On a regular charged black hole with a nonlinear electric source

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Introduction

Regular BH models – to avoid the singularity problem.

J. Bardeen (Proceedings of GR5, Tiflis, USSR (1968)) – the 1st regular BH model – no known physical sources associated to it (E. A. – Beato and A. Garcia, (gr-gc/9911084, 0009077)).
They interpreted Bardeen's sources as a nonlinear magnetic monopol of a self-gravitating magnetic field (see also K. A. Bronnikov, gr-gc/0006014; A. Borde, gr-gc/9612057).

Balart and Vagenas (1401.2136 [gr-gc], 1408.0306 [gr-gc]) built a static, charged, regular BH in the framework of Einstein – non-linear ED theory, where the Lagrangean L(F) is a nonlinear function of the EM scalar $F = (1/4)F^{ab}F_{ab}$.

2. Regular Reissner-Nordstrom metric

We shall employ a modified version of Schwarzschild (KS) geometry (H.-C., gr-qc/1305.5964)

$$ds^2 = -\left(1-\frac{2m}{r}e^{-\frac{k}{r}}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}e^{-\frac{k}{r}}} + r^2 d\Omega^2, \quad k>0$$

to obtain a regular, charged BH.
Therefore, we take $k = q^2/2m$. Hence,

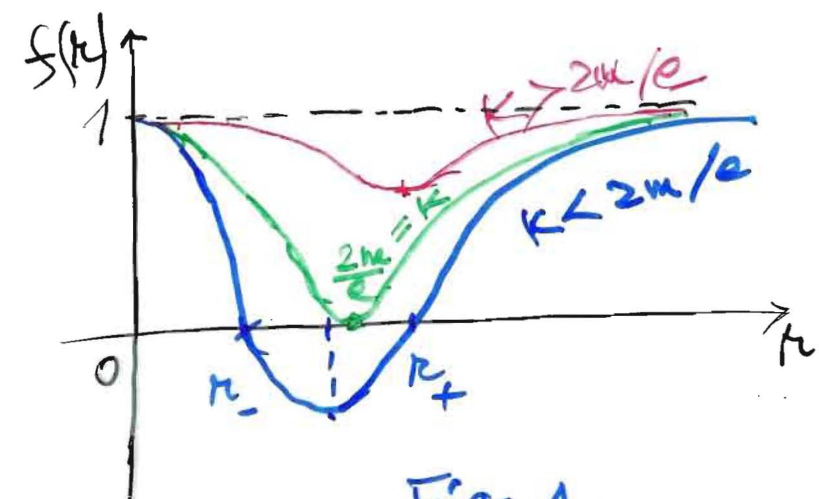
$$-g_{tt} = f(r) = 1 - \frac{2m}{r}e^{-\frac{q^2}{2mr}}$$

Asymptotically ($r \gg q^2/2m$), f tends to the RN value, $f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$.
We have

$$f'(r) = \frac{2m}{r^2}\left(1 - \frac{q^2}{2mr}\right)e^{-\frac{q^2}{2mr}},$$

with the root $r = q^2/2m$. Therefore,

$$f_{min} = f\left(\frac{q^2}{2m}\right) = 1 - \left(\frac{2m}{q\sqrt{e}}\right)^2.$$



r_- – the Cauchy inner horizon
r_+ – the event horizon

Fig. 1.

$|q| = 2m/\sqrt{e}$ ($\kappa = 2m/e$) – the extremal BH, $f(r_+) = 0$, $f'(r_+) = 0$ (degenerate horizon, see H.C., gr-qc/1305.5964).

For $|q| < 2m/\sqrt{e}$, Balart and Vagenas have given the expressions of r_+, r_- in terms of Lambert W-function.

Take now a static observer with a velocity vector field

$$u^b = (1/\sqrt{f}, 0, 0, 0)$$

The acceleration 4-vector $a^b = u^a \nabla_a u^b$

$$a^b = \left(0, \frac{m\left(1 - \frac{q^2}{2mr}\right)}{r^2}, 0, 0\right)$$

$a^r = 0$, when $r \to 0$ and at $r = q^2/2m$.

The gravitational field becomes repulsive for $r < q^2/2m$, when $a^r < 0$.

The surface gravity on the event horizon H, where $f(r_+) = 0$, yields (T. Jacobson, 1212.6821 [gr-qc])

$$\kappa = \sqrt{g^{ab} |\xi|_{,a} |\xi|_{,b}}\ \Big|_H$$

where $\xi^a = (1, 0, 0, 0)$ is the timelike Killing vector field, with

$$|\xi| = \sqrt{\xi^a \xi_a} = 1 - \frac{2m}{r} e^{-\frac{q^2}{2mr}}.$$

One obtains

$$x = \frac{1}{2r_+}\left(1 - \frac{q^2}{2mr_+}\right).$$

It vanishes ~~for~~ the extremal BH, when $r_+ = q^2/2m = 2m/e$.

3. Anisotropic stress tensor

The action of the GR coupled to non-linear ED can be written as

$$S = \int \left(\frac{R}{16\pi} - \frac{1}{4\pi} L(F)\right)\sqrt{-g}\, d^4x$$

The Lagrangean $L(F)$ – a nonlinear function of the EM scalar F, with

$$T_{ab} = L(F) g_{ab} - \frac{dL}{dF} F_{ac} F^c{}_b,$$

and

$$\nabla_a \left(\frac{dL}{dF} F^{ab}\right) = 0$$

For the electrostatic case, $F = -E^2(r)/2$, and $E(r)\, dL/dF = -q^2/(4\pi r^2)$.

One obtains (Balart and Vagenas, 1408.0306)

$$G^t{}_t = G^r{}_r = 8\pi\left(L + E^2 \frac{dL}{dF}\right)$$

$$G^\theta{}_\theta = G^\varphi{}_\varphi = 8\pi L$$

whence, in principle, we could find L and dL/dF, once the metric is given.

For our modified RN metric, we have, from $G_{ab} = 8\pi T_{ab}$:

$$T^t{}_t = -\rho = -\frac{mK}{4\pi r^4} e^{-\frac{K}{r}}, \quad T^r{}_r = p_r = -\rho,$$

$$T^\theta{}_\theta = T^\varphi{}_\varphi = p_\theta = p_\varphi = \frac{mK}{4\pi r^4}\left(1 - \frac{K}{2r}\right) e^{-\frac{K}{r}}$$

When $K = 2m/e$, we retrieve the components of $T^a{}_b$ from H.+C., 1305.5964.

$\rho > p_\theta$ always and $p_r = -\rho$, as for DE.

All the components of $T^a{}_b$ are non-singular at $r=0$ and at infinity.

For $r < K/2 = q^2/4m$, the strong energy condition is not satisfied:

$$\rho + \Sigma p_i = 2 p_\theta < 0$$

For $r \gg q^2/2m$, $T^a{}_b$ becomes the Maxwell stress tensor

$$T_{(e)}{}^a{}_b = \frac{q^2}{8\pi r^4}(-1, -1, 1, 1)$$

We assume that $T^a{}_b$ is valid for any $r > 0$, both in gravity and electrostatics, with K chosen accordingly:

- $K = 2m/e$, the KS spacetime is obtained asymptotically.
- $K = q^2/2m$, $T^a{}_b$ depends both on q and m, and gravity is mixed with electrostatics.

- $r \gg q^2/2m$, $T^a{}_b = T_{(e)}{}^a{}_b$ and the standard electrostatics is recovered.
- $T^a{}_b = 0$, both at $r=0$ and $r\to\infty$.

4. Energetic considerations

We evaluate the gravitational Komar energy

$$W = 2\int \left(T_{ab} - \frac{1}{2} g_{ab} T^c{}_c\right) u^a u^b N \sqrt{\gamma}\, d^3x$$

N – the lapse function, u^a – for a static observer. We get

$$W(r) = \int_0^r \frac{q^2}{r'^4}\left(1 - \frac{q^2}{4mr'}\right) e^{-\frac{q^2}{2mr'}} r'^2 dr'.$$

The substitution $x = 1/r'$ leads to

$$W(r) = m\left(1 - \frac{q^2}{2mr}\right) e^{-\frac{q^2}{2mr}}$$

(we took the limit at $r'=0$).

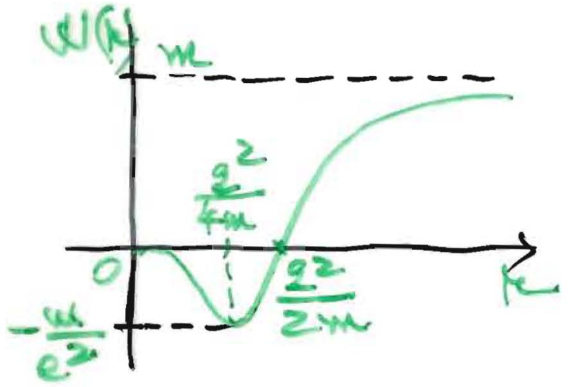


Fig. 2.

$W<0$, for $r < q^2/2m$. $W_{min} = -m/e^2$ (a steady state, rooted from negative pressures contribution).

$\lim_{r\to 0} W(r) = 0$; $\lim_{r\to\infty} W(r) = m$.

For $r \gg q^2/2m$, we have

$$W(r) \simeq m\left(1-\frac{q^2}{2mr}\right)^2 \simeq \underbrace{m}_{W(\infty)} \underbrace{-\frac{q^2}{r}}_{\text{the Coulombian term}} + O\left(\frac{q^4}{mr^2}\right)$$

Using $W(r)$, we could compute the horizon entropy $S = |W|/2T$ (T. Padmanabhan, gr-qc/0308070), where $T = \kappa/2\pi$. With r_+ from $f(r_+)=0$, one obtains

$$S_H = \frac{m\left(1-\frac{q^2}{2mr_+}\right)e^{-q^2/2mr_+}}{\frac{1}{2\pi r_+}\left(1-\frac{q^2}{2mr_+}\right)} = \pi r_+^2,$$

i.e. the relation $S_H = A_H/4$ is obtained, as for a BH.

Thanks to the expression of the electric field $E = F_{tr}$ obtained by Balart and Vagenas

$$E(r) = \frac{q}{r^2}\left(1-\frac{q^2}{8mr}\right)e^{-\frac{q^2}{2mr}},$$

in their nonlinear EM model, (In fact, $E(r; q, m, c)$. However, $r \gg q^2/2m$ leads to the Coulombian expression q/r^2) we get the electrostatic potential energy from

$$\phi(r) = -\int E(r)\,dr.$$

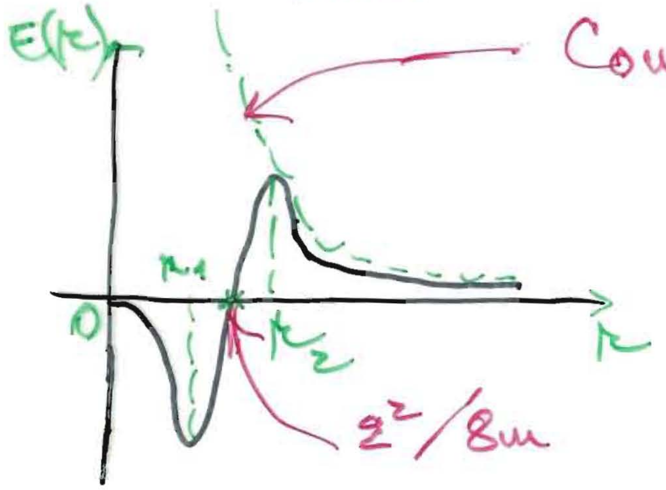
Example: Compute the electric field

close to an electron, at, say,
$r = r_e \simeq 10^{-15}\,m = q_e^2/m_e c^2$. We get

$$E_e \approx \frac{q_e}{r_e^2}\left(1-\frac{1}{8}\right)\frac{1}{\sqrt{e}} = \frac{7}{8\sqrt{e}}\cdot E_{coulomb}.$$

From $E'(r)=0$, we find that $E(r)$ has two extrema at

$$r_1 = \frac{q^2}{32m}\left(7-\sqrt{17}\right), \quad r_2 = \frac{q^2}{32m}\left(7+\sqrt{17}\right)$$



Coulombian field

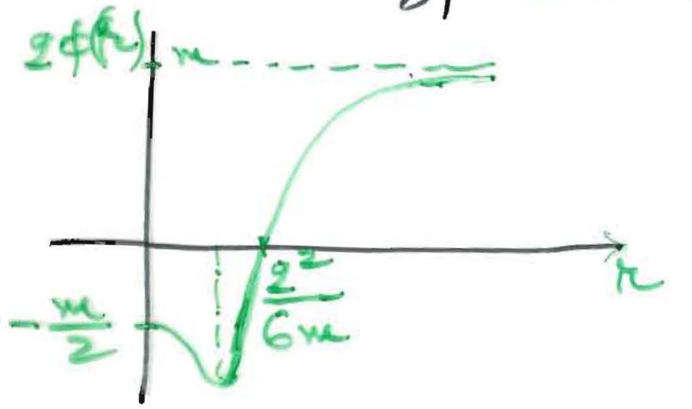
$$\lim_{r\to 0} E(r) = 0.$$

Fig. 3.

Let us find now the expression of $\phi(r)$. After an integration by parts, we have

$$\phi(r) = \frac{3m}{2q}\left(1-\frac{q^2}{6mr}\right)e^{-\frac{q^2}{2mr}} - \frac{m}{2q},$$

where we have chosen the constant of integration such that the electrostatic energy $q\phi(r) = m$ at infinity



$W(r) > q\,\phi(r)\,,\ \forall r>0$

$\phi(r; m, q, c)$

Fig. 4.

Far from the charge, i.e for $r \gg q^2/2m$, a power series development gives

$$\phi(r) \simeq \frac{m}{q} - \frac{q}{r}\left(1 - \frac{5}{8}\cdot\frac{q^2}{2mr} + \frac{1}{4}\left(\frac{q^2}{2mr}\right)^2 - \frac{1}{16}\left(\frac{q^2}{2mr}\right)^3 + \dots\right)$$

and we recognize the 1st two terms $m - q^2/r$ of $q\,\phi(r)$, exactly as for $W(r)$.

5. Conclusions

- singularity-free solution of Einstein's equation coupled to non-linear ED.
- asymptotically – RN spacetime and the Maxwell stress tensor as source.
- the regular charged BH – from a modified version of the KS metric.
- $E(r; q, m, c)$; $\phi(r; q, m, c)$ – are regular at $r = 0$ and when $r \to \infty$.
- for $r \gg q^2/2m$,
  $E(r) \simeq q\phi(r) \simeq mc^2 - \frac{q^2}{r}$.
- with $S = |W|/2T$,
  $S_H = \pi r_+^2 = A_H/4$, as for a BH.