VII Black Holes Workshop

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Non-minimal black holes with regular electric field

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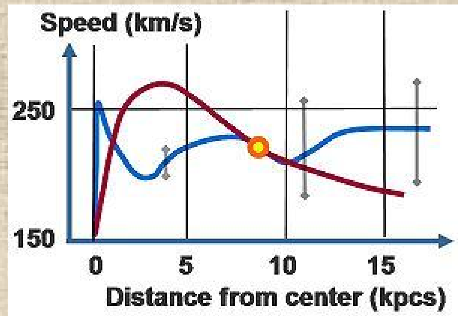
Plan of the talk

- Motivation and problems
- Mathematical formalism
- Examples of exact solutions
- Summary

Basic motives for the non-minimal extensions of the Field Theory

- **Non-minimal interactions as an *alternative to Dark Energy*: is it possible to explain accelerated expansion of the Universe in terms of curvature coupling?**

- **Non-minimal interactions as an *alternative to Dark Matter*: is it possible to explain flat rotation curves of spiral galaxies in terms of curvature coupling?** **(Navarro–Frenk–White)**



- **Non-minimal interactions and causal structure of space-time: can non-minimal interactions eliminate or transform *singularities and horizons*?**

$$U = \frac{GM}{r} \quad ??? \quad \Rightarrow \quad ??? \quad U^* = \frac{GM}{\sqrt{r^2 + a^2}}$$

Bardeen, 1975

$$U^*(0) = \frac{GM}{a}$$

Our dream is to find

exact static spherically symmetric solutions to the master equations of the non-minimal Einstein-Maxwell theory (or its extension), which satisfy the following three conditions:

1. **The solutions for electromagnetic and gravitational fields have to be analytical (explicit or in quadratures) .**
2. **The solutions for magnetic and electric fields have to be regular in the center.**
3. **The solutions for the metric functions and the corresponding curvature invariants have to be regular in the center.**

The main *problem* of the Non-minimal Field Theory seems to be connected with a large number of coupling constants introduced phenomenologically:

1. **NM Einstein-Maxwell theory contains 3 new coupling parameters.**
2. **NM Einstein-Yang-Mills-Higgs theory includes 8 new parameters.**

How one can reduce the number of parameters, e.g., to one (non-minimal radius)?

A) To use the requirements of *regularity* of the metric !

B) To reduce new coupling parameters to *known* constants!

C) To introduce geometric analogs for NM *susceptibilities* !

Non-minimal Einstein-Maxwell model

$$S_{\mathrm{NMEM}} = \int d^4x\sqrt{-g}\left[\frac{R}{8\pi} + \frac{1}{2}F_{ik}F^{ik} + \frac{1}{2}\mathcal{R}^{ikmn}F_{ik}F_{mn}\right]$$

Non-minimal susceptibility tensor

$$\mathcal{R}^{ikmn} = q_1 R g^{ikmn} + q_2 \Re^{ikmn} + q_3 R^{ikmn}$$

$$g^{ikmn} \equiv \tfrac{1}{2}(g^{im}g^{kn} - g^{in}g^{km}),$$

$$\Re^{ikmn} \equiv \tfrac{1}{2}(R^{im}g^{kn} - R^{in}g^{km} + R^{kn}g^{im} - R^{km}g^{in})$$

Non-minimally extended Maxwell equations

$$\nabla_k H^{ik} = I^i \qquad H^{ik} \equiv F^{ik} + \mathcal{R}^{ikmn}F_{mn} \qquad \nabla_k F^{*ik} = 0$$

Three examples of relations between non-minimal coupling constants

I. Drummond-Hathrell -type relations, based on QED one-loop calculations:

$$q_1 \equiv -5\tilde{Q},\ q_2 = 13\tilde{Q},\ q_3 = -2\tilde{Q}.$$

$$\tilde{Q} \equiv \frac{\alpha \lambda_e^2}{180\pi}$$

Compton wave-length of electron

Fine structure constant

II. Gauss-Bonnet type requirements: **NM susceptibility tensor is proportional to the double dual Riemann tensor**

$$q_1 + q_2 + q_3 = 0 \ + \ 2q_1 + q_2 = 0 \rightarrow \mathcal{R}_{ikmn} = \gamma^* R^*_{ikmn}$$

III. NM susceptibility tensor is proportional **to the difference between Riemann and Weyl tensors**

$$3q_1 + q_2 = 0 \ + \ q_3 = 0 \rightarrow \mathcal{R}_{ikmn} = \Omega[R_{ikmn} - \mathcal{C}_{ikmn}]$$

Static spherically symmetric models with electric field

Balakin A.B., Bochkarev V.V. and Lemos J.P.S. Phys. Rev. D, 2008.
Balakin A.B., Lemos J.P.S. and Zayats A.E. Phys. Rev. D, 2010

$$ds^2 = \sigma^2 N\,dt^2 - \frac{dr^2}{N} - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

$$\sigma(\infty) = 1 \qquad N(\infty) = 1$$

Equation for the electric field is coupled to metric functions and their derivatives

$$E(r)\left\{r^2\left[1 + (q_1 + q_2 + q_3)\left(N'' + 3N'\frac{\sigma'}{\sigma} + 2N\frac{\sigma''}{\sigma}\right)\right] + 2r(2q_1 + q_2)\left(N' + N\frac{\sigma'}{\sigma}\right) + 2q_1(N-1)\right\} = Q$$

$$E^2(r) = -\tfrac{1}{2}F_{ik}F^{ik}$$

$$\frac{[r(1-N)]'}{\kappa r^2} = -(E^2)'' N(q_1+q_2+q_3) + (E^2)'\Big[-\frac{1}{2}(q_1+q_2+q_3)\Big(N' + \frac{8N}{r}\Big) + \frac{N}{r}(2q_1+q_2)\Big] + E^2\Big[\frac{1}{2} + (q_1+q_2+q_3) \times \Big(N'' + 3N'\frac{\sigma'}{\sigma} + 2N\frac{\sigma''}{\sigma} - \frac{N'}{r} - 2\frac{N}{r^2}\Big) + (2q_1+q_2)\Big(2\frac{N'}{r} + 2\frac{N}{r}\frac{\sigma'}{\sigma} + \frac{N}{r^2}\Big) + q_1\frac{(N-1)}{r^2}\Big], \quad (16)$$

Two master equations
for the gravity field

(we write them simply to demonstrate the scale of problems with integration procedure for arbitrary set of coupling parameters)

$$\frac{2\sigma'}{\kappa r\sigma} = -(E^2)''(q_1+q_2+q_3) + (E^2)'\Big[(q_1+q_2+q_3)\Big(\frac{\sigma'}{\sigma} - \frac{4}{r}\Big) + (2q_1+q_2)\frac{2}{r}\Big] + E^2\Big[(q_1+q_2+q_3)\frac{2\sigma'}{r\sigma} - \frac{2q_3}{r^2}\Big]. \quad (17)$$

Guiding parameters of the model and convenient dimensionless quantities

Reissner-Nordstrom radius

$$r_Q \equiv \sqrt{G}|Q|$$

Schwarzschild radius

$$r_M \equiv 2GM$$

Non-minimal radius

$$r_q = \sqrt{2|q|}$$

$$a \equiv \frac{2q}{r_Q^2} = \pm \frac{r_q^2}{r_Q^2}$$

Dimensionless guiding parameter

$$x = \frac{r}{r_Q}$$

$$E_Q \equiv \frac{Q}{r_Q^2}$$

$$Z(x) = \frac{E(r)}{E_Q}$$

Dimensionless electric field

I. Exactly integrable model with $q_1 = -q_2$, $q_3 = 0$

!!! Special case, when the non-minimal and Reissner-Nordstrom's radii coincide $a = 1$

Equation for the electric field transforms into cubic algebraic equation:

$$(Z - 1)[(1 + x^2)Z^2 + (x^2 - 1)Z - 1] = 0$$

$$Z_{\text{Coulomb}}(x, 1) = \frac{1}{2(1 + x^2)}\left[1 - x^2 + \sqrt{x^4 + 2x^2 + 5}\right]$$

$Z(x, 1) \to \frac{1}{x^2}$ $\quad E(r) \to \frac{Q}{r^2}$

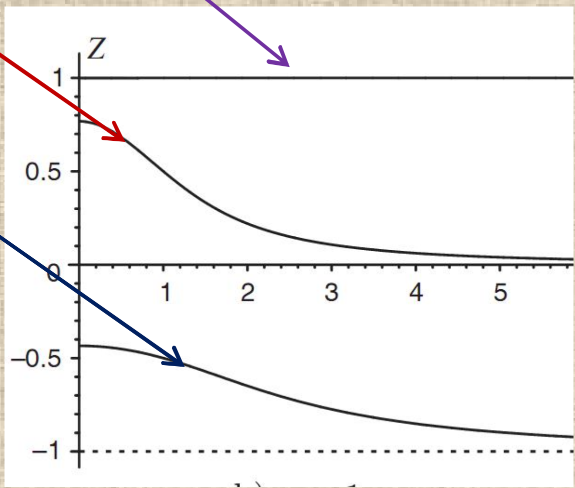
Regular electric field

$Z(0, 1) = \frac{\sqrt{5}+1}{2} = 1.618\ldots.$ "golden section" ϕ

$Z_{\text{const}}(x, 1) = 1$ $\quad E(r) = E_Q$

$$Z_{\text{nonCoulomb}}(x, 1) = \frac{1}{2(1+x^2)}\left[1 - x^2 - \sqrt{x^4 + 2x^2 + 5}\right]$$

$Z_{\text{nonCoulomb}}(0, 1) = -\frac{2}{1 + \sqrt{5}} = -\frac{1}{\phi}$ $\quad Z_{\text{nonCoulomb}}(r \to \infty, 1) \to -1 - \frac{1}{x^4}$



I. Exactly integrable model with $q_1 = -q_2 \; , \; q_3 = 0$

Regular metric functions

$$\sigma(x) = \exp\left\{-\frac{3 + (1 - x^2)\sqrt{x^4 + 2x^2 + 5} + x^4}{2(1 + x^2)^2}\right\} \qquad \sigma(\infty) = 1$$

$$\sigma(0) = \exp\left\{-\frac{1}{2}(3 + \sqrt{5})\right\} \rightarrow \exp\{-(1 + \phi)\}$$

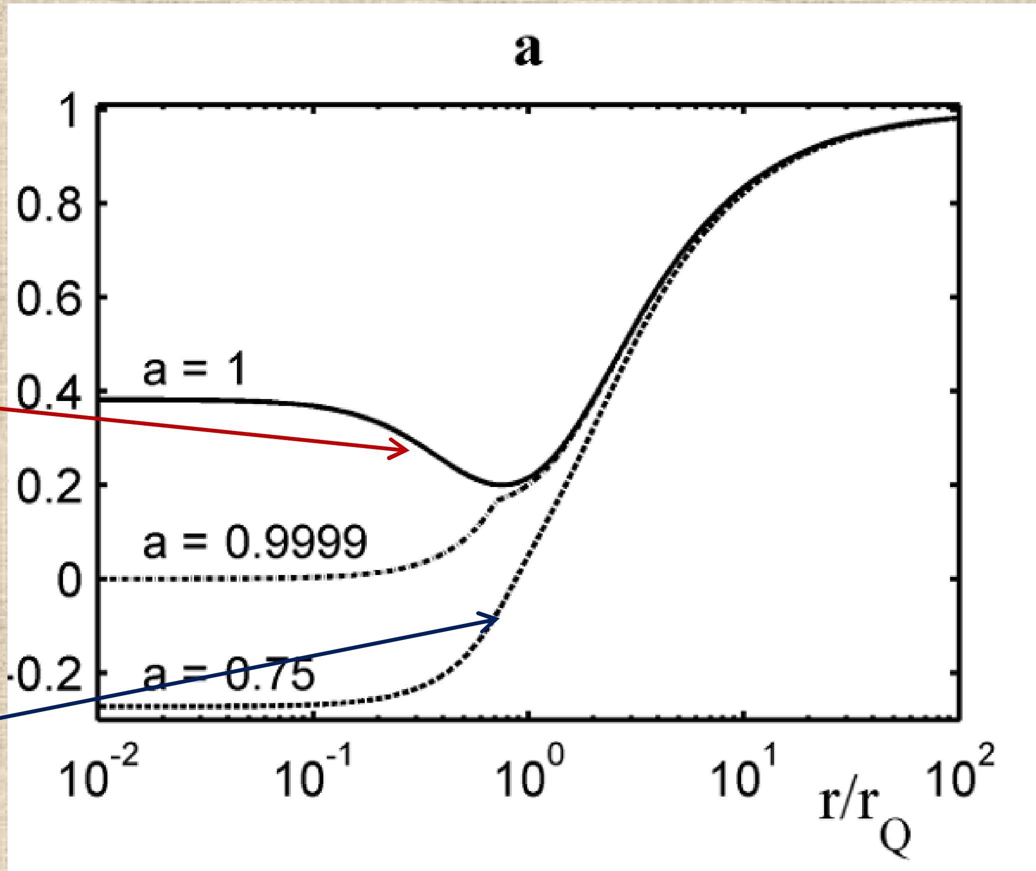
$$N(x) = \frac{1}{2x\sigma(x)} \int_0^x d\xi \sigma(\xi)[\xi^2 + 3 - \sqrt{\xi^4 + 2\xi^2 + 5}]$$

$$N(0) = \frac{3-\sqrt{5}}{2} \qquad N(\infty) = 1$$

Metric function N(r) is regular and positive everywhere.
There are no horizons, i.e., this exact solution does not describe black holes.
We deal with the so-called regular Fibonacci soliton…

Fibonacci soliton

Example of black hole



N(r)

Asymptotic mass

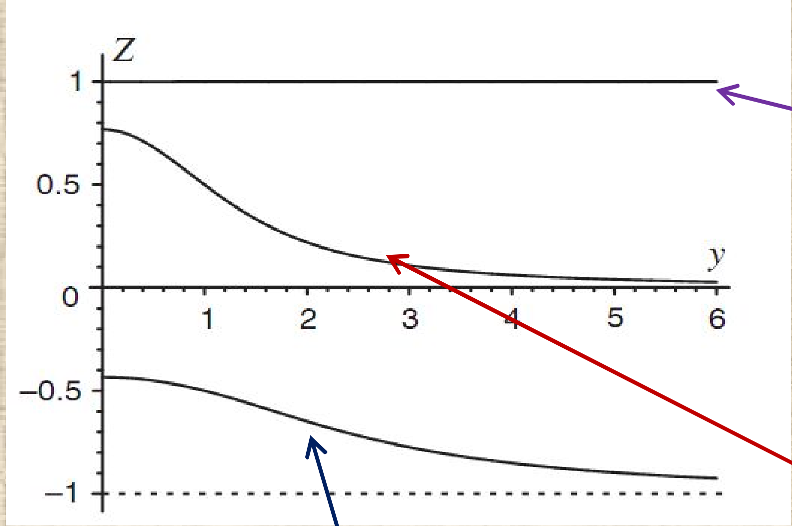
$$M = \frac{|Q|}{4\sqrt{G}} \int_0^\infty d\xi \Big[\frac{1}{\sigma(\xi)} - \xi \frac{\sigma'(\xi)}{\sigma^2(\xi)} - \frac{1}{2}\sigma(\xi)\big(\xi^2 + 3 - \sqrt{\xi^4 + 2\xi^2 + 5}\big) \Big]$$

$$M \simeq 0.442 \frac{|Q|}{\sqrt{G}}$$

II. Exactly integrable model with $q_1 = -q,\ q_2 = 3q\ \ q_3 = 0$

$$a = \frac{2q}{r_Q^2}, \qquad \boldsymbol{a = 1}$$

Behavior of the electric field



$$Z = 1$$

Coulombian-type solution

$$Z_*(y, 1) = \frac{\sqrt{13 + 2y^2 + y^4} + 1 - y^2}{2(3 + y^2)}$$

$$Z_*(0, 1) = \frac{1 + \sqrt{13}}{6}$$

Non-Coulombian-type solution

$$Z_-(y, 1) = \frac{-\sqrt{13 + 2y^2 + y^4} + 1 - y^2}{2(3 + y^2)}$$

$$Z_-(0, 1) = \frac{1 - \sqrt{13}}{6}$$

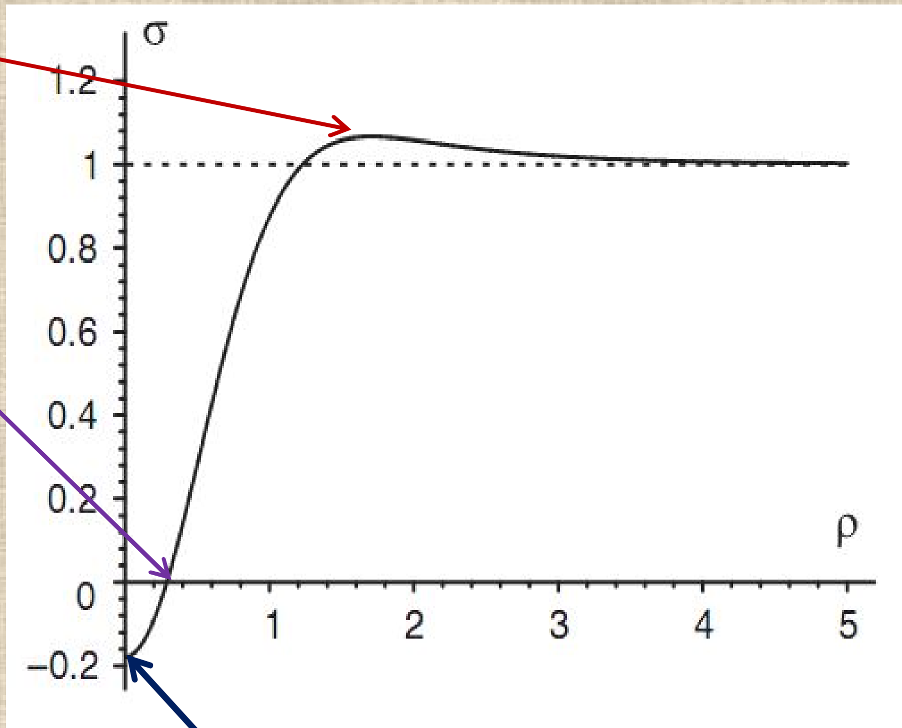
Behavior of the metric function $\sigma(\rho)$

$$\sigma(\rho) = \frac{(6 + 34\rho^2)\sqrt{13 + 2\rho^2 + \rho^4} - 39 + 82\rho^2 + 10\rho^4 + 10\rho^6 + \rho^8}{(3 + \rho^2)^3\sqrt{13 + 2\rho^2 + \rho^4}}.$$

$\sigma_{\max} = 1.067$

$\rho = 1.706$

$\rho_s = 0.289$



$\sigma(0, 1) = -(\sqrt{13} - 2)/9 \approx -0.178$

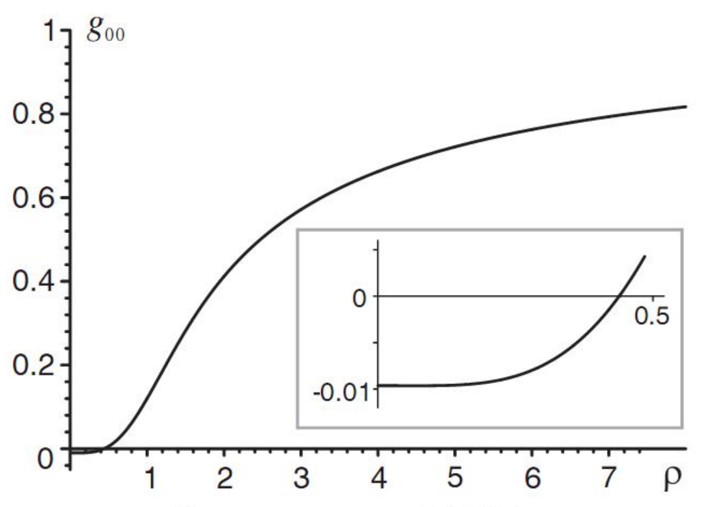
Behavior of the metric function $N(\rho)$

$$N = \frac{1}{\sigma^2}\left[1 - \frac{2m - J(\rho)}{\rho}\right]$$

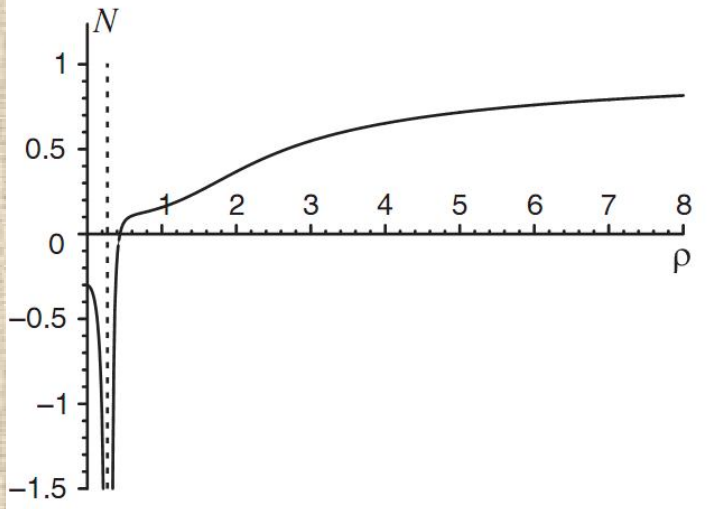
$$J(\rho) = \int_\rho^\infty d\rho\left[1 - \frac{\sigma}{2(3+\rho^2)^2}(19 + 23\rho^2 + 5\rho^4 + \rho^6 - (5 + 2\rho^2 + \rho^4)\sqrt{13 + 2\rho^2 + \rho^4})\right]$$

First specific value of the asymptotic mass

$$m_0 = \frac{1}{2}J(0) = 0.7924$$

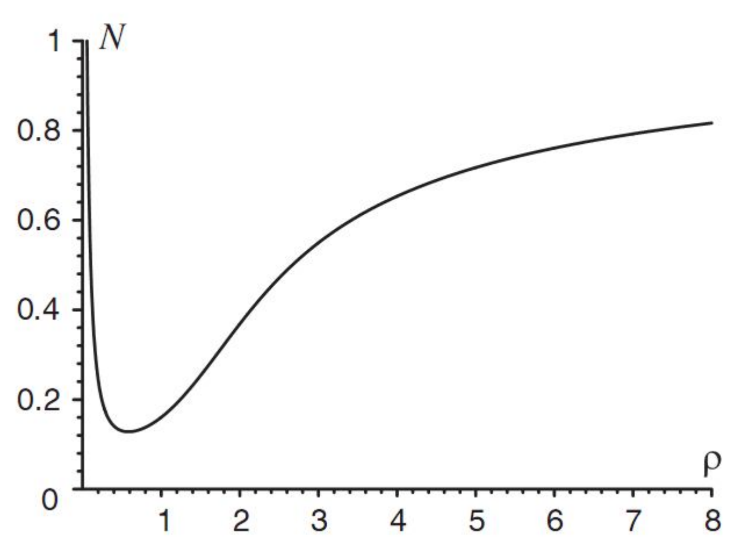


d) $m = m_0 = 0.7924$

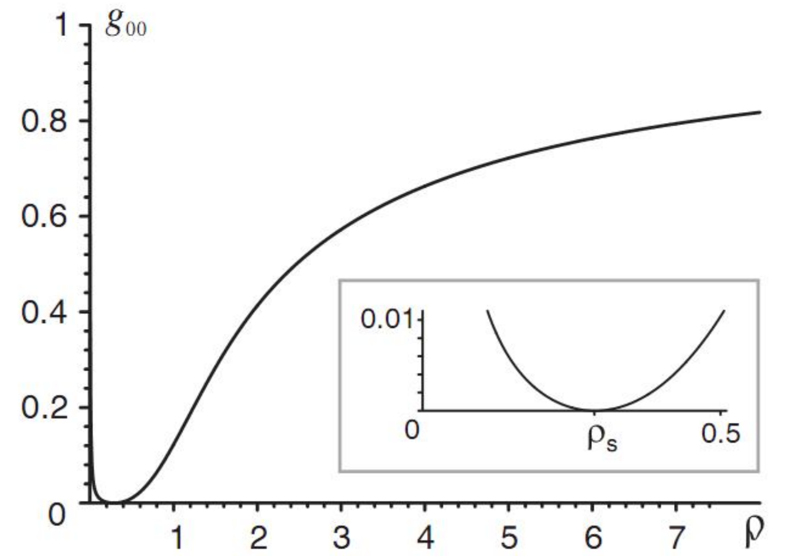


d) $m = m_0 = 0.7924$

Second specific value of the asymptotic mass



b) $m = m^* = 0.7913$



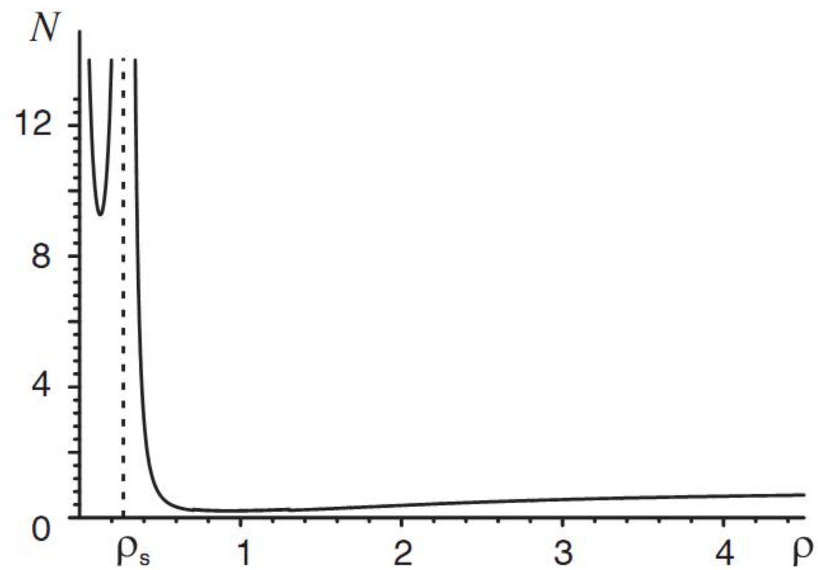
b) $m = m^* = 0.7913$

N(r) is positive for this asymptotic mass

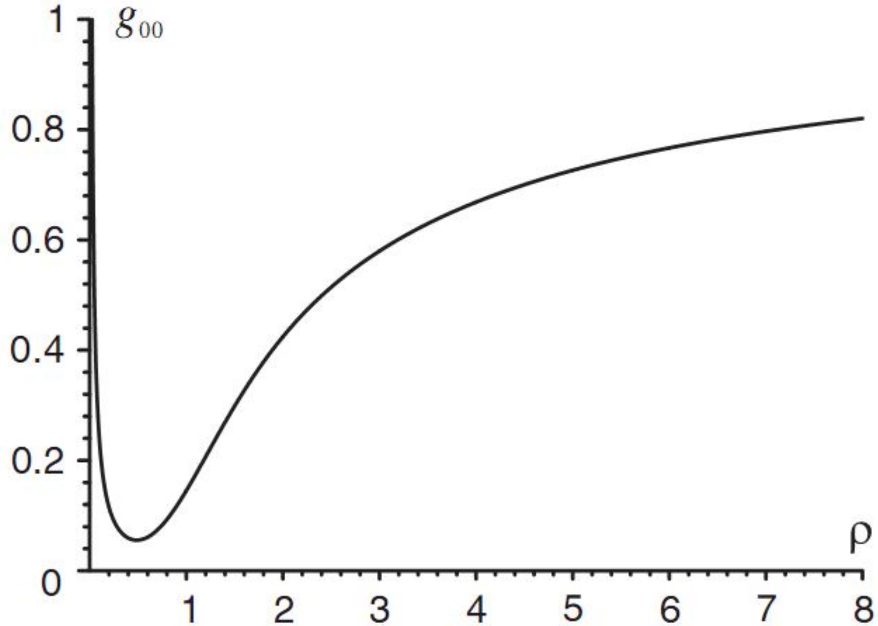
This metric function is non-negative and touches the horizontal axis at

$$\rho = \rho_s = 0.289$$

Asymptotic masses as guiding parameters of the model:
I. No horizons. Central singularity. Infinite barrier.

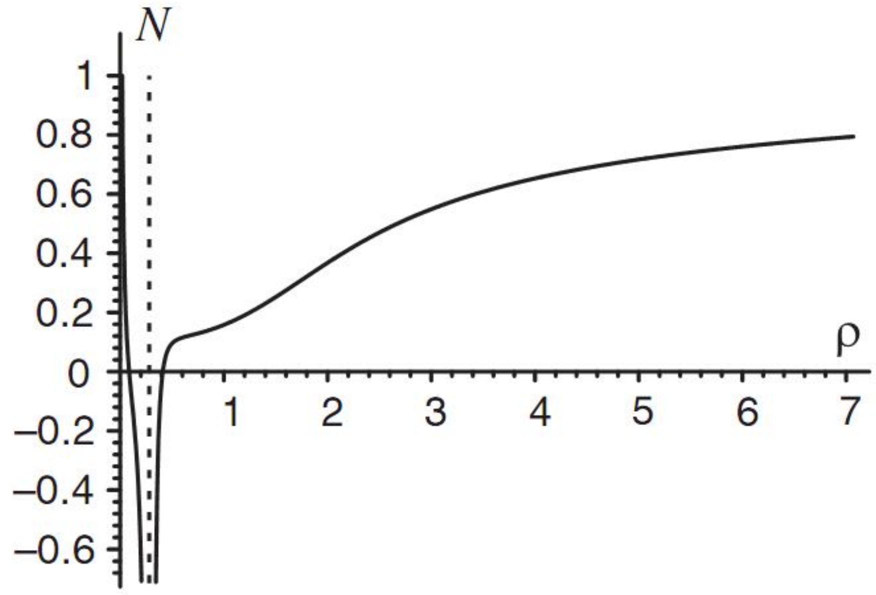


a) $0 < m < m^*$

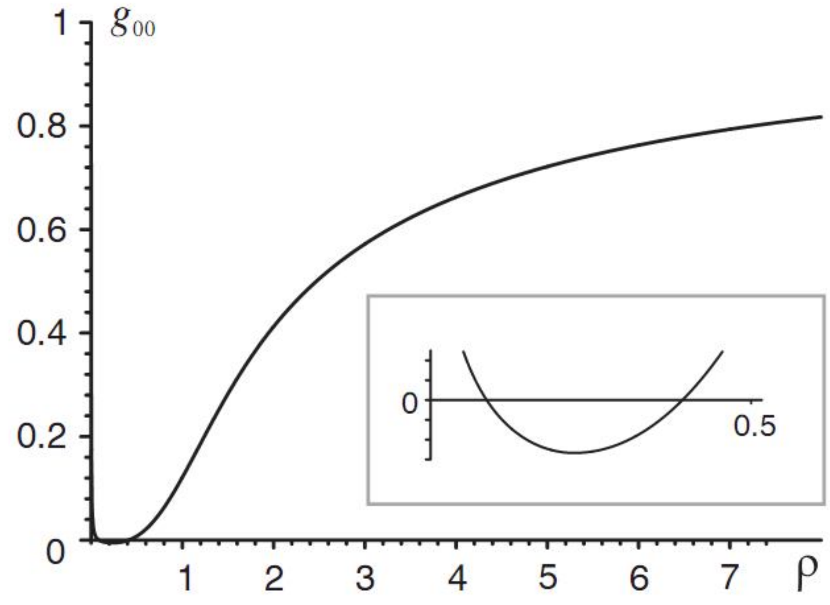


a) $0 < m < m^*$

Asymptotic masses as guiding parameters of the model:
II. Two horizons. Central singularity. Infinite well. Infinite barrier.

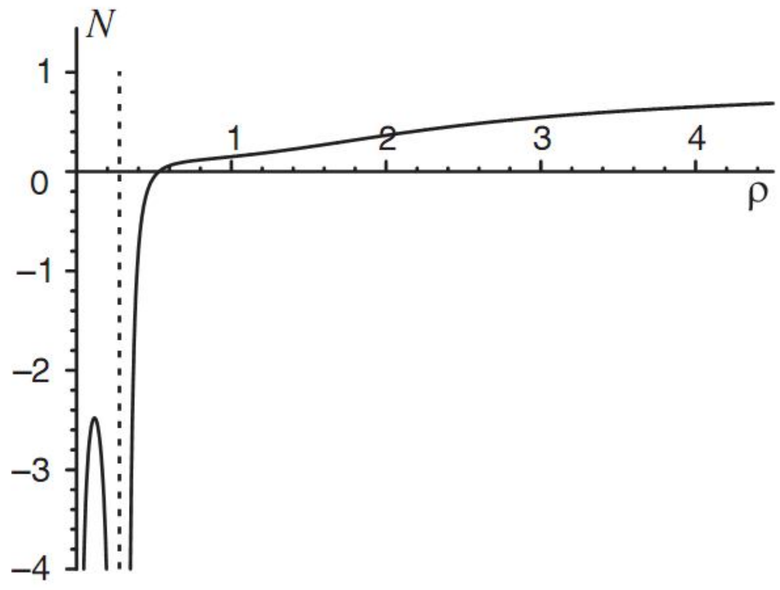


c) $m^* < m < m_0$

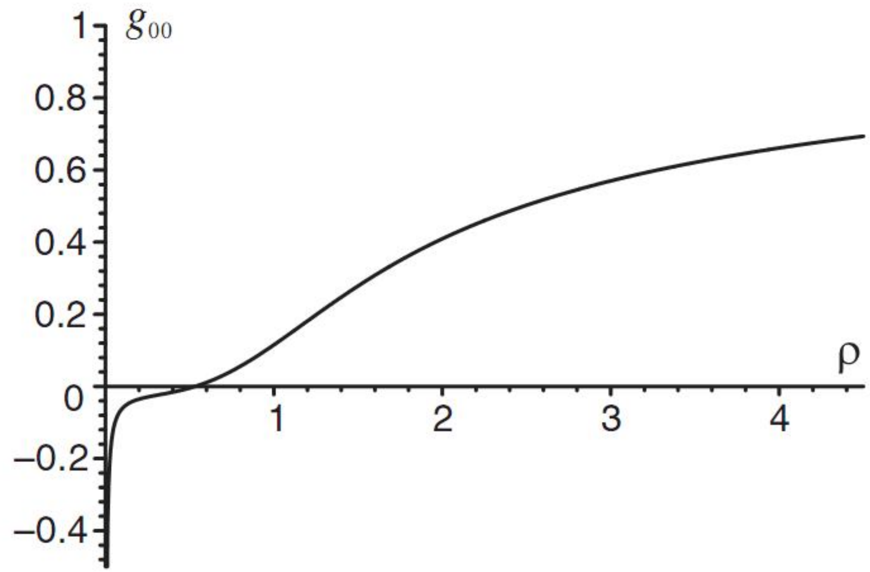


c) $m^* < m < m_0$

Asymptotic masses as guiding parameters of the model:
III. One horizon. Central singularity. Infinite well.



e) $m > m_0$



e) $m > m_0$

Magnetic analog: nom-minimal Wu-Yang monopole with regular gravitational field and magnetic field singular in the center

A.B. Balakin and A.E. Zayats. Phys. Lett. B , 2007.

$q_1 = -q < 0$ $\quad q_2 = 4q$ $\quad q_3 = -6q$

$$B = \frac{\nu}{\mathcal{G} r^2}$$

$$\sigma(r) = 1$$

$$N(x) = 1 + \frac{x^2(1 - mx)}{1 + x^4}$$

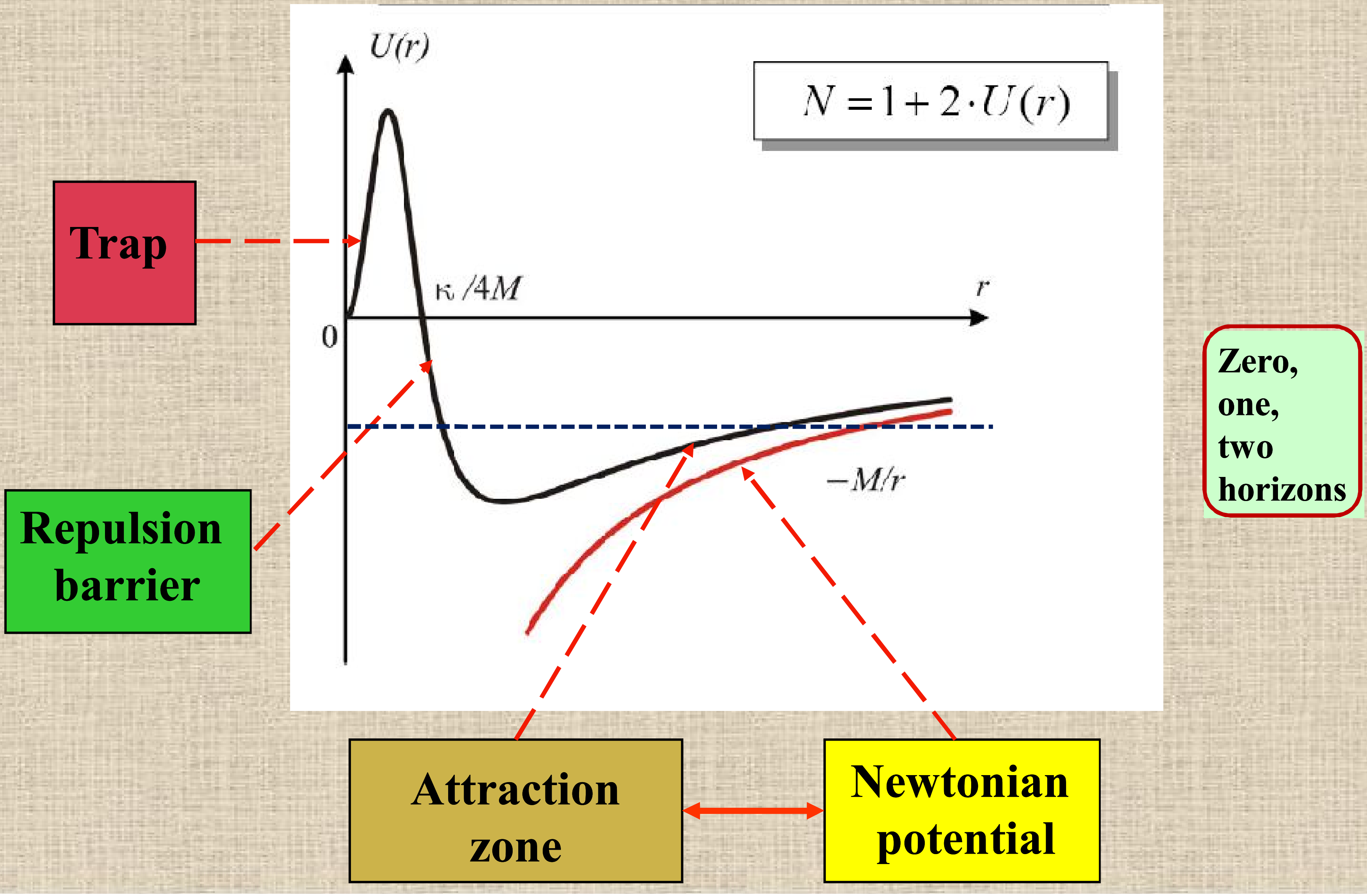
$$N(\infty) = 1$$

$$N(0) = 1$$

$\kappa = 8\pi\nu^2/\mathcal{G}^2$ → $r_Q^2 = \dfrac{\kappa}{2}$ → $m = \dfrac{2M}{r_Q}$

$$2q = r_Q^2$$

Effective gravitational potential near the non-minimal regular Wu-Yang monopole



Summary

• We considered three special non-minimal models, for which the Reissner-Nordstrom and non-minimal radii coincide; the obtained exact solutions for the electric, magnetic and gravitational fields are analytical.

• For the first model the electric field and both metric functions are regular in the center , but curvature invariants are singular.

•For the second model the electric field is again regular in the center, but one of the metric functions is singular.

•For the third model the metric functions and curvature invariants are regular in the center, but the magnetic field is singular.

Is there no perfection in this world?

We intend to continue to search for completely regular exact solutions using the model of non-minimal dyons and the model of axion electrostatics !!!

THANK YOU FOR THE ATTENTION !