Renormalised vacuum polarisation of rotating black holes

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(i) Technical complexity, due to the lack of spherical symmetry;

(ii) Non-existence of generalizations of the (globally defined, regular and isometry-invariant) Hartle-Hawking state defined in static black hole spacetimes;

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Approach:

(i) Focus on a rotating black hole in 2+1 dimensions;

(ii) Introduce a mirror at fixed radial coordinate such that quantum state can be defined;

(iii) Use a 'quasi-Euclidean' method to obtain the complex Riemannian section of the black hole spacetime.

Outline

1.1. Klein-Gordon equation. A scalar field Φ of squared mass m^2 in the exterior region satisfies

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1.2. Hadamard renormalisation. The Feynman propagator G^{F}, evaluated for Hadamard states, is a bidistribution of Hadamard type

$$G^{\mathrm{F}}(x, x') = \frac{i}{4\sqrt{2}\pi}\left(\frac{U(x, x')}{\sqrt{\sigma(x, x') + i\epsilon}} + W(x, x')\right), \qquad \epsilon \to 0+\ . \tag{1.2}$$

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It can be shown that

- $U(x,x')$ only depends on the geometry along the geodesic joining x and x';
- $W(x,x')$ contains the quantum state dependence.

Hadamard singular part. The singular, state-independent part of G^{F} is

$$G_{\mathrm{Had}}(x,x') := \frac{i}{4\sqrt{2}\pi}\,\frac{U(x,x')}{\sqrt{\sigma(x,x')+i\epsilon}}\,. \tag{1.3}$$

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Vacuum polarisation. The renormalised vacuum polarisation is

$$\langle \Phi^2(x) \rangle := -i \lim_{x' \to x} \left[G^{\mathrm{F}}(x, x') - G_{\mathrm{Had}}(x, x') \right] \, . \tag{1.4}$$

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Remarks:

(1) Can $G^{\mathrm{F}}(x,x')$ be obtained as a sum over mode solutions, using some type of 'Euclidean methods'?

(2) $G_{\mathrm{Had}}(x,x')$ is given in closed form (1.3). How to do the subtraction?

Outline

2.1. Euclidean methods. For a *static* (analytic) spacetime (M, g), if t is a global timelike coordinate, one can obtain the *real Riemannian* $(M^{\mathrm{R}}, g^{\mathrm{R}})$ by a *Wick rotation*

$$g = g_{tt}\, dt \otimes dt + h_{ij}\, dx^i \otimes dx^j \xrightarrow{t \to -i\tau} g^{\mathrm{R}} = g_{\tau\tau}\, d\tau \otimes d\tau + h_{ij}\, dx^i \otimes dx^j \,, \tag{2.1}$$

where $\tau \in \mathbb{R}$, $g_{tt} < 0$, $g_{\tau\tau} > 0$ and h is a Riemannian metric.

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2.2. Quasi-Euclidean method. Consider a (compact) *stationary* spacetime (M, g) with coordinates (t, x^i), such that $g_{tt} < 0$. Perform a Wick rotation:

$$\begin{aligned} g &= g_{tt}\, dt \otimes dt + h_{ij}\, \left(dx^i + N^i dt\right) \otimes \left(dx^j + N^j dt\right) \\ &\xrightarrow{t \to -i\tau} g^{\mathbb{C}} = g_{\tau\tau}\, d\tau \otimes d\tau + h_{ij}\, \left(dx^i - iN^i d\tau\right) \otimes \left(dx^j - iN^j d\tau\right)\,. \qquad (2.2) \end{aligned}$$

This is the *complex Riemannian section* $(M^{\mathbb{C}}, g^{\mathbb{C}})$ of a complex manifold for which (M, g) is a real Lorentzian section.

Gibbons, Hawking (1977), Frolov (1982), Brown, Martinez, York (1991), Moretti (2000)

2.3. Green's function. The Green's function G associated with the Klein-Gordon equation satisfies the distributional equation

$$\left(\nabla^2 - m^2\right) G(x,x') = -\frac{\delta^3(x,x')}{\sqrt{g(x)}}\,, \tag{2.3}$$

where $g(x) := |\det(g^{\mathbb{C}}_{\mu\nu})|$ and $\nabla^2 := (g^{\mathbb{C}})^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian operator.

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The *unique* solution is given by

$$G(x, x') = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (\text{BH modes})\,, \tag{2.4}$$

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$$G_{\text{Had}}(x,x') = \frac{1}{4\sqrt{2}\pi}\frac{1}{\sqrt{\sigma(x,x')}} + \mathcal{O}(\sigma^{1/2})\,. \tag{3.1}$$

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3.2. Minkowski modes. In order to subtract the divergences of $G(x,x')$ in the coincidence limit, $G_{\text{Had}}(x,x')$ is rewritten as a sum over Minkowski modes.

$$G_{\text{Had}}(x,x') = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}(\text{Minkowski modes}) + (\text{finite term})\,. \tag{3.3}$$

We can write

$$G(x,x') - G_{\mathrm{Had}}(x,x') = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[(\text{BH modes}) - (\text{Minkowski modes}) \right] + (\text{finite term})\,. \tag{3.4}$$

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Minkowski's parameters. The Minkowski Green's function has free parameters:

$$T_{\mathbb{M}}\,, \quad \Omega_{\mathbb{M}}\,, \quad m^2_{\mathbb{M}}\,. \tag{3.5}$$

They are chosen such that the double sum is finite in the coincidence limit.

They are obtained by comparing the leading terms in the asymptotic expansions of the summands for large values of n and k.

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Theorem: *If the parameters $T_{\mathbb{M}}$ and $\Omega_{\mathbb{M}}$ are chosen as*

$$T_{\mathbb{M}} = \frac{\kappa_+}{2\pi}\,, \qquad \Omega_{\mathbb{M}} = N^{\theta}(r) + \Omega_{\mathcal{H}}\,, \tag{3.6}$$

then the double sum is finite.

Outline

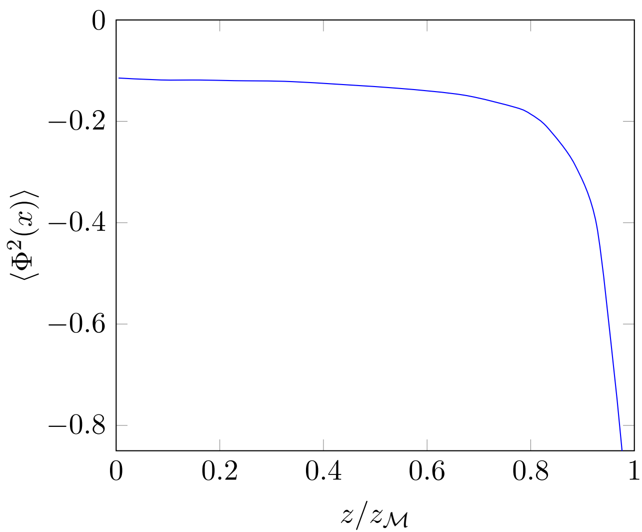

Figure 1: Vacuum polarisation for the scalar field in a warped AdS_3 black hole as a function of $z/z_{\mathcal{M}}$ for $\nu = 1.2$, $r_+ = 15$, $r_- = 1$, $r_{\mathcal{M}} = 62$ and $m = 1$. $z/z_{\mathcal{M}} = 0$ corresponds to the horizon, whereas $z/z_{\mathcal{M}} = 1$ corresponds to the mirror.

Outline

Conclusions

- We have computed $\langle \Phi^2(x) \rangle$ for a massive scalar field Φ in the Hartle-Hawking state on a rotating black hole. To our knowledge, this is the first general computation of a renormalised vacuum polarisation in the exterior of a rotating black hole spacetime.

- We have employed a 'quasi-Euclidean' method to obtain a complex Riemannian section of the original spacetime, at the level of which the Hadamard renormalisation procedure was applied.

- The divergences of the Green's function in the coincidence limit were subtracted by a sum over Minkowski modes with the same singularity structure.

- The implementation of our method in Kerr seems feasible in principle, given that only the asymptotic properties of the solutions in the limit of large quantum numbers are needed to perform the renormalisation.

THANK YOU FOR YOUR ATTENTION!