Gravity as gauge theory squared: from amplitudes to black holes

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VII Black Holes Workshop
Aveiro, 19 December 2014

Based on work with D. O'Connell and C. D. White

Gravity vs gauge theory

Free fields

- looks like $h_{\mu\nu} \sim A_\mu \, \tilde{A}_\nu$

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Interactions

- n-particle gluon amplitude: $\mathcal{A}_n(\{k_i^\mu, \epsilon_i^\mu, a_i\})$
- n-particle graviton amplitude: $\mathcal{M}_n(\{k_i^\mu, \epsilon_i^{\mu\nu}\})$

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- relations between scattering amplitudes: **double copy**

Colour-Kinematics Duality

[Bern, Carrasco, Johansson '08]

Gauge theory amplitude depends on kinematics (k_i, ϵ_i) and colour (a_i):

$$\boxed{\mathcal{A}_n = \sum_{\alpha \in \text{cubic}} \frac{n_\alpha c_\alpha}{D_\alpha}}$$

- colour factors - $c_\alpha = f^{\cdots} f^{\cdots} \cdots f^{\cdots}$ $\qquad f^{abc} = \mathrm{tr}([T^a, T^b] T^c)$
- propagators - $D_\alpha(k_i)$
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Kinematic algebra for numerators?

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Tree level: Similar relations connect many theories of massless particles.
[Cachazo, He, Yuan '14] [Kawai, Lewellen, Tye '86]

Loop level conjecture: Allows study of supergravity divergences. [Bern et al '08-'14]

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<u>Prescription $\mathcal{A} \leftrightarrow \mathcal{M}$</u>

- same diagrams and propagators
- kin. numerator $\leftrightarrow$ colour factor

Self-dual Sectors of Gauge Theory and Gravity

Scalar equations of motion with cubic vertex.

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Use light-cone coords: $ds^2 = -du\,dv + dw\,d\bar{w}$

$$u = t - z, \; v = t + z, \; w = x + iy, \; \bar{w} = x - iy$$

define $\hat{k}_\mu = (0, \partial_w, 0, \partial_u)$

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Self-dual gauge theory

[Bruschi, Levi, Ragnisco '82]

$$F_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,F^{\alpha\beta}$$

$$A_\mu = \hat{k}_\mu \Phi, \quad \Phi = \phi^a T^a$$

$$\Box\Phi + [\partial_w\Phi, \partial_u\Phi] = 0$$

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Self-dual gravity

[Plebanski '75]

$$R_{\mu\nu\lambda\rho} = \frac{i}{2}\,\varepsilon_{\mu\nu\alpha\beta}\,R^{\alpha\beta}{}_{\lambda\rho}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \hat{k}_\mu\hat{k}_\nu\phi$$

$$\Box\phi + \{\partial_w\phi, \partial_u\phi\} = 0$$

where $\{f, g\} = \partial_w f\,\partial_u g - \partial_u f\,\partial_w g$

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Colour-kinematics duality and double copy manifest. [RM, O'Connell '11]

$$[\cdot,\cdot]\{\cdot,\cdot\} \quad \rightarrow \quad \{\cdot,\cdot\}^2$$

Kerr-Schild Spacetimes

Consider Kerr-Schild metric [Kerr, Schild '65]

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu}$$

where k_μ is null and geodesic wrt $\eta_{\mu\nu}$ ($\eta^{\mu\nu} k_\mu k_\nu = 0$, $\eta^{\mu\nu} k_\mu \partial_\nu k_\lambda = 0$), therefore also wrt $g_{\mu\nu}$.

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Leads to linearisation of the Einstein equations:

- $g^{\mu\nu} = \eta^{\mu\nu} - \phi\, k^\mu k^\nu$
- $R^\mu{}_\nu = \frac{1}{2}\partial_\alpha \left[\partial^\mu \left(\phi k^\alpha k_\nu\right) + \partial_\nu \left(\phi k^\alpha k^\mu\right) - \partial^\alpha \left(\phi k^\mu k_\nu\right)\right]$ $\qquad \partial^\mu \equiv \eta^{\mu\nu}\partial_\nu$

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Stationary Einstein equations give ($k_0 = 1$):

$$R^0{}_0 = \frac{1}{2}\nabla^2\phi = 0 \qquad\qquad R^i{}_0 = \frac{1}{2}\partial_j \left[\partial^i \left(\phi k^j\right) - \partial^j \left(\phi k^i\right)\right] = 0$$

"Square Root" Gauge Theory

[RM, O'Connell, White '14]

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu \quad \longrightarrow \quad \boxed{A^a_\mu = \phi\, k_\mu\, c^a} \qquad (c^a \text{ const})$$

- keep spatial propagator ϕ
- keep one copy of k_μ
- colour? Linearisation in gravity $\to$ Abelianisation in gauge theory

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Also for Kerr, black branes. Similar for plane waves.

Conclusions

- New structures connecting perturbative gauge theory and gravity. Related to unexpected UV cancellations in supergravity.

- Simple manifestation in self-dual theories.

- Perturbative relations extend to exact solutions in examples.