RECONSTRUCTING SPHERICALLY SYMMETRIC METRICS IN GENERAL RELATIVITY

VII Black Holes Workshop

Gr@v, Aveiro

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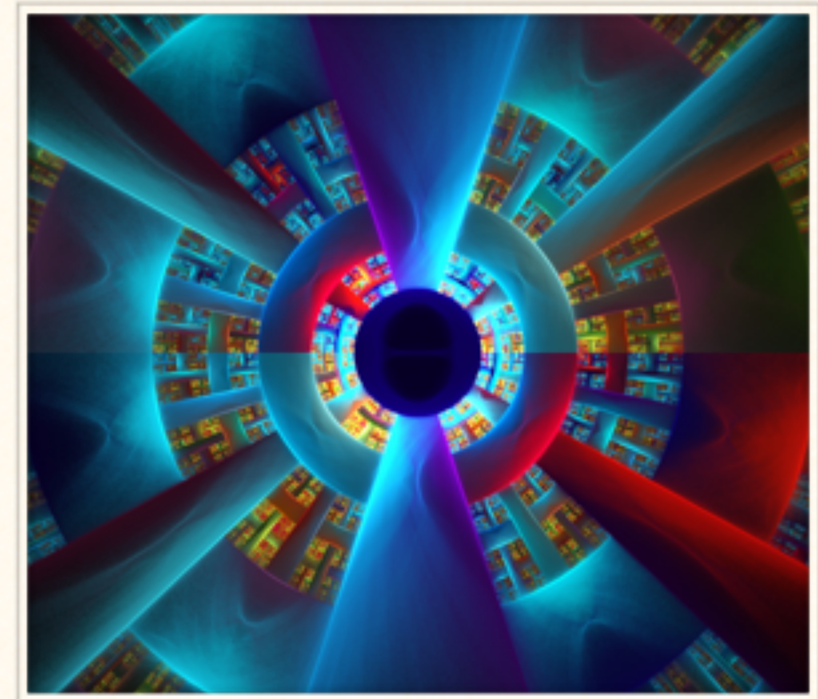
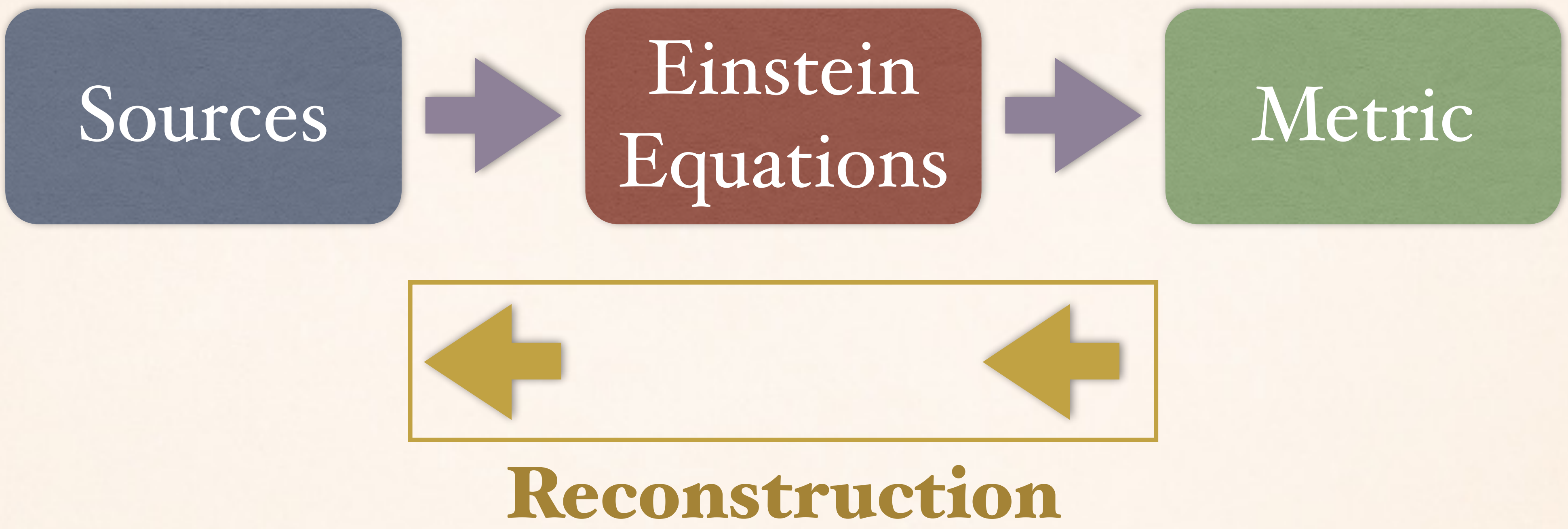


RECONSTRUCTION

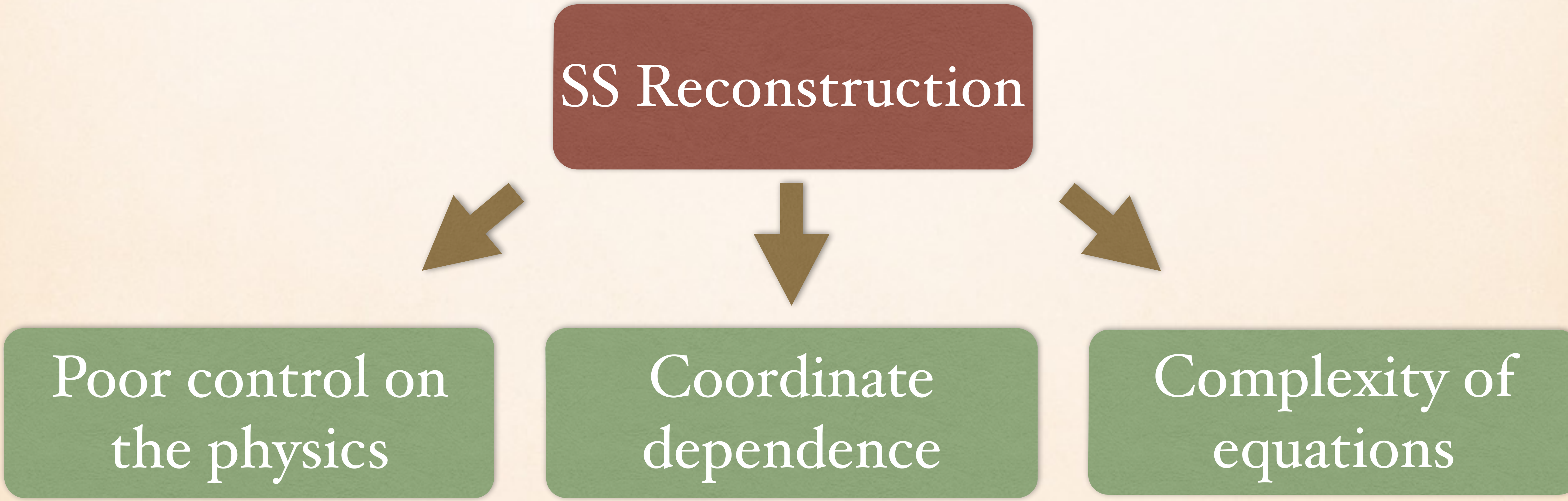


- The reconstruction technique is common in cosmology
- It was aimed to the determination of free functions in a cosmological model that produce a certain expansion law of the universe.
- Very successful for inflationary models and also modifications of GR

RECONSTRUCTION

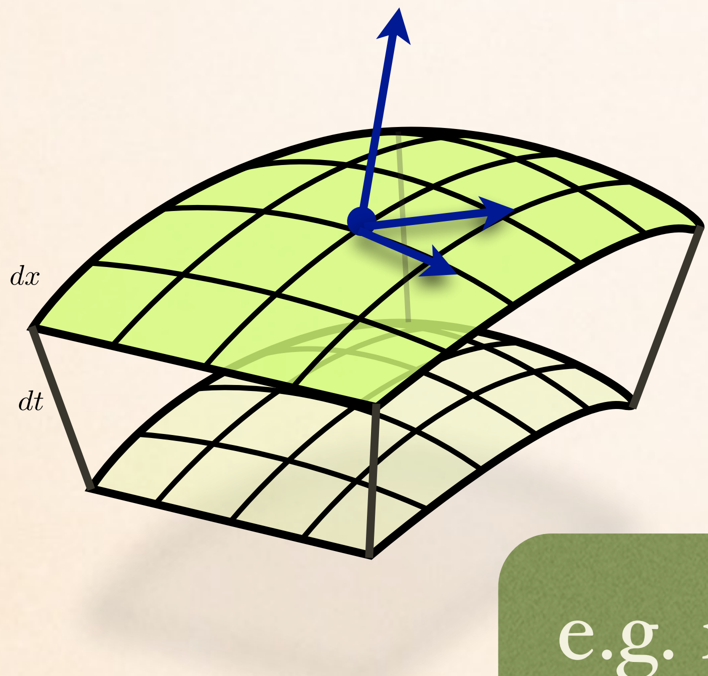
Can we apply this technique to the spherically symmetric case?

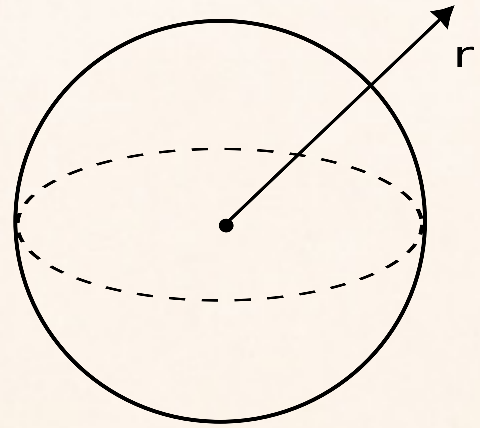
...yes but...

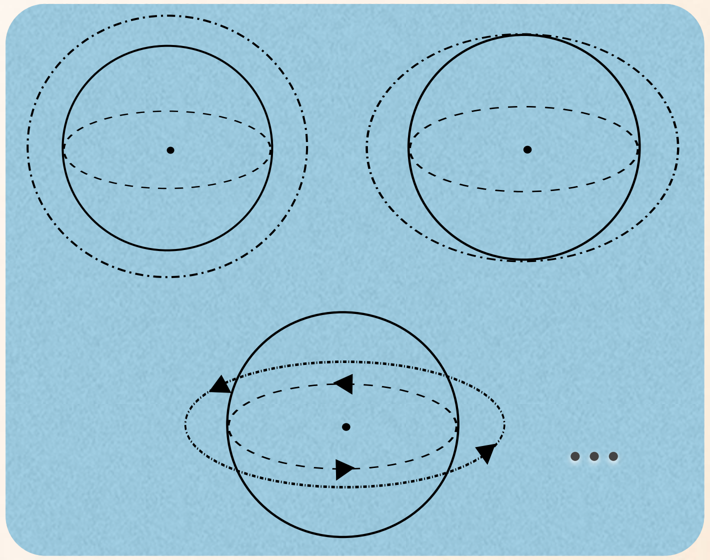


1+1+2 COVARIANT APPROACH

Given a congruence of geodesics...







e.g. in the case of spherical symmetry

Thermodynamics

1+1+2 COVARIANT APPROACH

In a general LRS-II spacetime the non zero variables are

$$\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q$$

Reducing to the case of static and spherically symmetric metrics we have

$\mathcal{A}, \phi, \mathcal{E}, K$

Geometry

μ, p, Π

Thermodynamics

All these quantities have a precise relation with the metric coefficients in a given coordinate system.

1+1+2 COVARIANT APPROACH

The equations for static spherically symmetric metrics can be written as

$$\phi\,\phi_{,\rho} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}\ ,$$

$$\mathcal{E}_{,\rho} - \frac{1}{3}\mu_{,\rho} + \Pi_{,\rho} = -\frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\ ,$$

$$\phi\ (p_{,\rho} + \Pi_{,\rho}) = -\left(\frac{3}{2}\phi + \mathcal{A}\right)\Pi - (\mu + p)\,\mathcal{A}\ ,$$

$$\phi\,\mathcal{A}_{,\rho} = -\left(\mathcal{A} + \phi\right)\mathcal{A} + \frac{1}{2}\left(\mu + 3p\right)\ ,$$

$$K_{,\rho} = -K,$$

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2,$$

$$0 = \mathcal{A}\,\phi - \frac{1}{3}\left(\mu + 3p\right) + \mathcal{E} - \frac{1}{2}\Pi.$$

THE NEW VARIABLES

Defining

$$X = \frac{\phi_{,\rho}}{\phi}, \qquad Y = \frac{\mathcal{A}}{\phi}, \qquad \mathcal{K} = \frac{K}{\phi^2}, \qquad E = \frac{\mathcal{E}}{\phi^2}$$

$$M = \frac{\mu}{\phi^2}, \qquad P = \frac{p}{\phi^2}, \qquad \mathbb{P} = \frac{\Pi}{\phi^2}$$

We obtain

$$Y_{,\rho} = M + 3P - 2Y(X + Y + 1),$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X),$$

$$P_{,\rho} + \mathbb{P}_{,\rho} = -2Y(M + \mathbb{P}) - 2P(2X + Y) - \mathbb{P}(4X + 3),$$

$$\boxed{0 = 2M + 2P + 2\mathbb{P} + 2X - 2Y + 1,}$$

$$0 = 1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P},$$

$$\boxed{0 = 2M + 6P + 3\mathbb{P} - 6Y - 6E,}$$

GR IN VACUUM

In vacuum GR the above system can be solved exactly to give

$$Y = -\frac{C_0}{4C_0 - e^{\rho/2}}, \qquad X = -\frac{C_0}{4C_0 - e^{\rho/2}} - \frac{1}{2} \qquad \mathcal{K} = \frac{1}{2} - \frac{C_0}{4C_0 - e^{\rho/2}}$$

which corresponds to

$$\phi = C_2 e^{-3\rho/4}\sqrt{e^{\rho/2} - 4C_1} \qquad \mathcal{A} = \frac{C_2 C_1 e^{-\frac{3\rho}{4}}}{\sqrt{e^{\rho/2} - 4C_1}}, \qquad K = \frac{K_0}{4}\, e^{-\rho}.$$

and in terms of the metric components

$$A = C_2\left(1 - \frac{4e^{C_1}}{r}\right) \qquad B = \frac{C_3}{1 - \frac{4e^{C_1}}{r}}, \qquad C = \frac{4}{K_0}\, r^2.$$

GR PLUS MATTER

In the non vacuum case the general system can be written as

$$
\begin{aligned}
M &= \mathcal{K} - X - \frac{3}{4}, \\
\mathbb{P} &= \frac{1}{3}\left[-X(2Y+1) - 2\left(\mathcal{K} + Y_{,\rho}\right) - 2Y^2 + Y\right], \\
P &= \frac{1}{12}\left[-4\mathcal{K} + X(8Y+4) + 8Y_{,\rho} + 8Y(Y+1) + 3\right], \\
P_{,\rho} + \mathbb{P}_{,\rho} &= \mathcal{K}(1+2X) + Y_{,\rho}, \\
\mathcal{K}_{,\rho} &= -\mathcal{K}(1+2X).
\end{aligned}
$$

with the constraint

$$1 - 4\mathcal{K} - 4P + 4Y - 4\mathbb{P} = 0$$

WEC SOLUTION

We can use the new variables to construct solutions able to satisfy the weak energy condition. In the 1+1+2 formalism the WEC is

$$\mu \geq 0, \quad \mu + p + \Pi \geq 0, \quad \mu + p - \frac{1}{2}\Pi \geq 0$$

which means

$$M \geq 0 \quad M + P + \mathbb{P} \geq 0 \quad M + P - \frac{1}{2}\mathbb{P} \geq 0$$

using the general equations one obtains

$$Y \geq \frac{1}{2}(2X+1) \qquad \mathcal{K} \geq \frac{1}{4}(4X+3)$$

$$Y_{,\rho} \geq \frac{1}{2}\left(-2\mathcal{K} - 2XY + X - 2Y^2 - Y + 1\right)$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1+2X).$$

WEC SOLUTION

Choosing

$$\mathcal{K} = \frac{1}{4}(4X + 3 + \alpha) \qquad Y = \frac{1}{2}(2X + 1 + \beta), \qquad \alpha = 3\beta,$$

$$Y_{,\rho} = \frac{1}{2}\left(-2\mathcal{K} - 2XY + X - 2Y^2 - Y + 1\right) + \gamma \qquad \gamma = \frac{1}{4}\left(\beta + 3\right)$$

substituting in the general equations one has

$$X = \frac{e^{\frac{3\beta\rho}{2} + \frac{\rho}{2}} - 3(\beta + 1)}{2\left(2 - e^{\frac{3\beta\rho}{2} + \frac{\rho}{2}}\right)}, \qquad Y = \frac{\beta e^{\frac{3\beta\rho}{2} + \frac{\rho}{2}} + \beta + 1}{2\left(e^{\frac{3\beta\rho}{2} + \frac{\rho}{2}} - 2\right)} \qquad \mathcal{K} = \frac{K_0 \exp\left(\frac{\rho(3\beta+1)}{2}\right)}{4\log\left(2 - e^{\frac{1}{2}(3\beta+1)\rho}\right)}$$

which means

$$\phi = \pm 2 e^{-\frac{3}{4}(\beta+1)\rho}\sqrt{2 - e^{\frac{1}{2}(3\beta+1)\rho}}, \qquad \mathcal{A} = \frac{\beta e^{\frac{3}{4}(\beta+1)\rho} + (\beta + 1) e^{-\frac{3}{4}(\beta+1)\rho}}{\sqrt{2 - e^{\frac{1}{2}(3\beta+1)\rho}}}.$$

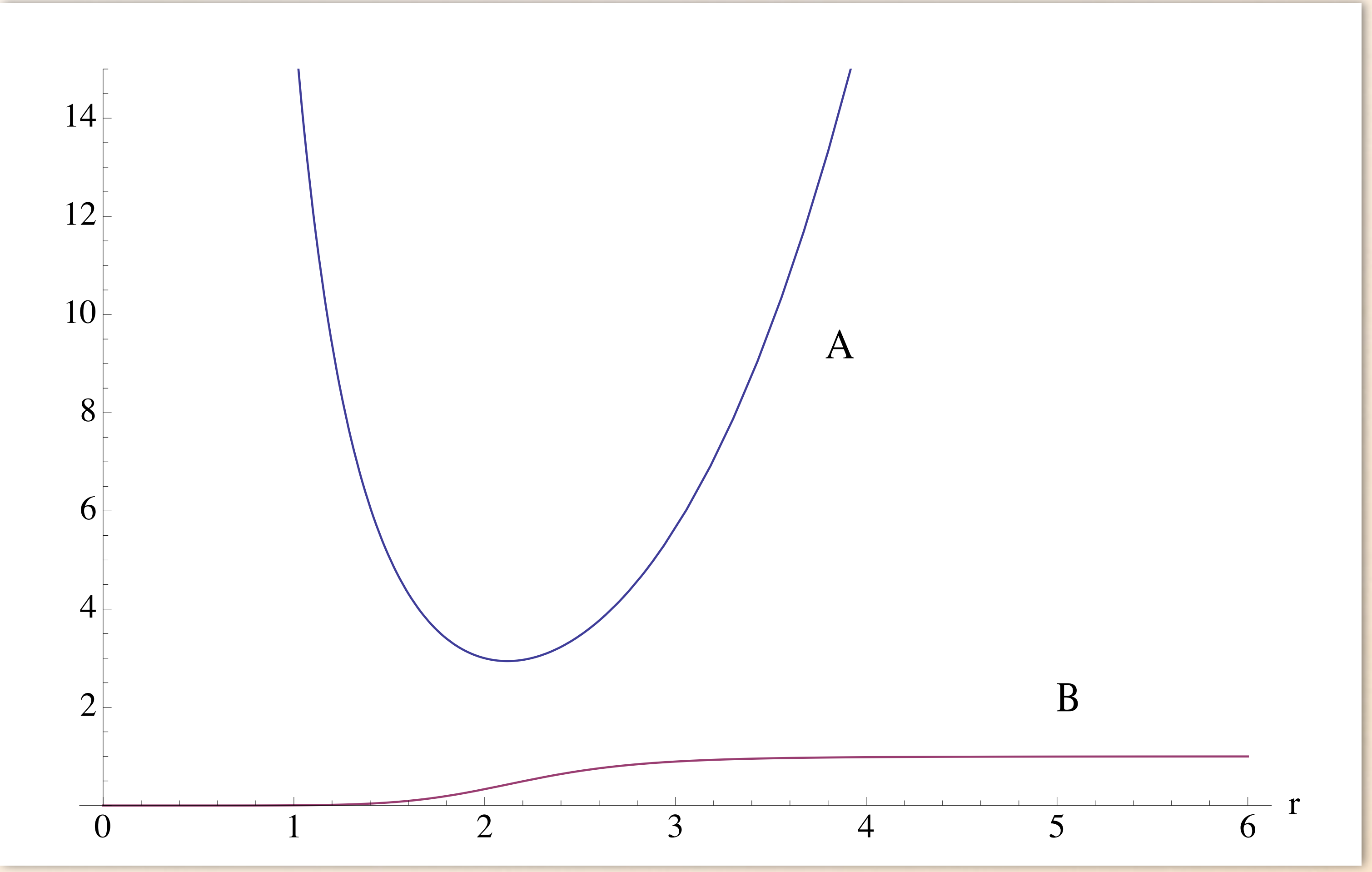
WEC SOLUTION

$$ds^2 = -Adt^2 + Bdr^2 + C\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

$$A = \frac{A_0}{r^{\beta+1}}\left(2 + r^{3\beta+1}\right),$$

$$B = \frac{4K_0}{\mathcal{K}_0}\left(2 + r^{3\beta+1}\right)^{-1},$$

$$C = \frac{4r^2}{K_0},$$

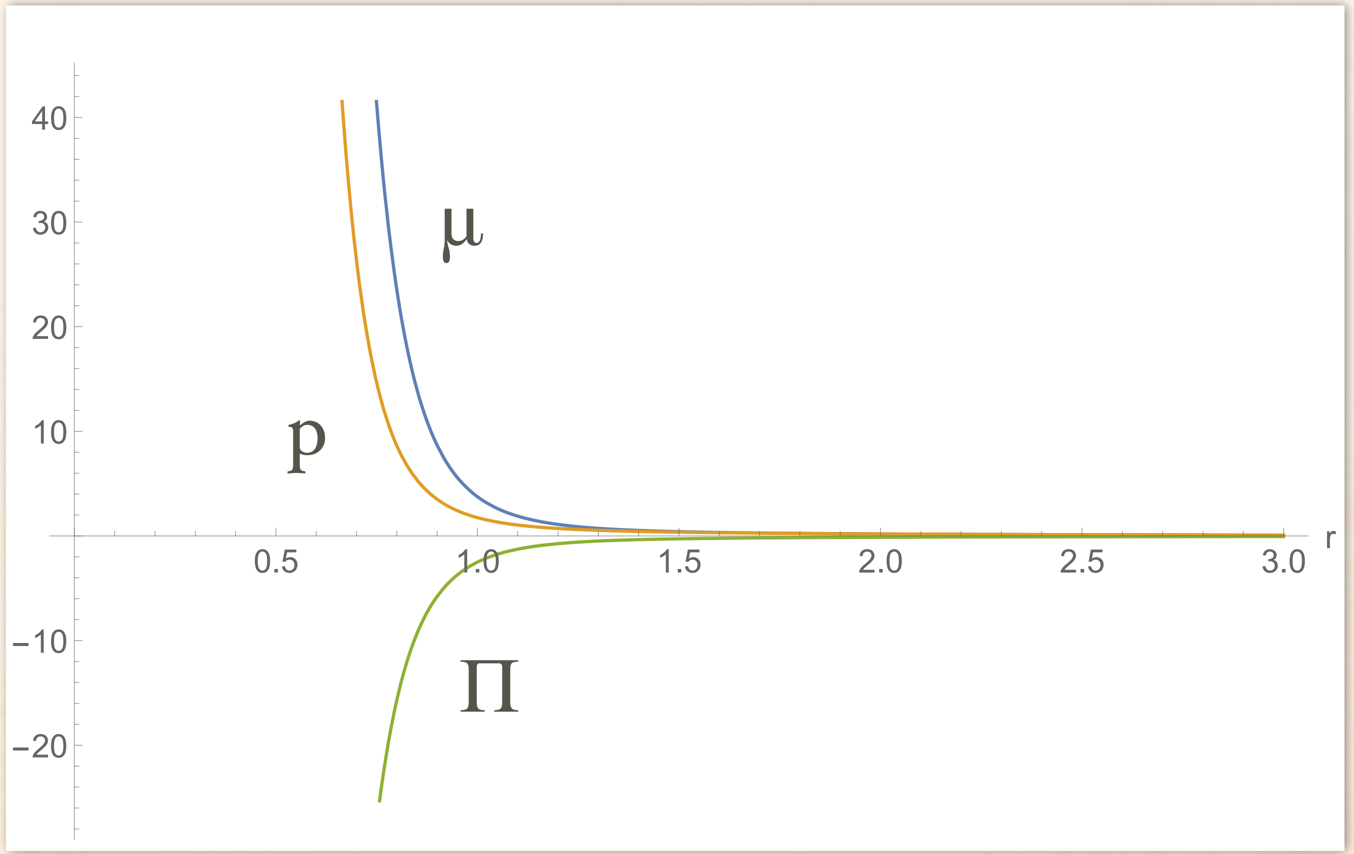


WEC SOLUTION

$$\mu = \frac{K_0}{4\mathcal{K}_0 r^3}\left[6\beta r^{-3\beta} - (1-4\mathcal{K}_0)\, r\big)\right],$$

$$p = \frac{K_0 r^{-3\beta-3}}{12\mathcal{K}_0}\left\{\left[2\beta(\beta+1) - 4\mathcal{K}_0 + 1\right] r^{3\beta+1} + 2\beta(2\beta-1)\right\},$$

$$\Pi = -\frac{K_0 r^{-3\beta-3}}{6\mathcal{K}_0}\left\{\left[(\beta-2)\beta + 4\mathcal{K}_0 - 1\right] r^{3\beta+1} + 2\beta(\beta+1)\right\}.$$



ASYMPTOTIC FLATNESS

asymptotically flat metric ⟺

$$g_{\mu\nu} \to \eta_{\mu\nu}$$
$$(R_{\mu\nu\rho\sigma} = 0)$$

in one coordinate system

In terms of our variables this implies

$$\{Y, \mu, p, \Pi\} \to 0 \qquad \mathcal{K} \to \frac{K_0}{\phi_0^2} = \frac{1}{4} \qquad X \to -\frac{1}{2}$$
$$\rho \to \infty$$

ASYMPTOTIC FLATNESS

Setting for example

$$Y = \frac{1}{e^{\alpha^2\rho} + Y_0} \qquad X = X_0 \exp\left(-\beta^2\rho\right) - \frac{1}{2}$$

$$\mathcal{K} = \mathcal{K}_0 \exp\left(\frac{2X_0 e^{-\beta^2\rho}}{\beta^2}\right)$$

One has

$$\phi = \pm 2\sqrt{\frac{K_0}{\mathcal{K}_0}} \exp\left(\frac{\rho}{2} - \frac{X_0 e^{-\beta^2\rho}}{\beta^2}\right) \; ,$$

$$\mathcal{A} = -\sqrt{\frac{K_0}{\mathcal{K}_0}}\, \frac{e^{-\frac{\rho}{2} - \frac{X_0 e^{-\beta^2\rho}}{\beta^2}}}{e^{\alpha^2\rho} + Y_0} \, .$$

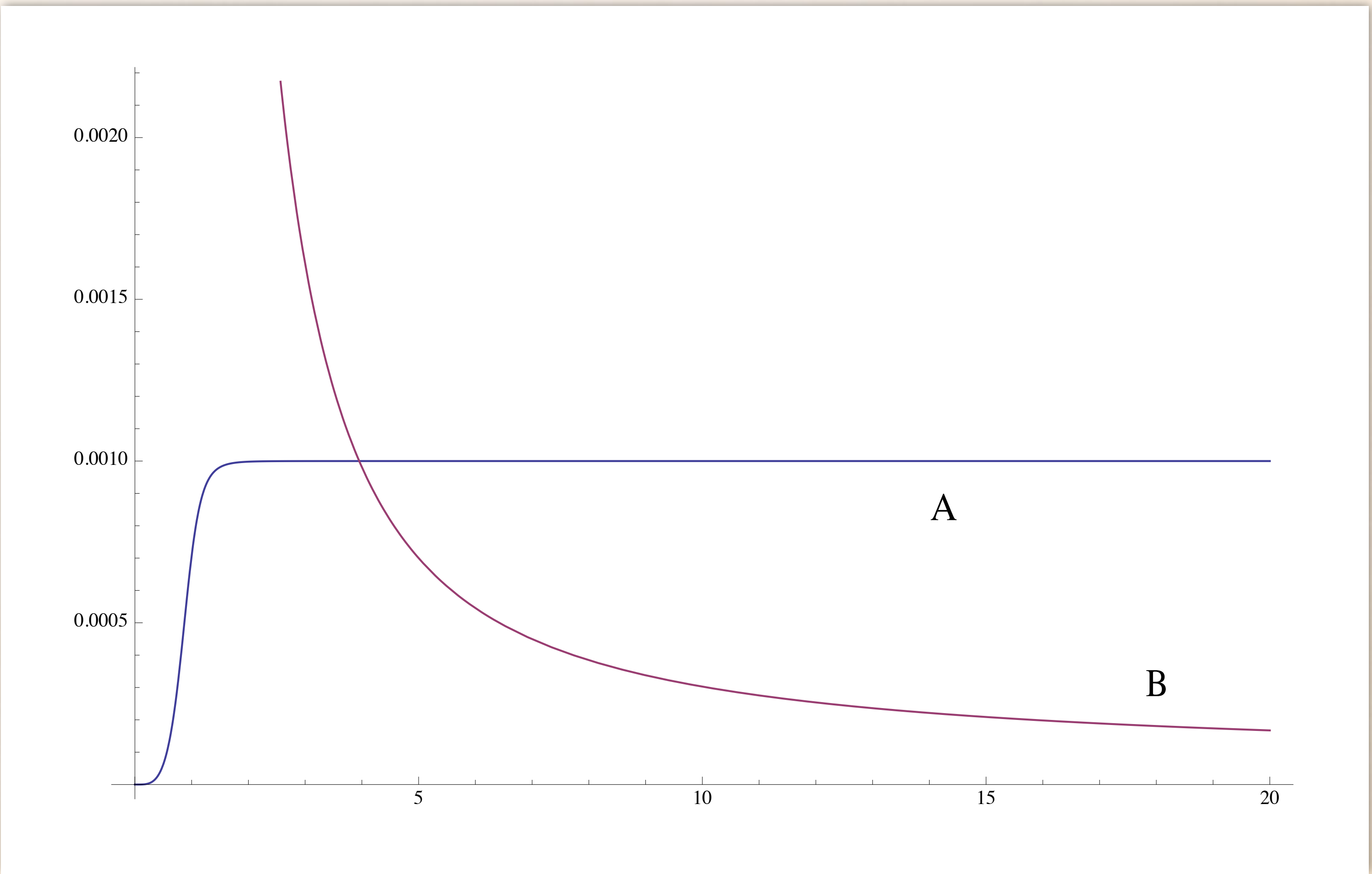
ASYMPTOTIC FLATNESS

$$ds^2 = -Adt^2 + Bdr^2 + C\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

$$A = A_0 r^{\frac{4}{Y_0}}\left(r^{2\alpha^2} + Y_0\right)^{-\frac{2}{\alpha^2 Y_0}},$$

$$B = \frac{4\mathcal{K}_0}{K_0}\exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right),$$

$$C = \frac{r^2}{K_0},$$



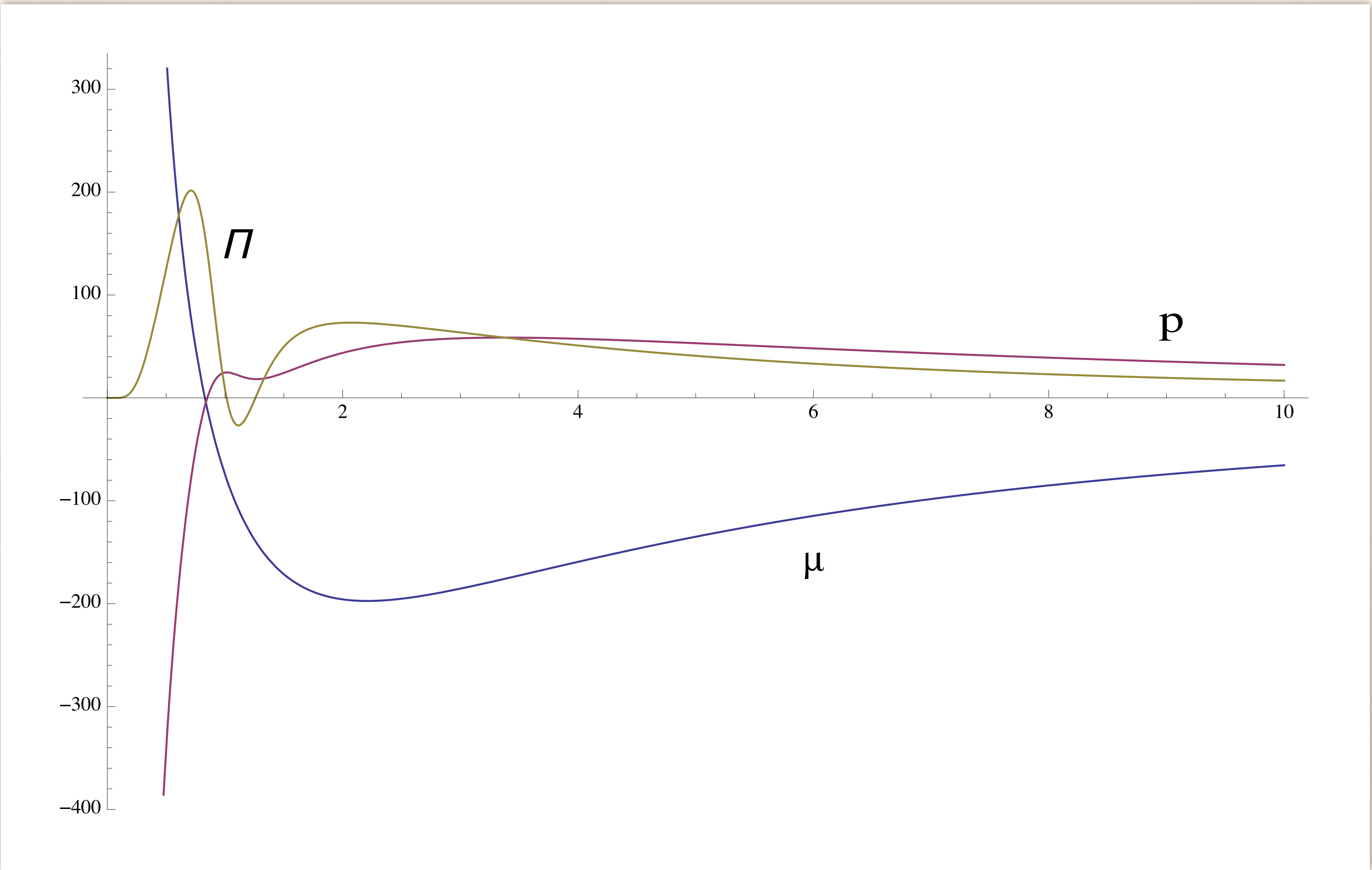
ASYMPTOTIC FLATNESS

$$\mu = \frac{K_0}{4\mathcal{K}_0 r^2}\left[4\mathcal{K}_0 - \left(4X_0 r^{-2\beta^2}+1\right)\exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right)\right],$$

$$p = \frac{K_0}{3\mathcal{K}_0 r^2}\left[X_0 r^{-2\beta^2}\left(\frac{2}{r^{2\alpha^2}+Y_0} - \mathcal{K}_0\exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right)+1\right) - \frac{\left(2\alpha^2-1\right)r^{2\alpha^2}}{\left(r^{2\alpha^2}+Y_0\right)^2} + \frac{\left(Y_0+2\right)}{\left(r^{2\alpha^2}+Y_0\right)^2}\right.$$
$$\left.+\frac{1}{4}\right]\exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right),$$

$$\Pi = \frac{K_0}{6\mathcal{K}_0 r^2}\left[4\mathcal{K}_0\exp\left(\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right) + 2X_0 r^{-2\beta^2}\left(1+\frac{2}{r^{2\alpha^2}+Y_0}\right) + \frac{4\left(\alpha^2 Y_0+1\right)}{\left(r^{2\alpha^2}+Y_0\right)^2} - \frac{4\left(\alpha^2+1\right)}{r^{2\alpha^2}+Y_0}\right.$$
$$\left.-1\right]\exp\left(-\frac{2X_0 r^{-2\beta^2}}{\beta^2}\right).$$

ASYMPTOTIC FLATNESS



GR PLUS SCALAR FIELD

When the matter is a scalar field the thermodynamics is defined as

$$\mu^\sigma = \frac{1}{2}\phi^2\sigma_\rho^2 + V(\sigma)$$

$$p^\sigma = -\frac{1}{6}\phi^2\sigma_\rho^2 - V(\sigma)$$

$$\Pi^\sigma = \frac{2}{3}\phi^2\sigma_\rho$$

and the general equations above specialise to

$$2\sigma_\rho^2 + 2X - 2Y + 1 = 0,$$

$$\sigma_\rho\left(\sigma_{\rho\rho} - \mathbb{V}_\sigma + \sigma_\rho(X + Y + 1)\right) = 0,$$

$$\mathbb{V} + Y(X + Y + 1) + Y_\rho = 0,$$

$$-4\mathcal{K} - 2\sigma_\rho^2 + 4\mathbb{V} + 4Y + 1 = 0$$

GR PLUS SCALAR FIELD

Again one of the above equations is redundant. We eliminate the second one an we recast the system to obtain

$$\sigma_\rho = \sqrt{\frac{-2X + 2Y - 1}{2}},$$

$$\mathbb{V} = \frac{1}{2}(2\mathcal{K} - X - Y - 1),$$

$$0 = 2Y_\rho + 2Y^2 + Y + 2\mathcal{K} + X(2Y - 1) - 1$$

$$\mathcal{K}_\rho = -\mathcal{K}(1 + 2X)$$

EXAMPLE

Let us consider now a solution that ensure that the scalar field is real:

$$X = -1 + Y$$

we obtain

$$Y = \frac{e^{\rho}\rho}{2e^{\rho}(\rho - 1) - 2} \quad \mathcal{K} = \frac{e^{\rho}}{2\left(1 - e^{\rho}\rho + e^{\rho}\right)}$$

Which corresponds to

$$\phi = \pm e^{-\rho}\sqrt{2K_0[1 - e^{\rho}(\rho - 1)]}, \qquad \mathcal{A} = -\sqrt{\frac{\rho^2 K_0}{2 - 2e^{\rho}(\rho - 1)}}.$$

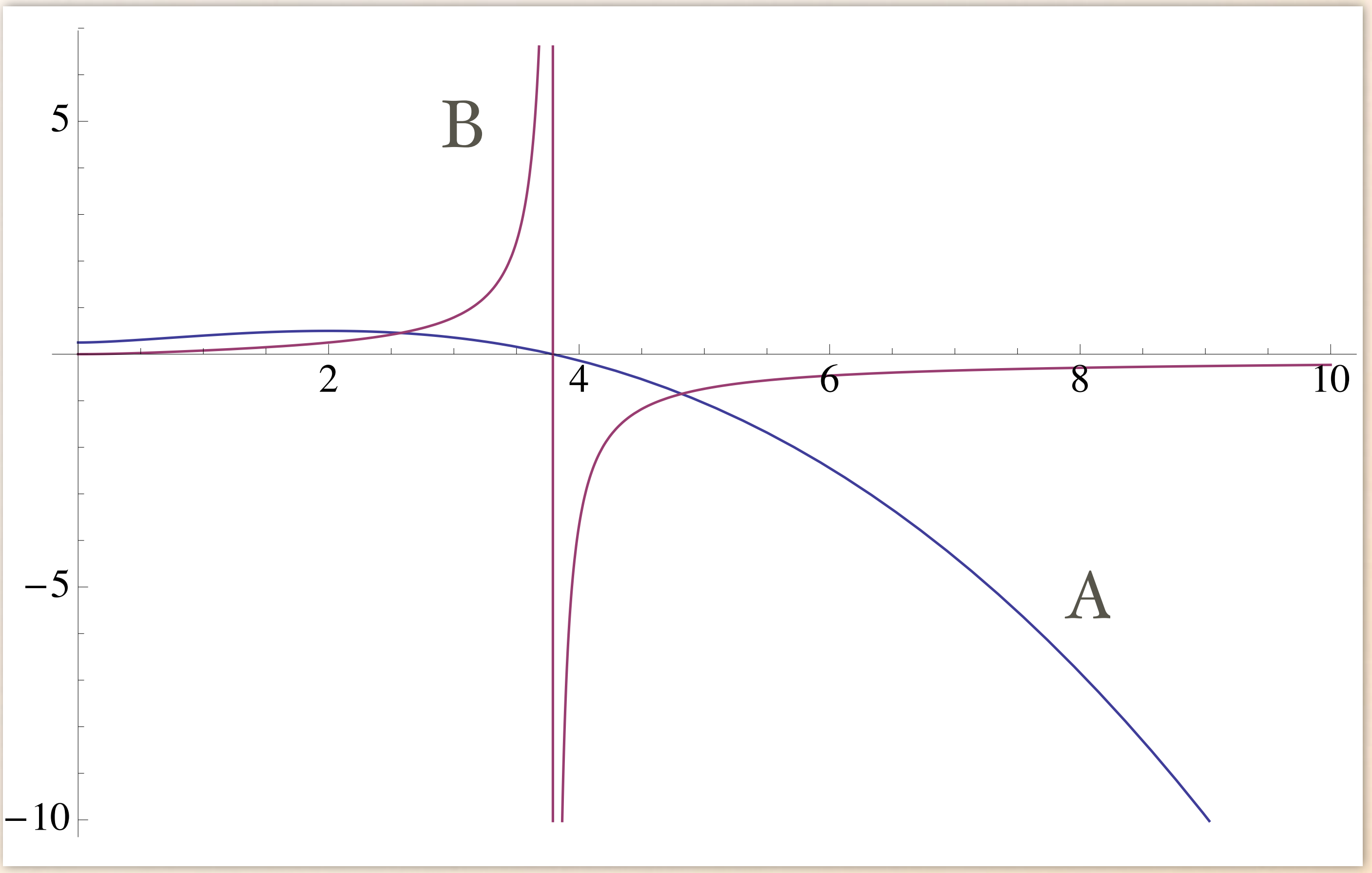
EXAMPLE

$$ds^2 = -Adt^2 + Bdr^2 + C\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

$$A = \frac{A_0}{4}\left[r^2 - 2r^2\log\left(\frac{r}{2}\right) + 4\right],$$

$$B = \frac{r^2}{2K_0\left[r^2 - 2r^2\log\left(\frac{r}{2}\right) + 4\right]},$$

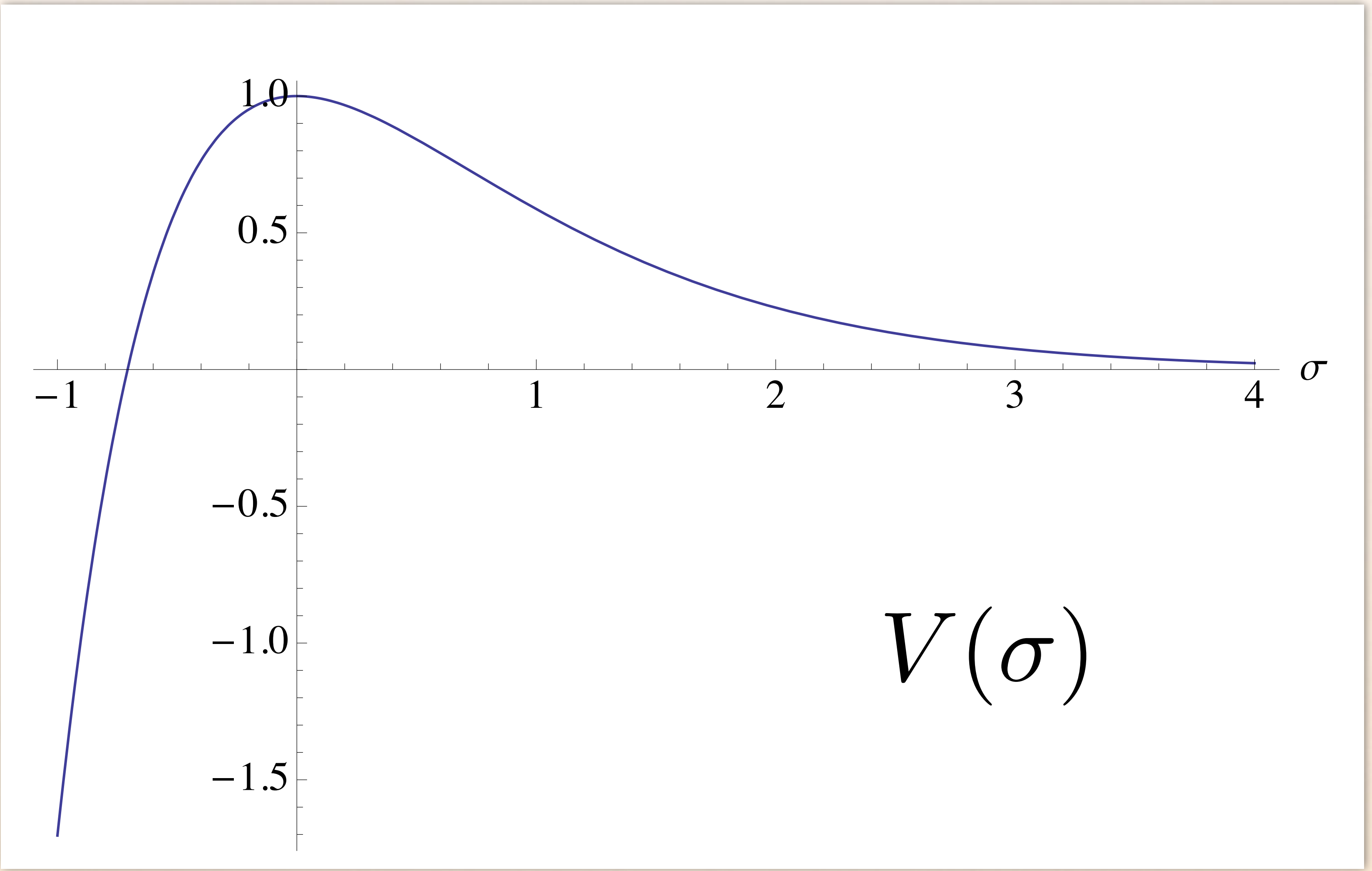
$$C = \frac{r^2}{4K_0}.$$



EXAMPLE

$$\sigma = \sqrt{2}\log\left(\frac{r}{2}\right),$$

$$V = K_0 e^{-\sqrt{2}\sigma}\left(\sqrt{2}\sigma + 1\right)$$



CONCLUSIONS

- We have presented a new method to reconstruct solutions of Einstein's equations using 1+1+2 scalar variables
- The new method allows to find a number of new exact spherically symmetric solutions in General Relativity
- We can impose some restrictions on the type of solution i.e. the implementation of the WEC and Asymptotic Flatness
- The new technique is also useful to find exact solutions in the presence scalar fields.
- Extensions?