustrated in Fig. 6.1 for $d = 5, 6, 7$. For any $d > 4$, the solutions exist as long as the coupling constant is sufficiently large, *i.e.* for

$$\frac{d-3}{8(d-4)} < \alpha < +\infty \,, \tag{6.2.26}$$

the minimal value corresponding to the $T_H \to 0$ limit of the RN background.

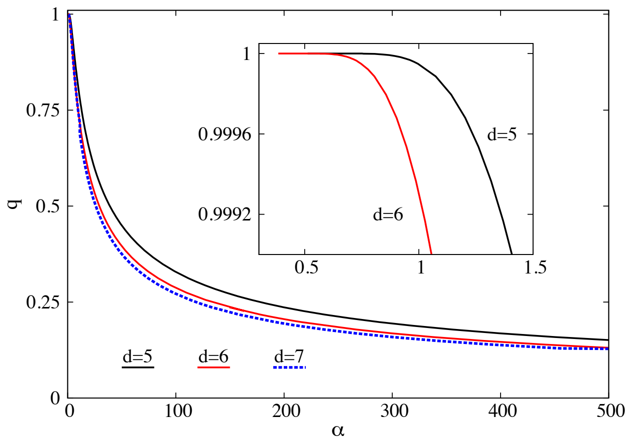


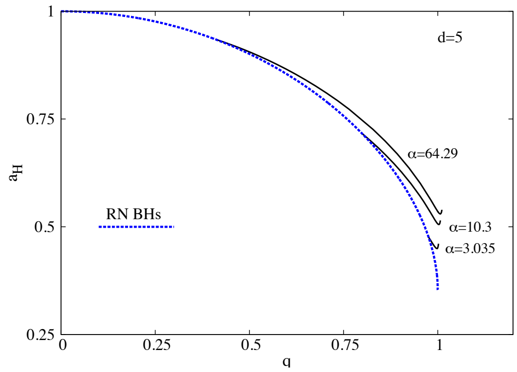
Figure 6.1: Existence lines for the $d = 5, 6, 7$ RN BH in a α *vs.* q diagram. The inset zooms around the minimal value of α.

6.2.3 An explicit construction: scalarised $d = 5$ RN BHs

The scalarised BH solutions obeying the asymptotic behaviours (6.2.15) and (6.2.16) are found numerically, by using a standard ordinary differential equations (ODE) solver. Here we shall report the $d = 5$ case that we have studied more systematically. We have also verified, however, the existence of scalarised solutions for $d = 6$ and we conjecture the existence of such configurations for any $d \geqslant 5$. Moreover, the properties of the five dimensional solutions appear to be generic. Also, only nodeless solutions (in the scalar field) were studied so far, corresponding to the fundamental states; but solutions with nodes should also exist, corresponding to excited states.

The basic properties of the $d = 5$ scalarised RN BHs can be summarised as follows. Given a value of the coupling constant α, the spherically symmetric BHs bifurcate from the RN solution supporting the corresponding scalar cloud, as discussed in the previous subsection. Keeping constant the parameter α, this branch has a finite extent, ending in a critical configuration. This limiting solution appears to be singular, as found when evaluating the Kretschmann scalar at the horizon, although its horizon area and global charges remain finite. This is illustrated[5] in Fig. 6.2, wherein the reduced charge *vs.*

[5] We emphasize that all numerical results in this work were found by solving a time independent problem. As such, the sequences of solutions in Fig. 6.2 should not be interpreted as dynamical evolutionary sequences.



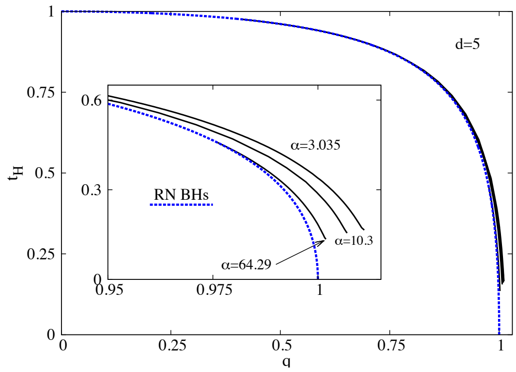


Figure 6.2: Sequences of scalarised $d = 5$ RN BHs, with several values of α, in a charge *vs.* horizon area (left panel) and a charge *vs.* Hawking temperature (right panel) diagram. The quantities are the reduced ones, *i.e.* given in units of mass.

horizon area (left panel) and *vs.* Hawking temperature (right panel) diagrams are exhibited, normalized *w.r.t.* the mass, for several values of the coupling constant α. As these constant α sequences of scalarised BHs emerge from RN BHs, the ratio $q \sim Q/M$ increases and becomes slightly larger than one, in a region close to the critical configuration where the sequence ends. In this sense, the scalarised BHs can be *overcharged*, that is, they can support more charge to mass ratio than RN BHs. To summarize, in an (α, q)-diagram, the domain of existence of the scalarised solutions is delimited by two curves: i) the existence line (RN BHs) and ii) the critical line, which is the set of all critical solutions discussed above.

6.2.4 Einstein frame picture and the relation to Einstein-Maxwell scalar models

The model (6.2.6) is formulated in the so called *Jordan frame*, wherein the scalar field is non-minimally coupled to the Ricci scalar. But it possesses an equivalent formulation in the *Einstein frame*, with a minimally coupled scalar field to the Ricci scalar but non-minimally coupled to the Maxwell invariant. That is, performing the conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^{\frac{4}{d-2}} g_{\mu\nu} \ , \qquad \Omega^2 = 1 - \alpha\phi^2 \ , \tag{6.2.27}$$

together with a redefinition of the scalar field

$$d\psi = \frac{\sqrt{1 - \alpha[1 - \frac{8\alpha(d-1)}{d-2}]\phi^2}}{1 - \alpha\phi^2} d\phi \ , \tag{6.2.28}$$

transforms (6.2.6) into the Einstein frame action (see Appendix B)

$$\mathcal{S} = \frac{1}{16\pi} \int d^d x \sqrt{-\bar{g}} \left[\bar{R} - \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - f(\psi) \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \right] \ , \tag{6.2.29}$$

with the coupling function

$$f(\psi) = \Omega^{-\frac{2(d-4)}{d-2}}(\psi) = \left(1-\alpha\phi^2(\psi)\right)^{-\frac{d-4}{d-2}} \ . \tag{6.2.30}$$

The new, Einstein frame, variables are the metric $\bar{g}_{\mu\nu}$ and the scalar field ψ. The transformation given by eqs. (6.2.27) and (6.2.28) therefore maps a solution of the field equations (6.2.8)-(6.2.11), to a solution that extremizes (6.2.29). The transformation is independent of any assumption of symmetry, and in this sense is covariant; one can easily infer that the transformation is one-to-one in general.

This transformation leads to an interesting twist: in the Einstein frame, the spontaneous scalarisation of electrovacuum BHs results from the nonstandard coupling of the new scalar field ψ to the Maxwell term (notice the analogy with the case in the recent work [8]). That is, the scalarised solutions of the scalar-tensor model can be interpreted as scalarised solutions of an Einstein-Maxwell-scalar model.

One can use this mapping to extract information about scalarisation (or lack thereof) of the corresponding Einstein-Maxwell-scalar model. The Einstein-frame scalar field, as resulting from (6.2.28) reads

$$\begin{aligned}\psi = &-\frac{\sqrt{8(d-1)(\alpha-\alpha_c)}}{\sqrt{(d-2)\alpha}}\operatorname{arcsinh}\left(\sqrt{\frac{8(d-1)\alpha(\alpha-\alpha_c)}{d-2}}\phi\right) \\ &+\frac{2\sqrt{2(d-1)}}{\sqrt{(d-2)}}\operatorname{arctanh}\left(\frac{2\sqrt{2(d-1)}\alpha\phi}{\sqrt{d-2+8(d-1)(\alpha-\alpha_c)\alpha\phi^2}}\right) \ ,\end{aligned} \tag{6.2.31}$$

with

$$\alpha_c \equiv \frac{1}{8}\frac{d-2}{d-1} \ . \tag{6.2.32}$$

α_c is a special value of α corresponding to a d-dimensional *conformally coupled* scalar field in the Jordan frame. Choosing $\alpha = \alpha_c$, the coupling function to the Maxwell invariant in the Einstein frame can be computed in closed form, yielding

$$f(\psi) = \cosh^{\frac{2(d-4)}{d-2}}\left(\frac{1}{2}\sqrt{\frac{d-2}{2(d-1)}}\psi\right) \ . \tag{6.2.33}$$

Unfortunately, it is simple to verify that the value $\alpha = \alpha_c$ of the coupling constant does *not* obey (6.2.26). Thus, a conformally coupled scalar will not allow the scalarisation of the RN BH in the scalar-tensor model.

However, the scalarisation becomes possible for large enough values of α. In fact, as long as $\Omega^2 > 0$ all solutions of the initial model (6.2.6) are mapped to BHs of the Einstein frame model (6.2.29). The corresponding expression of the coupling function results by inverting (6.2.31) and replacing in (6.2.30). Although $f(\psi)$ cannot be found in closed form (unless $\alpha = \alpha_c$), its expression for a small enough scalar field reads

$$f(\psi) \simeq 1 + \beta\psi^2 + \mathcal{O}(\psi^4) \ , \qquad \text{where} \quad \beta \equiv \frac{\alpha(d-4)}{d-2} \ . \tag{6.2.34}$$

One remarks that $d = 4$ scalarised RN BHs with the above form of the coupling function have been studied in [81, 105], and they capture the basic properties of the generic case.

6.3 Scalarised vacuum BHs in $d > 4$ extended scalar-tensor models

6.3.1 The framework

For our second sub-class of models we consider an *extended* scalar-tensor model. Thus, we take (6.1.1) with (6.1.3) and $\mathcal{L}_{\rm mat} = 0$. The explicit expressions of the first terms in the hierarchy of $\mathcal{L}_{(p)}$ are

$$\mathcal{L}_{(0)} = 1 \ , \qquad \mathcal{L}_{(1)} = R \ , \qquad \mathcal{L}_{(2)} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ , \tag{6.3.35}$$

$$\begin{aligned}\mathcal{L}_{(3)} = {} & R^3 - 12RR_{\mu\nu}R^{\mu\nu} + 16R_{\mu\nu}R^{\mu}{}_{\rho}R^{\nu\rho} + 24R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} + 3RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ & - 24R_{\mu\nu}R^{\mu}{}_{\rho\sigma\kappa}R^{\nu\rho\sigma\kappa} + 4R_{\mu\nu\rho\sigma}R^{\mu\nu\eta\zeta}R^{\rho\sigma}{}_{\eta\zeta} - 8R_{\mu\rho\nu\sigma}R^{\mu}{}_{\eta}{}^{\nu}{}_{\zeta}R^{\rho\eta\sigma\zeta} \ .\end{aligned} \tag{6.3.36}$$

In constructing higher dimensional generalisations of the scalarised BHs in [85–87], we use the observation that, in even dimensions, the contribution to the action of the $d/2$-th order $\mathcal{L}_{(p)}$ becomes a topological invariant, and alone does not contribute to the field equations. This ceases to be the case when a nontrivial coupling function, $f(\phi)$ is present: the term $\mathcal{L}_{(d/2)}$ becomes dynamical. As an example, for $d = 4$ one takes $p = 2$ (*i.e.* the GB term) and the geometrical scalarisation model in [85–87] is recovered.

In what follows, we investigate solutions of the model (6.1.1) with (6.1.3), $\mathcal{L}_{\rm mat} = 0$ and

$$d = 2p \ , \qquad \text{where} \qquad p \geqslant 2 \ , \tag{6.3.37}$$

and show that the properties of the four dimensional solutions are generic. As in the previous section, for ease of notation we drop the subscript label in the coupling constant: $\alpha_{\rm L} \to \alpha$. Thus, the considered action reads

$$\mathcal{S} = -\frac{1}{16\pi}\int d^{2p}x\sqrt{-g}\left\{R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha f(\phi)\mathcal{L}_{(p)}\right\} \ . \tag{6.3.38}$$

6.3.2 The equations of motion and general results

In obtaing the equations of motion it is useful to observe that, for general p, the variation of the Euler density term is (see Appendix C)

$$\frac{\delta(f(\phi)\mathcal{L}_{(p)})}{\delta g^{\mu\nu}} = -2pP^{(p)}_{\mu\rho\nu\beta}\nabla^\rho\nabla^\beta f(\phi) \ , \tag{6.3.39}$$

where the $P^{(p)}_{\mu\nu\alpha\beta}$ tensor is naturally defined in $2p$ dimensions using the Levi-Civita tensor in that dimension:

$$P^{(p)\mu\nu\alpha\beta} = -\frac{1}{2^p}\epsilon^{\mu\nu\mu_1\nu_1\ldots\mu_{p-1}\nu_{p-1}}\epsilon^{\alpha\beta\alpha_1\beta_1\ldots\alpha_{p-1}\beta_{p-1}}R_{\mu_1\nu_1\alpha_1\beta_1}\ldots R_{\mu_{p-1}\nu_{p-1}\alpha_{p-1}\beta_{p-1}} \ . \tag{6.3.40}$$

We remark that this tensor shares some of the symmetries and properties of the Riemann tensor:

$$P^{(p)}_{\mu\nu\alpha\beta} = -P^{(p)}_{\nu\mu\alpha\beta} = -P^{(p)}_{\mu\nu\beta\alpha} \ , \qquad P^{(p)}_{\mu\nu\alpha\beta} = P^{(p)}_{\alpha\beta\mu\nu} \ , \qquad \nabla^{\mu} P^{(p)}_{\mu\nu\alpha\beta} = 0 \ . \tag{6.3.41}$$

Taking the same ansatz as before for the metric and scalar field (6.2.7), a straightforward (but cumbersome) computation leads to the following equations for the metric functions and the scalar field[6]

$$\begin{aligned}(d-2)N' - (d-2)(d-3)\frac{(1-N)}{r} + \frac{r}{2}N\phi'^2 + \frac{2\alpha}{r^{d-3}}(1-N)^{\frac{d-4}{2}} \\ \times \left[(1-N)Nf''(\phi) - \frac{1}{2}\left\{(d-1)N-1\right\}N'f'(\phi)\right] = 0 \ ,\end{aligned} \tag{6.3.42}$$

$$(d-2)\sigma' - \frac{r}{2}\sigma\phi'^2 + \frac{\alpha}{r^{d-3}}(1-N)^{\frac{d-4}{2}}\left[(1-N)\sigma f''(\phi) + \left\{(d-1)N-1\right\}\sigma' f'(\phi)\right] = 0 \ , \tag{6.3.43}$$

$$(N\sigma r^{d-2}\phi')' - \alpha\frac{df(\phi)}{d\phi}\left\{(1-N)^{\frac{1}{2}(d-2)}\left(\sigma N' + 2N\sigma'\right)\right\}' = 0 \ . \tag{6.3.44}$$

These equations can also be derived from the effective Lagrangian[7]

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{E}} + \mathcal{L}_s + \alpha\mathcal{L}_{(p)} \ , \tag{6.3.45}$$

with $\mathcal{L}_{\text{E}}$ and $\mathcal{L}_s$ given by (6.2.13) and

$$\mathcal{L}_{(p)} = \frac{dT_{(p)}}{dr} \ , \qquad T_{(p)} \equiv -(1-N)^{\frac{1}{2}(d-2)}\left(\sigma N' + 2N\sigma'\right) \ . \tag{6.3.46}$$

As in the last section, we are interested in BH solutions, with a horizon at $r = r_h > 0$. Restricting to non-extremal configurations, the near horizon expansion of the solutions reads

$$\begin{aligned}N(r) = N_1(r-r_h) + \dots \ , \qquad \sigma(r) = \sigma_h + \sigma_1(r-r_h) + \dots \ , \\ \phi(r) = \phi_h + \phi_1(r-r_h) + \dots \ .\end{aligned} \tag{6.3.47}$$

All coefficient are determined by the essential parameters r_h, $\phi(r_h)$ and $\sigma(r_h)$; for example, one finds

$$N_1 = \frac{(d-2)(d-3)}{r_h[d - 2_{\alpha}\phi_1 f'(\phi_h)]} \ . \tag{6.3.48}$$

The coefficient $\phi'(r_h)$ satisfies a second order algebraic equation of the form

$$\phi_1^2 + p\phi_1 + q = 0 \ , \tag{6.3.49}$$

[6] There is yet another second order equation, which, however, can be expressed as a linear combination of Eqs. (6.3.42), (6.3.43) and their first derivatives, together with (6.3.44).

[7] Here, as well as in the equations (6.3.42)-(6.3.44), to simplify the relation, we have absorbed in α a factor of $\frac{1}{2}(d-2)!$.

where (p,q) are non-trivial functions of r_h, ϕ_h. Consequently, a real solution for ϕ_1 of (6.3.49) exists only if $\Delta = p^2 - 4q \geqslant 0$, a condition which translates into the following inequality

$$1 - \left(\frac{\alpha}{r_h^{d-2}}\right)^2 \frac{4(d-1)(d-3)}{d-2} \left(\left.\frac{df(\phi)}{d\phi}\right|_{\phi_h}\right)^2 \left\{1 - \left(\frac{\alpha}{r_h^{d-2}}\right)^2 \frac{d(d-4)}{4(d-2)} \left(\left.\frac{df(\phi)}{d\phi}\right|_{\phi_h}\right)^2\right\} > 0 \ , \tag{6.3.50}$$

which implies the existence of a minimal horizon size, denoted as $r_h^{(\mathrm{min})}$, determined by α and the value of the scalar field at the horizon.

The far field expansion of the solutions reads

$$N(r) = 1 - \frac{m}{r^{d-3}} + \dots \ , \qquad \sigma(r) = 1 - \frac{(d-3)}{4(d-2)} \frac{Q_s^2}{r^{2(d-3)}} + \dots \ , \qquad \phi(r) = \frac{Q_s}{r^{d-3}} + \dots \ , \tag{6.3.51}$$

in terms of two constants: the scalar 'charge', Q_s, and m, which fixes the ADM mass M as in (6.2.17).

The horizon data, corresponding to the Hawking temperature and horizon area are still given by (6.2.18) (with vanishing electric charge). In terms of all these quantities, the solutions satisfy the Smarr-like relation:

$$M = \frac{(d-2)}{(d-3)} T_H S + M_{(\phi)} \ , \tag{6.3.52}$$

where S is the BH entropy as computed from Wald's formula [152]

$$S = S_{\mathrm{EH}} + S_{(p)} \ , \qquad S_{\mathrm{EH}} = \frac{1}{4} A_H \ , \qquad S_{(p)} = \frac{1}{4} \alpha V_{d-2} f(\phi(r_h)) \ , \tag{6.3.53}$$

and $M_{(\phi)}$ is the mass stored in the scalar field

$$M_{(\phi)} = -\frac{d-2}{d-3} \frac{1}{16\pi} \int_\Sigma d^{d-1}x \sqrt{-g} \frac{f(\phi)}{\dot{f}(\phi)} \Box \phi \ , \tag{6.3.54}$$

where the integral is taken over a spacelike surface Σ and $\dot{f} \equiv df/d\phi$.

We define the *reduced* horizon area and Hawking temperature as in (6.2.19) by normalising the corresponding quantities *w.r.t.* the total mass of solutions. Analogously, the reduced entropy is defined as:

$$s \equiv \frac{4S}{M^{\frac{d-2}{d-3}}} c_a \ , \tag{6.3.55}$$

where c_a is given by (6.2.20)

The scalar-free solution in this model is the Schwarzschild-Tangherlini BH [150], which has a vanishing scalar field, $m = r_h^{d-3}$, $\sigma = 1$, while its reduced quantities are simply $a_H = s = t_H = 1$.

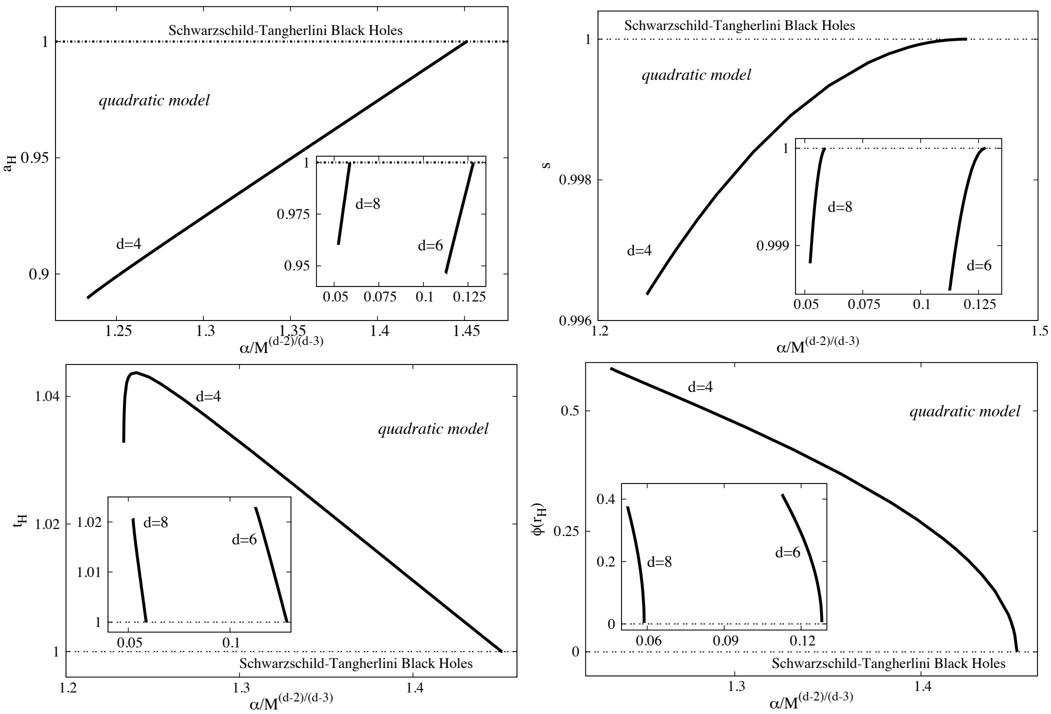


Figure 6.3: Reduced area (top left panel), reduced entropy (top right panel), reduced temperature (bottom left panel) and the scalar field at the horizon (bottom right panel) *vs.* the coupling (normalized by the mass) for scalarised BHs in the extended scalar-tensor model in $d = 4, 6, 8$ dimensions.

6.3.3 The scalarised BHs in $d = 4, 6, 8$ with a quadratic coupling

As in the previous section we shall illustrate the BHs in the d-dimensional extended scalar-tensor models by considering the simplest function which satisfies the condition (1.5.24):

$$f_{\rm L}(\phi) = \phi^2 \ , \tag{6.3.56}$$

which was initially considered for $d = 4$ solutions in [86]. The numerical construction of the solutions in $d = 4, 6, 8$ follows a strategy similar to one used in the last section. Some properties of the solutions are shown in Fig. 6.3 and can be summarised as follows. Firstly, the qualitative features of the $d = 4$ solutions still hold in higher d, namely: (i) the branching off from the Schwarzschild-Tangherlini BH wherein the latter supports a scalar cloud; (ii) the limited range wherein solutions exist; (iii) and the trends of the different quantities when α is varied. Quantitatively, however, one can see a smaller domain of existence in terms of α in higher dimensions, likely due to the faster fall-off of the gravitational interaction. Secondly, the model possesses a (presumably infinite) tower of scalarised spherically symmetric solutions which are indexed by the number of nodes n of the scalar field. As in the previous study of the scalar-tensor model, here we are focusing on the fundamental $n = 0$ solutions. Thirdly, all solutions can be obtained continuously in the parameter space: they form a line, starting from the smooth GR limit ($\phi \to 0$), and ending at some limiting solution. Once the limiting configuration is reached, the solutions cease to exist in the parameter space. The existence of these 'critical' configurations can be understood from the condition (6.3.50), with the determinant Δ vanishing at that point.

It would be interesting to study the linear stability of these scalarised BHs. Since the $d = 4$ solutions are radially unstable [90, 97], it is possible that their higher dimensional generalisations are also unstable.

6.3.4 A linear coupling detour: the shift-symmetric model in d-dimensions

If instead of the choice (6.3.56) one chooses the coupling function

$$f_{\mathrm{L}}(\phi) = \phi \ , \tag{6.3.57}$$

the scalarisation condition (1.5.24) is not obeyed. This case corresponds to a linear coupling or a 'shift symmetric' model, which is interesting for different reasons and has been extensively studied for $d = 4$ - see *e.g.* [153–156]. Although scalarisation is absent, the model possesses a variety of interesting properties. Here, we shall use it to contrast with the picture found for the quadratic coupling in the previous subsection.

Since the condition (1.5.24) is not satisfied in the linear model (6.3.57) for $\alpha \neq 0$, the Schwarzschild-Tangerlini BH is not a solution. Also, the equations of motion are invariant under the transformation

$$\phi \to \phi + \phi_0 \ , \tag{6.3.58}$$

with ϕ_0 an arbitrary constant, which results from the fact that the $\mathcal{L}_{(p)}$ term alone is a total divergence. This implies the existence of a current, whose conservation leads to the following interesting relation between the 'scalar charge' and the Hawking temperature

$$Q_s = \frac{4\pi\alpha}{(d-3)} T_H \ , \tag{6.3.59}$$

which is a unique property of this class of models (see also the discussion in [157] for $d = 4$ and [158] for the issue of BH temperature in Horndeski models).

In the probe limit, that is considering the scalar field equation of the model on the Schwarzschild-Tangherlini background, we find the following general exact solution[8] valid for all range of r:

$$\phi(r) = \frac{\alpha}{r_h^{d-2}} \left\{ B\left[\left(\frac{r_h}{r}\right)^{d-3}; \frac{d^2-d-4}{2(d-3)}, 0 \right] + \log\left(1 - \left[\frac{r_h}{r}\right]^{d-3}\right) \right\} \ , \tag{6.3.60}$$

[8]Note that a constant of integration has been fixed in the expression by imposing $\phi(r) \to 0$ as $r \to \infty$

where $B[x;a,b]$ is the incomplete β-function. Simple expressions exist for $d = 4, 6$ only (with $x \equiv r_h/r$):

$$d=4: \qquad \phi(r) = \frac{\alpha}{r_h^2}\left(x + \frac{x^2}{2} + \frac{x^3}{3}\right) , \tag{6.3.61}$$

$$d=6: \qquad \phi(r) = \frac{3\alpha}{r_h^4}\left[x + \frac{x^4}{4} + \frac{x^7}{7} + \frac{x^{10}}{10} - \frac{1}{2}\log(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}x}{2+x}\right)\right] . \tag{6.3.62}$$

In principle, this solution can be used to construct a closed form perturbative solution as a power series in the parameter α/r_h^{d-3}, see *e.g.* the $d = 4$ results in [156]. As discussed therein, this analytical solution provides a good approximation to the numerical results.

A feature which, however, cannot be captured within a perturbative approach is the existence of a minimal horizon size. The condition (6.3.50) on the near horizon data takes a simple form for the choice (6.3.57) of the coupling function, with

$$\frac{\alpha}{r_h^{d-2}} < \left[\frac{2(d-1)(d-3)}{(d-2)} + \sqrt{3(d-1)(d-3)}\right]^{-1/2} . \tag{6.3.63}$$

This requirement translates into a coordinate independent condition imposing a minimal size for the horizon size in terms of the coupling constant α only,

$$A_H > c_0\alpha , \qquad \text{where} \qquad c_0 \equiv V_{(d-2)}\sqrt{\frac{2(d-1)(d-3)}{(d-2)} + \sqrt{3(d-1)(d-3)}} . \tag{6.3.64}$$

Some results of the numerical integration for non-perturbative solutions are shown in Fig. 6.4. Again, the solutions stop existing at the point where the condition (6.3.63) fails to be satisfied.

6.4 Summary and overview

The main purpose of this chapter was to discuss higher dimensional generalisations of $d = 4$ spontaneous scalarisations models, in its various guises, via the existence of the corresponding scalarised BH solutions. As the broader take-home message, the study herein shows the phenomenon of 'spontaneous scalarisation' is not peculiar to $d = 4$, but qualitative and quantitative differences occur in higher d.[9]

Concerning the case of the scalar-tensor model studied in section 6.2, we have established that, since the conformal invariance of the Maxwell action is lost in $d > 4$, the higher dimensional electrovacuum BHs possess scalarised generalisations in these models. This is a qualitative difference with respect to the $d = 4$ case. Moreover, by a conformal mapping,

[9]In various aspects $d = 4$ BH physics has unique properties; recent research has revealed that as d increases, the BH's phase structure becomes increasingly intricate and diverse [151]. This further motivates the analysis herein.

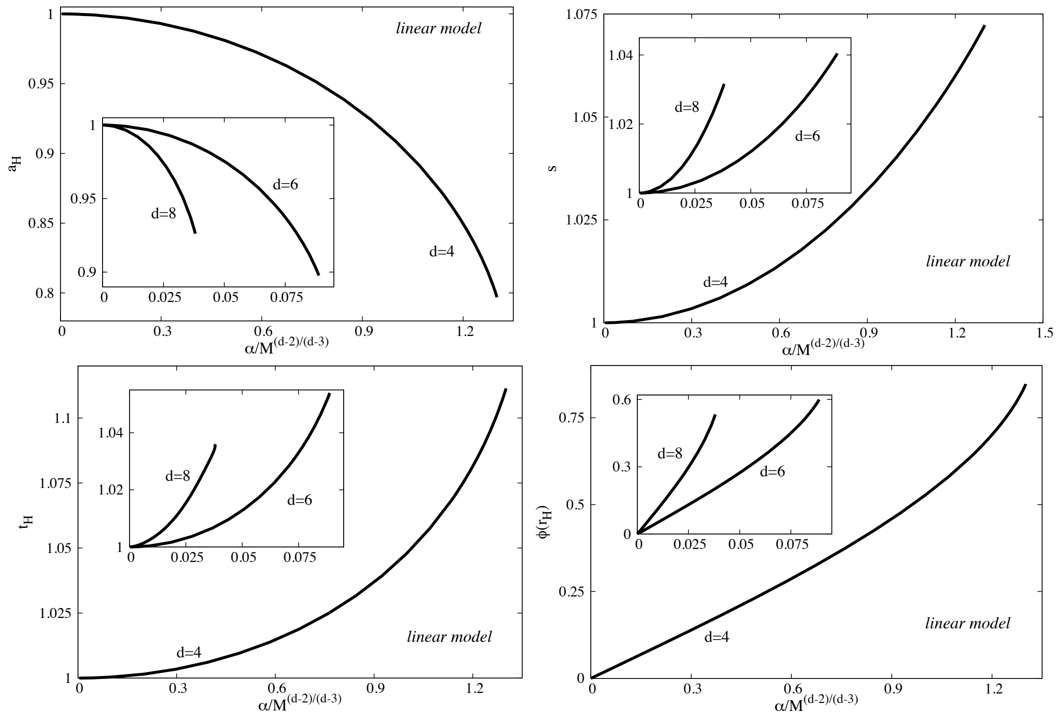


Figure 6.4: Reduced area (top left panel), reduced entropy (top right panel), reduced temperature (bottom left panel) and the scalar field at the horizon (bottom right panel) for BHs in the shift symmetric model in $d = 4, 6, 8$ spacetime dimensions.

these solutions can be interpreted as Einstein-Maxwell-scalar solutions, bridging between these two guises of scalarisation.

Concerning the case of the extended scalar-tensor model studied in section 6.3, our construction generalised the 'geometric scalarisation' in [85–87] to any *even* dimension. In $d = 4$, Einstein's gravity can be deduced by assuming general coordinate covariance and the absence of higher derivative terms larger than the second order in the Lagrangian. In $d > 4$, the same assumptions lead to Lovelock gravity [88]. All Euler densities, $\mathcal{L}_{(p)}$, starting with the Ricci scalar and the Gauss-Bonnet curvature squared combination, can be written as the divergences of genuine vector densities in the critical dimensions $d = 2p$, with $p = 1, 2, \ldots$ (while they vanish for $d < 2p$). However, such a density can be made dynamical by coupling it to a scalar field, which results in the term $\alpha_{\rm L} f_{\rm L}(\phi)\mathcal{L}_{(p)}$ in the action (6.1.1). Thus, there is a hierarchy of models, with the $d = 4$ ($p = 2$) case in [85–87] being a special case. Here, we have found that the properties of the solutions of the latter are generic, being shared by the higher dimensional $d = 2p$ counterparts, but with quantitative differences.

As to provide a comparative benchmark, we have also generalised the $d = 4$ 'shift symmetric' Horndeski model in [153–155] to any $d = 2p \geqslant 4$ even dimension. Again, the properties of the $d = 4$ solutions are generic. Although these configurations do not qualify for scalarised BHs (in particular the condition (1.5.24) is not satisfied), they possess a variety of interesting properties (*e.g.* the existence of a conserved current, which implies that the Hawking temperature is fixed by the scalar charge; also an exact solution is found in the probe limit).

All configurations in this work are spherically symmetric and asymptotically flat, being regular on and outside the horizon (which possesses a spherical topology). Rotating generalisations should exist, following the $d = 4$ studies in, *e.g.* [99,156,159].

Let us close this chapter with some remarks concerning the status of the extended scalar-tensor model for the (lower) dimension $d = 2$. Einstein gravity alone is trivial in two dimensions; however, as in the generic $d = 2p$ case, $\mathcal{L}_{(1)} = R$ can contribute to the equations of motion by coupling it with a scalar field. This suggests us to consider the following $d = 2$ version of the generic model (6.1.1):

$$\mathcal{S} = -\frac{1}{16\pi} \int d^2x \sqrt{-g} \left\{ \alpha f(\phi) R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + U(\phi) \right\} \ , \tag{6.4.65}$$

with $U(\phi)$ a scalar potential. Interestingly, this corresponds to the generic form of the Jackiw-Teitelboim gravity [160,161]. This model has received considerable interest recently in connection with BH dynamics (see, *e.g.* [162–165]). Near extremal BHs/branes have a near horizon 'throat region' corresponding to an AdS_2 spacetime [166] and so, upon compactification, the action (6.4.65) appears naturally, with the scalar field representing the modulus associated to the transverse directions (the volume of the transverse sphere). Moreover, it was shown [165] that the Jackiw-Teitelboim model is a good approximation for the low-temperature dynamics and thermodynamics of a large class of spinning/charged BHs, including the near extremal Kerr BH. It would be interesting to study solutions of the model (6.4.65) for various choices of the coupling function, in particular for a $f(\phi)$ allowing for scalarisation.

Chapter 7

Conclusions

In this thesis we have studied various properties of the Einstein-Maxwell-scalar model (1.1.2). The distinguishing property of this model is the non-minimal coupling $f(\phi)$ between the electromagnetic and scalar terms of the action that directly couples these two fields. The main topics of interest in this model that were studied in this thesis include solitonic solutions, duality symmetries and scalarisation. These topics were introduced in chapter 1 and then thoroughly discussed in the following chapters.

Chapter 2 was dedicated to the study of no go theorems in the EMS model. The Einstein-scalar model was studied as a valid truncation of the full EMS model and a no go theorem was shown for asymptotically flat stationary and axisymmetric solitons. A similar result was then shown for asymptotically flat static and strictly stationary solitons in the full EMS model. This last result was discussed once again in chapter 3, where the axion term was added to the EMS model. It was shown in that chapter that the axion term does not change the results considered in chapter 2. These no go theorems can be viewed as important directions of what conditions to avoid if we want to find solitonic solutions in this model.

Following this line of thought, we reach chapter 4 where a solitonic solution to the EMS model is found by circumventing a condition imposed by the theorems of the previous two chapters. The condition circumvented is the finiteness of the non-minimal coupling function $f(\phi)$, meaning that our soliton solution has a diverging coupling, even though every physical field and energy density is regular. The solution is a soliton composed of a pure electric field and a scalar field which is constructed perturbatively and numerically.

In chapter 5, we discuss the duality properties of a generalised EMS model with two non-minimal couplings: $f(\phi)$ coupling the scalar field to the Maxwell term and $g(\phi)$ coupling to the axionic term (discussed in chapter (3)). We then show that we can establish what we denote as "duality orbits": a map between solutions of a model with specific coupling functions $f(\phi)$ and $g(\phi)$ to another model with different coupling functions. These duality transformations preserve the scalar field but change the electric and magnetic fields. Various examples of solutions obtained through this map were presented, including a magnetic variant of the purely electric soliton solution found in the preceding chapter 4. Chapter 5 ends with various examples of possible avenues of generalisation of this procedure.

Lastly, in chapter 6, we discuss the topic of scalarisation. The basic idea of scalarisation was introduced in chapter 1 (section 1.5) and in chapter 6 we present the idea of scalarisation in higher dimensions. Scalarisation of higher dimensional Reissner-Nordström BHs in scalar-tensor models was considered and solutions were explicitly constructed for the $d = 5$ case. A conformal transformation that maps between two different models, (one with a scalar field non-minimally coupled to the Ricci scalar and another with a scalar field non-minimally coupled to the Maxwell term) which have different scalarisation methods is also discussed, relating these two different scalarisation methods. Finally, we consider the non-minimal coupling in even $2p$ dimensions of the scalar field with the p^{th} Euler density. These are extended-scalar-tensor-Lovelock gravity models that present scalarisation of the Schwarzschild-Tangherlini BHs. Examples are constructed for the 6-dimensional and 8-dimensional cases which show that they possess the same qualitative properties of the original 4-dimensional case, albeit with quantitative differences. A comparison is also made with the hairy BHs in shift-symmetric Horndeski theory which are also constructed.

The Einstein-Maxwell-scalar is clearly a very rich model and hopefully the subjects of study in this thesis can help pave the way for more interesting work in this kind of model. Further advances in the theory behind soliton solutions, duality symmetry, or even scalarisation in this model can likely be considered based on the concepts introduced in this thesis.

Appendix A

Asymptotic behavior of the vector W^μ

In this appendix, we want to find the behavior of the vector W^μ, as it was defined in equation (2.3.76).

The integral (2.3.78) assumes that W^μ decays fast enough (with a leading term of r^{-n}, $n > 2$) for it to vanish at the 2-surface $\partial\Sigma$ at infinity. To show this, we start with the fact that the surface $\partial\Sigma$ has two normal vectors: the timelike unit vector and the radial unit vector. As $W^t = 0$, the only component that matters is W^r. To obtain the asymptotical behavior of this component first note that, at infinity we have

$$\frac{1}{V} = -\frac{1}{g_{\mu\nu}k^\mu k^\nu} = -\frac{1}{g_{tt}} \to 1 \tag{A.0.1}$$

because the metric will be approximately Minkowski so $g_{tt}^{-1} \to g^{tt} \to -1$ as the cross terms of the metric will decay to zero. This means that we only need to care about the asymptotic behaviour of the functions $U_E, U_B, E^r, B^r, \varphi, \psi$ and ω^r. As most of these functions are related by their definitions, we just really need to know the behaviors of ω^r and two other functions (one related to the electric field and the other to the magnetic field).

A.1 The twist vector limit

The twist vector ω^μ was defined in (2.3.50) as

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}k_\nu\nabla_\alpha k_\beta \ . \tag{A.1.2}$$

where[1] $\varepsilon^{\mu\nu\alpha\beta} = \sqrt{-g}\epsilon^{\mu\nu\alpha\beta}$ and $\nabla_\alpha k_\beta = \partial_\alpha k_\beta$ as it is antisymmetrized. We only need the radial component ω^r and we assume that the only rotation cross term is $g_{t\varphi}$, so ω^r can be

[1]Note the distinction between the Levi-Civita *tensor* $\varepsilon^{\mu\nu\alpha\beta}$ and the Levi-Civita *symbol* $\epsilon^{\mu\nu\alpha\beta}$.

obtained as

$$\omega^r = \frac{1}{2\sqrt{-g}}\epsilon^{1\nu\alpha\beta}k_\nu\nabla_\alpha k_\beta \tag{A.1.3}$$

$$= \frac{1}{2\sqrt{-g}}\epsilon^{1023}k_0\partial_2 k_3 + \frac{1}{2\sqrt{-g}}\epsilon^{1320}k_3\partial_2 k_0 \tag{A.1.4}$$

$$= -\frac{1}{2\sqrt{-g}}k_t\partial_\theta k_\varphi + \frac{1}{2\sqrt{-g}}k_\varphi\partial_\theta k_t \ . \tag{A.1.5}$$

We have that $k_t = g_{tt}$ and $k_\varphi = g_{\varphi t}$ in the asymptotic limit. As $g_{tt} \to -1$ we have that

$$\omega^r \to -\frac{1}{2r\sqrt{-g}}\partial_\theta g_{\varphi t} + \frac{1}{2r\sqrt{-g}}g_{\varphi t}\partial_\theta g_{tt} \tag{A.1.6}$$

We know that $\sqrt{-g}$ grows as r^2 at infinity. Now we want to know the leading terms of $g_{\varphi t}$ and $\partial_\theta g_{tt}$. As $g_{\varphi t}$ vanishes at infinity, then we can at least assume that it has a leading term r^{-1} and the same thing for $g_{tt}+1$. So we can expand them as:

$$g_{\varphi t} \to \frac{\alpha(\theta)}{r} + \mathcal{O}(r^{-2}) \tag{A.1.7}$$

$$g_{tt} + 1 \to \frac{\beta(\theta)}{r} + \mathcal{O}(r^{-2}) \tag{A.1.8}$$

where α, β, δ are functions of θ whose form is not important in this context. If the leading term has a power of r lower than -1, it will just make the leading term of ω^r decay even faster, so if we can prove that W^r decays fast enough for this kind of behaviour, then we do not need to know which are the leading terms of these metric components. We can consider the example of the Kerr metric in Boyer-Lindquist coordinates:

$$g_{\varphi t} = -\frac{2Mra\sin^2\theta}{r^2 + a^2\cos^2\theta} \tag{A.1.9}$$

$$g_{tt} = -1 + \frac{2Mr}{r^2 + a^2\cos^2\theta} \tag{A.1.10}$$

which as we see can be expanded asymptotically just like in equations (A.1.7) and (A.1.8). So this means that

$$\partial_\theta g_{\varphi t} \to \frac{\alpha'(\theta)}{r} + \mathcal{O}(r^{-2}) \tag{A.1.11}$$

$$\partial_\theta g_{tt} \to \frac{\beta'(\theta)}{r} + \mathcal{O}(r^{-2}) \tag{A.1.12}$$

As $\sqrt{-g} \to \gamma(\theta)r^2$, we get for ω^r

$$\omega^r \to \frac{\alpha'(\theta)}{2\gamma(\theta)r^3} + \mathcal{O}(r^{-2}), \tag{A.1.13}$$

where we can see that the leading term has a r^{-3} decay.

A.2 The W^r term

The behaviour of the rest of the fields that compose W^μ can be deduced by considering that the leading term of the vector potential A^μ is a r^{-1} term. From this we can obtain

$$\varphi \to \frac{\varphi_0(\theta)}{r} + \mathcal{O}(r^{-2}) \qquad \psi \to \frac{\psi_0(\theta)}{r} + \mathcal{O}(r^{-2}) \tag{A.2.14}$$

$$E^r \to \frac{E_0^r(\theta)}{r^2} + \mathcal{O}(r^{-3}) \qquad B^r \to \frac{B_0^r(\theta)}{r^2} + \mathcal{O}(r^{-3}) \tag{A.2.15}$$

$$U_E \to \frac{U_E^0(\theta)}{r^2} + \mathcal{O}(r^{-3}) \qquad U_B \to \frac{U_B^0(\theta)}{r^2} + \mathcal{O}(r^{-3}) \tag{A.2.16}$$

so we finally have

$$W^r = 2(U_E + U_B)\frac{\omega^r}{V^2} - 2\frac{\psi B^r + f\varphi E^r}{V} \to \frac{W_0^r(\theta)}{r^3} + \mathcal{O}(r^{-4}) \tag{A.2.17}$$

As W^r decays faster than r^{-2}, we have that the integral (2.3.78) does vanish at the surface $\partial\Sigma$.

Appendix B

Einstein frame formulation of non-minimally coupled gravity

B.1 The Model

We consider the action described in (6.2.6) in d dimensions

$$\mathcal{S} = -\frac{1}{16\pi}\int d^dx\sqrt{-g}\big[\Omega^2(\phi)R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - F_{\mu\nu}F_{\mu\nu}\big] \ , \tag{B.1.1}$$

where $\Omega^2(\phi) = 1 - \alpha\phi^2$. We want to make the following conformal transformation

$$\bar{g}_{\mu\nu} = \Omega^n g_{\mu\nu}, \tag{B.1.2}$$

where n is a constant. With this transformation, after redefining the scalar field to $\psi(\phi)$, we want to obtain the following Einstein-Maxwell-scalar action

$$\mathcal{S} = -\frac{1}{16\pi}\int d^dx\sqrt{-\bar{g}}\left[\bar{R} - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\psi\partial^\mu\psi - f(\psi)\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}F_{\mu\alpha}F^{\nu\beta}\right] . \tag{B.1.3}$$

B.1.1 The n constant

We know we can write the Ricci scalar R of the metric g with respect to the Ricci scalar $\bar{R}$ of the metric $\bar{g}$ as follows

$$R = \Omega^n\bar{R} + \Omega^n A(\phi, \partial\phi, \partial^2\phi) \ , \tag{B.1.4}$$

where A can be written as

$$A = n(d-1)\bar{\Box}^2\ln\Omega - \frac{n^2}{4}(d-1)(d-2)(\bar{\nabla}\ln\Omega)^2 \ , \tag{B.1.5}$$

where $\bar{\Box}^2 = \bar{\nabla}^2 = \bar{\nabla}_\mu\bar{\nabla}^\mu = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu$. The determinant of the metric $\bar{g}$ can be calculated as

$$g = \Omega^{-dn}\bar{g} \ . \tag{B.1.6}$$

This means that

$$\sqrt{-g}\,\Omega^2 R = \sqrt{-\bar{g}}\,\Omega^{2-\frac{dn}{2}+n}\bar{R} + \dots \,. \tag{B.1.7}$$

Now, to have this term coincide with the new Ricci scalar term in (B.1.3), we need the exponent of Ω in the above expression to vanish. This allows us to obtain the n constant

$$n = \frac{4}{d-2}\,, \tag{B.1.8}$$

which results in the conformal transformation $\bar{g}_{\mu\nu} = \Omega^{\frac{4}{d-2}} g_{\mu\nu},$. Note that $n = 2$ for $d = 4$.

B.1.2 The A function

Using $n = 4/(d-2)$ we can write (B.1.5) as

$$A = 4\frac{d-1}{d-2}\left[\bar{\Box}^2 \ln\Omega - (\bar{\nabla}\ln\Omega)^2\right]\,. \tag{B.1.9}$$

As $\ln\Omega = \frac{1}{2}\ln(1-\alpha\phi^2)$ we can calculate the derivatives as follows

$$\nabla\ln\Omega = -\frac{\alpha\phi\nabla\phi}{1-\alpha\phi^2}\,, \tag{B.1.10}$$

$$\begin{aligned}\bar{\Box}^2\ln\Omega &= -\frac{\alpha[(\bar{\nabla}\phi)^2 + \phi\bar{\Box}^2\phi)]}{1-\alpha\phi^2} - \frac{2\alpha^2\phi^2(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)^2}\\ &= -\alpha(\bar{\nabla}\phi)^2\frac{1+\alpha\phi^2}{(1-\alpha\phi^2)^2} - \alpha\frac{\phi\bar{\Box}^2\phi}{1-\alpha\phi^2}\,.\end{aligned} \tag{B.1.11}$$

Inserting these results in (B.1.9), we get

$$A = \frac{-4\alpha}{1-\alpha\phi^2}\frac{d-1}{d-2}\left[(\bar{\nabla}\phi)^2\frac{1+2\alpha\phi^2}{(1-\alpha\phi^2)} + \alpha\phi\bar{\Box}^2\phi\right]\,. \tag{B.1.12}$$

Now we can isolate the $\phi\bar{\Box}^2\phi$ term multiplied by $\sqrt{-g}$ and rewrite it in terms of $(\bar{\nabla}\phi)^2$ in the following way[1]

$$\begin{aligned}&\sqrt{-\bar{g}}\,\Omega^{\frac{-2d}{d-2}}\Omega^2\Omega^{\frac{4}{d-2}}\left(\frac{-4\alpha^2}{1-\alpha\phi^2}\frac{d-1}{d-2}\phi\bar{\Box}^2\phi\right)\\ =&\sqrt{-\bar{g}}\,\bar{\nabla}^\mu\left(\frac{-4\alpha^2}{1-\alpha\phi^2}\frac{d-1}{d-2}\phi\bar{\nabla}_\mu\phi\right) + \sqrt{-\bar{g}}\,\frac{4\alpha^2}{1-\alpha\phi^2}\frac{d-1}{d-2}\left((\bar{\nabla}\phi)^2 + \frac{2\alpha\phi^2}{1-\alpha\phi^2}(\bar{\nabla}\phi)^2\right)\\ =&\bar{\nabla}^\mu(\dots)_\mu + \sqrt{-\bar{g}}\,\frac{4\alpha^2}{(1-\alpha\phi^2)^2}\frac{d-1}{d-2}(1+\alpha\phi^2)(\bar{\nabla}\phi)^2\,.\end{aligned} \tag{B.1.13}$$

The divergence term will not contribute to the action integral, so we can just ignore it. We now write A as

$$\begin{aligned}A &= -\frac{4\alpha(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)^2}\frac{d-1}{d-2}\left(1+2\alpha\phi^2 - 1 - \alpha\phi^2\right) + \bar{\nabla}^\mu(\dots)_\mu\\ &= -\frac{4\alpha^2\phi^2(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)^2}\frac{d-1}{d-2} + \bar{\nabla}^\mu(\dots)_\mu\,.\end{aligned} \tag{B.1.14}$$

[1]Note that the Ω terms all cancel out.

B.1.3 The scalar kinetic term

The scalar field ϕ kinetic term with respect to the new metric $\bar{g}$ can be written as

$$\begin{aligned}\sqrt{-g}\,\frac{g^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi &= \sqrt{-\bar{g}}\,\Omega^{-2d/(d-2)}\frac{\Omega^{4/(d-2)}\bar{g}^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi \\ =&\sqrt{-\bar{g}}\,\frac{\Omega^{-2}}{2}(\bar{\nabla}\phi)^2 = \sqrt{-\bar{g}}\,\frac{(\bar{\nabla}\phi)^2}{2(1-\alpha\phi^2)}\ .\end{aligned} \tag{B.1.15}$$

With this, we can now rewrite the gravitational and scalar part of the action as

$$\begin{aligned}&\int d^dx\sqrt{-g}\big[\Omega^2(\phi)R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\big] \\ =&\int d^dx\sqrt{-\bar{g}}\Bigg[\bar{R} - \frac{4\alpha^2\phi^2(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)^2}\frac{d-1}{d-2} - \frac{1}{2}\frac{(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)} + \bar{\nabla}^\mu(...)_\mu\Bigg] \\ =&\int d^dx\sqrt{-\bar{g}}\Bigg[\bar{R} - \frac{1}{2}\frac{(\bar{\nabla}\phi)^2}{(1-\alpha\phi^2)^2}\left(1-\alpha\phi^2+8\alpha^2\phi^2\frac{d-1}{d-2}\right)\Bigg]\ .\end{aligned} \tag{B.1.16}$$

Now we use the following redefinition of the scalar field

$$d\psi = \frac{\sqrt{1-\alpha[1-8\alpha\phi^2(d-1)/(d-2)]\phi^2}}{1-\alpha\phi^2}d\phi\ , \tag{B.1.17}$$

to get

$$\int d^dx\sqrt{-\bar{g}}\left[\bar{R} - \frac{1}{2}(\bar{\nabla}\psi)^2\right]\ , \tag{B.1.18}$$

which is the expected form of the action presented in (B.1.3).

B.1.4 The Maxwell term

Now we consider the Maxwell term. We easily see that

$$\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} = \sqrt{-g}\,g^{\mu\alpha}g^{\nu\beta}F_{\mu\alpha}F^{\nu\beta} = \sqrt{-\bar{g}}\,\Omega^{-2d/(d-2)}\Omega^{8/(d-2)}\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}F_{\mu\alpha}F^{\nu\beta}\ . \tag{B.1.19}$$

This is the exact same Maxwell term as in (B.1.3) with

$$f(\psi) = \Omega^{\frac{8-2d}{d-2}}(\psi) = [1-\alpha\phi^2(\psi)]^{\frac{4-d}{d-2}}\ . \tag{B.1.20}$$

Adding this term inside the integral of (B.1.18), we recover (B.1.3) as expected.

Appendix C

Variation of the Euler density term for general dimension

C.1 The Model

In here we want to calculate the contribution of the general Euler density to the equations of motion when non-minimally coupled to a scalar field. We will first calculate the results for dimensions 4 and 6 before generalising.

Consider the action

$$\mathcal{S} = \frac{1}{4\pi}\int d^4x\sqrt{-g}\Big[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)\mathcal{L}_{(p)}\Big] \ , \tag{C.1.1}$$

where $\mathcal{L}_{(p)}$ is the p^{th} Euler density contribution which can be defined for dimension $2p$ as in (1.5.18). This term is non-minimally coupled to the scalar field through the function $f(\phi)$. The variation of these terms would usually vanish in 4 dimensions or less, but due to the non-minimal coupling, we will get another contribution to the equations of motion. We will now calculate the variation of $f(\phi)\mathcal{L}_{(2)}$ and $f(\phi)\mathcal{L}_{(3)}$ terms separately as the process used for the 4-dimensional term can be mostly replicated to the 6-dimensional term.

C.1.1 The $\mathcal{L}_{(2)}$ term

We can easily see that after some reordering of the indices, considering that the variation is with respect to the metric, we can write the second Euler density as

$$\delta(f\mathcal{L}_{(2)}) = f\delta\mathcal{L}_{(2)} = \frac{1}{4}\epsilon_{\mu}{}^{\nu}{}_{\gamma}{}^{\sigma}\epsilon^{\alpha\beta\delta\lambda}(\delta R^{\mu}{}_{\nu\alpha\beta}R^{\gamma}{}_{\sigma\delta\lambda} + R^{\mu}{}_{\nu\alpha\beta}\delta R^{\gamma}{}_{\sigma\delta\lambda})f \ . \tag{C.1.2}$$

Now we can simplify this expression by redefining $(\mu,\nu,\alpha\beta) \leftrightarrow (\gamma,\sigma,\delta,\lambda)$ in the second term. With this we can sum the terms and get

$$\delta(f\mathcal{L}_{(2)}) = \frac{1}{2}\epsilon_{\mu}{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}R_{\gamma\sigma\delta\lambda}\delta R^{\mu}{}_{\nu\alpha\beta}f \ . \tag{C.1.3}$$

The variation of the Riemann tensor is

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha(\delta\Gamma^{\mu}_{\beta\nu}) - \nabla_\beta(\delta\Gamma^{\mu}_{\alpha\nu}) \ , \tag{C.1.4}$$

We can now pass the derivatives to the other Riemann tensor and the function $f(\phi)$ as follows

$$\begin{aligned}\delta(f\mathcal{L}_{(2)}) = \nabla_\alpha(...) - \frac{1}{2}\epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}\delta\Gamma^{\mu}_{\beta\nu}(\nabla_\alpha R_{\gamma\sigma\delta\lambda} f + R_{\gamma\sigma\delta\lambda}\nabla_\alpha f) \\ - \nabla_\beta(...) + \frac{1}{2}\epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}\delta\Gamma^{\mu}_{\alpha\nu}(\nabla_\beta R_{\gamma\sigma\delta\lambda} f + R_{\gamma\sigma\delta\lambda}\nabla_\beta f) \ .\end{aligned} \tag{C.1.5}$$

The $\nabla(...)$ terms will be total divergences which will not contribute to the equations of motion and we will ignore their contribution. We can also see that the terms with $\nabla_\alpha R_{\gamma\sigma\delta\lambda}$ and $\nabla_\beta R_{\gamma\sigma\delta\lambda}$ will vanish. This is because both α and β are antisymmetrised with the indices (δ, λ), which results in the term vanishing due to the Bianchi identity. Lastly, we can redefine $\alpha \leftrightarrow \beta$ in one of the terms to sum them.

So we now have

$$\delta(f\mathcal{L}_{(2)}) = \epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}\delta\Gamma^{\mu}_{\alpha\nu} R_{\gamma\sigma\delta\lambda}\nabla_\beta f \ . \tag{C.1.6}$$

The variation of the Levi-Civita connection is

$$\delta\Gamma^{\mu}_{\alpha\nu} = \frac{1}{2}g^{\mu\rho}(\nabla_\alpha\delta g_{\nu\rho} + \nabla_\nu\delta g_{\alpha\rho} - \nabla_\rho\delta g_{\nu\alpha}) \ , \tag{C.1.7}$$

so we can use the same reasoning as before to obtain the following three terms[1]

$$\delta(f\mathcal{L}_{(2)}) = \frac{1}{2}\epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}R_{\gamma\sigma\delta\lambda}g^{\mu\rho}(-\delta g_{\nu\rho}\nabla_\alpha\nabla_\beta f - \delta g_{\alpha\rho}\nabla_\nu\nabla_\beta f + \delta g_{\nu\alpha}\nabla_\rho\nabla_\beta f) \ . \tag{C.1.8}$$

Antisymmetry between α and β implies that the first term will vanish so we get

$$\delta(f\mathcal{L}_{(2)}) = \frac{1}{2}\epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}R_{\gamma\sigma\delta\lambda}(\delta g_{\nu\alpha}\nabla^\mu\nabla_\beta f - g^{\mu\rho}\delta g_{\alpha\rho}\nabla_\nu\nabla_\beta f) \ . \tag{C.1.9}$$

Knowing that $\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}$ we can finally take the variation with respect to $\delta g^{\mu'\nu'}$:

$$\begin{aligned}\frac{\delta(f\mathcal{L}_{(2)})}{\delta g^{\mu'\nu'}} &= \frac{1}{2}\epsilon_\mu{}^{\nu\gamma\sigma}\epsilon^{\alpha\beta\delta\lambda}R_{\gamma\sigma\delta\lambda}(-g_{\nu\mu'}g_{\alpha\nu'}\nabla^\mu\nabla_\beta f + g^{\mu\rho}g_{\alpha\mu'}g_{\rho\nu'}\nabla_\nu\nabla_\beta f) \\ &= \frac{1}{2}R_{\gamma\sigma\delta\lambda}(\epsilon_{\mu'}{}^{\mu\gamma\sigma}\epsilon_{\nu'}{}^{\beta\delta\lambda}\nabla_\mu\nabla_\beta f + \epsilon_{\nu'}{}^{\nu\gamma\sigma}\epsilon_{\mu'}{}^{\beta\delta\lambda}\nabla_\nu\nabla_\beta f) \ .\end{aligned} \tag{C.1.10}$$

By redefining some of the indices and dropping the primes, we finally get

$$\frac{\delta(f\mathcal{L}_{(2)})}{\delta g^{\mu\nu}} = R_{\gamma\sigma\delta\lambda}\epsilon_\mu{}^{\rho\gamma\sigma}\epsilon_\nu{}^{\beta\delta\lambda}\nabla_\rho\nabla_\beta f = -4P_{\mu\rho\nu\beta}\nabla^\rho\nabla^\beta f \ , \tag{C.1.11}$$

where

$$P_{\mu\rho\nu\beta} = -\frac{1}{4}\epsilon_{\mu\rho\gamma\sigma}R^{\gamma\sigma\delta\lambda}\epsilon_{\nu\beta\delta\lambda} \ . \tag{C.1.12}$$

[1] Once again we discard total divergences and the derivatives of the Riemann tensor which vanish due to the Bianchi identities.

C.1.2 The $\mathcal{L}_{(3)}$ term

We now calculate the extra term that comes from when we consider a 6-dimensional spacetime. We have that

$$\delta(f\mathcal{L}_{(3)}) = \frac{1}{8}\epsilon_{\mu_1}{}^{\nu_1}{}_{\mu_2}{}^{\nu_2}{}_{\mu_3}{}^{\nu_3}\epsilon^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}\delta(R^{\mu_1}{}_{\nu_1\alpha_1\beta_1}R^{\mu_2}{}_{\nu_2\alpha_2\beta_2}R^{\mu_3}{}_{\nu_3\alpha_3\beta_3})f \ . \tag{C.1.13}$$

We can now look at the method for $\mathcal{L}_{(2)}$ to get to the result. The reasoning used for equation (C.1.3) gives us now a factor of 3:

$$\delta(f\mathcal{L}_{(3)}) = \frac{3}{8}\epsilon_{\mu_1}{}^{\nu_1\mu_2\nu_2\mu_3\nu_3}\epsilon^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}R^{\mu_2\nu_2\alpha_2\beta_2}R^{\mu_3\nu_3\alpha_3\beta_3}\delta R^{\mu_1}{}_{\nu_1\alpha_1\beta_1}f \ . \tag{C.1.14}$$

For the sake of simplification, we define a new $P_{\mu\rho\nu\beta}$ tensor right away

$$P^{(6)\nu_1\alpha_1\beta_1}_{\mu_1} = -\frac{1}{8}\epsilon_{\mu_1}{}^{\nu_1\mu_2\nu_2\mu_3\nu_3}\epsilon^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}R_{\mu_2\nu_2\alpha_2\beta_2}R_{\mu_3\nu_3\alpha_3\beta_3} \ , \tag{C.1.15}$$

So we can write

$$\delta(f\mathcal{L}_{(3)}) = -3P^{(6)\nu\alpha\beta}_{\mu}\delta R^{\mu}{}_{\nu\alpha\beta}f \ . \tag{C.1.16}$$

However we should always take into account the symmetries associated with the tensor $P_{\mu}{}^{\nu\alpha\beta}$ to apply the same methods we applied for the $\mathcal{L}_{(2)}$ term. For more details see section C.1.3.

We now write the variation of the Riemann tensor in terms of the variations of the Levi-Civita connections and the reasoning applied to equations (C.1.5) and (C.1.6) give us total divergences and derivatives of the Riemann tensors which we can again discard. Antisymmetry again gives us a factor of 2 when we sum the two connection terms:

$$\delta(f\mathcal{L}_{(3)}) = -6P^{(6)\nu\alpha\beta}_{\mu}\delta\Gamma^{\mu}_{\alpha\nu}\nabla_{\beta}f \ . \tag{C.1.17}$$

Now we write the variation of the connection with respect to the variations of the metric and apply the methods used in equations (C.1.8) and (C.1.9), obtaining two terms

$$\delta(f\mathcal{L}_{(3)}) = -3P^{(6)\nu\alpha\beta}_{\mu}(\delta g_{\nu\alpha}\nabla^{\mu}\nabla_{\beta}f - g^{\mu\rho}\delta g_{\alpha\rho}\nabla_{\nu}\nabla_{\beta}f) \ . \tag{C.1.18}$$

Once again taking the variation with respect to $\delta g^{\mu\nu}$ we finally get

$$\frac{\delta(f\mathcal{L}_{(3)})}{\delta g^{\mu\nu}} = -6P^{(6)}_{\mu\rho\nu\beta}\nabla^{\rho}\nabla^{\beta}f \ , \tag{C.1.19}$$

which is a similar result to equation (C.1.11) but now with a factor of 6 and a different definition for $P_{\mu\rho\nu\beta}$.

C.1.3 The P tensor and the $\mathcal{L}_{(p)}$ density

Now we try to generalise this procedure for the Euler density of any dimension. We first note that we could have done the exact same procedure by defining the P tensor right away as we did in the last section, but without having to resort to the $\mathcal{L}_{(2)}$ section by taking a look at its symmetries first.

We can easily see that

$$P_{\mu\nu\alpha\beta} = -P_{\nu\mu\alpha\beta} = -P_{\mu\nu\beta\alpha} \ , \tag{C.1.20}$$

$$P_{\mu\nu\alpha\beta} = P_{\alpha\beta\mu\nu} \ . \tag{C.1.21}$$

These symmetries are also respected by the Riemann tensor. The other symmetry we need to take into account is the Bianchi identity that simply translates to

$$\nabla^{\mu} P_{\mu\nu\alpha\beta} = 0 \ , \tag{C.1.22}$$

or, in other words, *any divergence of $P_{\mu\nu\alpha\beta}$ vanishes.*

Taking into account the definitions above for the P tensor, we can define $P_{\mu\nu\alpha\beta}$ for $2p$ dimensions:

$$P^{(p)\mu\nu\alpha\beta} = -\frac{1}{2^p}\epsilon^{\mu\nu\mu_1\nu_1\ldots\mu_{p-1}\nu_{p-1}}\epsilon^{\alpha\beta\alpha_1\beta_1\ldots\alpha_{p-1}\beta_{p-1}}R_{\mu_1\nu_1\alpha_1\beta_1}\ldots R_{\mu_{p-1}\nu_{p-1}\alpha_{p-1}\beta_{p-1}} \ , \tag{C.1.23}$$

and after reproducing the steps in the last section, the equivalent of equation (C.1.16) for $2p$ dimensions, by applying the Leibniz rule for p Riemann tensors, is

$$\delta(f\mathcal{L}_{(p)}) = -pP^{(p)\nu\alpha\beta}_{\mu}\delta R^{\mu}{}_{\nu\alpha\beta} f \ . \tag{C.1.24}$$

As all the other steps are the same we get the final result for the variation of the Euler density for $2p$ dimensions:

$$\frac{\delta(f\mathcal{L}_{(p)})}{\delta g^{\mu\nu}} = -2pP^{(p)}_{\mu\rho\nu\beta}\nabla^{\rho}\nabla^{\beta} f \ . \tag{C.1.25}$$

Appendix D

List of Publications

This thesis is based on a number of publications by the author. These publications are:

- *On the inexistence of solitons in Einstein-Maxwell-scalar models*

  Carlos Herdeiro, João Oliveira

  Class. Quant. Grav. **36** (2019) no.10, 105015, [arXiv:1902.07721 [gr-qc]],

  doi:10.1088/1361-6382/ab1859.

- *On the inexistence of self-gravitating solitons in generalised axion electrodynamics*

  Carlos Herdeiro, João Oliveira

  Phys. Lett. B **800** (2020), 135076, [arXiv:1909.08915 [gr-qc]],

  doi:10.1016/j.physletb.2019.135076.

- *A class of solitons in Maxwell-scalar and Einstein?Maxwell-scalar models*

  Carlos Herdeiro, João Oliveira, Eugen Radu

  Eur. Phys. J. C **80** (2020) no.1, 23, [arXiv:1910.11021 [gr-qc]],

  doi:10.1140/epjc/s10052-019-7583-9.

- *Electromagnetic dual Einstein-Maxwell-scalar models*

  Carlos Herdeiro, João Oliveira

  JHEP **07** (2020), 130, [arXiv:2005.05354 [gr-qc]],

  doi:10.1007/JHEP07(2020)130.

- *Higher dimensional black hole scalarization*

  Dumitru Astefanesei, Carlos Herdeiro, João Oliveira, Eugen Radu

  Accepted in JHEP.

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