Black hole entropy in d-dimensions through thin matter shells

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1 - Introduction

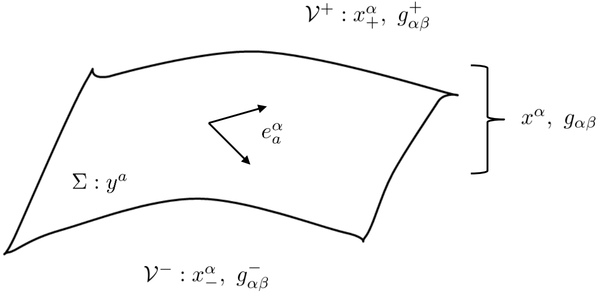
- Black holes are objects whose behavior involves both general relativity and quantum field theory. Consequently, it is important to fully understand these objects in the hopes finding some hints for a quantum gravity theory.
- Although black holes have been widely studied by now, there are still some open problems. One of those problems concerns the location of the degrees of freedom of the black hole entropy given by the Bekenstein-Hawking formula

$$S = \frac{1}{4}\frac{A}{l_p^2}\,.$$

- In this work we will try to answer this question by studying the thermodynamics of a thin matter shell in a d-dimensional spacetime, whose radius is taken to it's horizon limit.

The thin shell formalism

2.1 - Problem setup



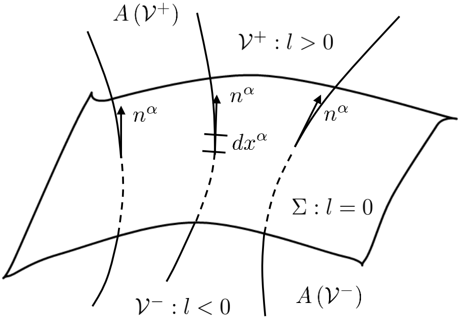
$$\text{Tangent vectors:} \quad e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

$$\text{Differential of a path:} \quad dx^\alpha = e^\alpha_a dy^a$$

$$\text{Metric at } \Sigma : \quad ds^2_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta = h_{ab} dy^a dy^b$$

where $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ is called the induced metric at Σ.

2.1 - Problem setup



Normal vectors: $dx^\alpha = n^\alpha dl \Rightarrow n_\alpha = \varepsilon \partial_\alpha l$

Jump of a quantity: $[A] \equiv A\left(V^+\right)\big|_\Sigma - A\left(V^-\right)\big|_\Sigma$

where $\varepsilon = n^\alpha n_\alpha$ can be $+1$ (timelike hypersurface) or -1 (spacelike hypersurface). Since x^α, y^a are continuous across Σ, we have

$$[n^\alpha] = [e^\alpha_a] = 0.$$

2.2 - First junction condition

The metric for the whole spacetime can be written as

$$g_{\alpha\beta} = \Theta(l) g^{+}_{\alpha\beta} + \Theta(-l) g^{-}_{\alpha\beta}$$

where $\Theta(l)$ is the Heaviside distribution. However, we need to guarantee that this is a valid solution of the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G_d T_{\alpha\beta}\,.$$

We do that by checking if the geometric quantities of interest make sense. The first problematic quantity is

$$g_{\alpha\beta,\gamma} = \Theta(l) g^{+}_{\alpha\beta,\gamma} + \Theta(-l) g^{-}_{\alpha\beta,\gamma} + \varepsilon \delta(l) \left[g_{\alpha\beta} \right] n_{\gamma},$$

since it will give rise to terms like $\Theta(l)\delta(l)$, unless we impose $[g_{\alpha\beta}] = 0$. A more useful form is

$$[h_{ab}] = 0$$

since is independent of the coordinates x^{α}. This is the first junction condition.

2.3 - Second junction condition

The remaining interesting term arises when calculating the Riemann tensor, which gives

$$R^{\alpha}{}_{\beta\gamma\delta} = \Theta(l) R^{+\alpha}{}_{\beta\gamma\delta} + \Theta(-l) R^{-\alpha}{}_{\beta\gamma\delta} + \delta(l) A^{\alpha}{}_{\beta\gamma\delta}$$

where

$$A^{\alpha}{}_{\beta\gamma\delta} = \varepsilon \left([\Gamma^{\alpha}{}_{\beta\delta}] \, n_{\gamma} - [\Gamma^{\alpha}{}_{\beta\gamma}] \, n_{\delta} \right)$$

is the singular part of the Riemann tensor, to which is assigned the stress-energy tensor

$$S_{ab} = -\frac{\varepsilon}{8\pi G_d} \left([K_{ab}] - [K] \, h_{ab} \right) \, .$$

Therefore, to eliminate the singular part of the curvature, we must have $S_{ab} = 0$, or equivantely,

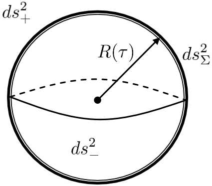
$$[K_{ab}] = 0.$$

This is called the second junction condition. If it not satisfied, than there is distribution of matter at Σ, called a thin matter shell, with stress-energy tensor S_{ab}.

The (d-1)-dimensional shell and the black hole limit

3.1 - The (d-1)-dimensional shell spacetime

$$ds_{\pm}^2 = -F_{\pm}(r)dt^2 + \frac{dr^2}{F_{\pm}(r)} + r^2 d\Omega_{d-2}^2, \qquad \mu = \frac{8\pi G_d}{(d-2)\Omega_{d-2}}$$



$$\begin{cases} F_+(r) = 1 - \frac{2m\mu}{r^{d-3}} & \Rightarrow \text{d-dimensional Schwarzschild} \\ F_-(r) = 1 & \Rightarrow \text{Minkowski} \end{cases}$$

$$ds_\Sigma^2 = -d\tau^2 + R^2 d\Omega_{d-2}^2 \quad \Rightarrow S_{d-2} \text{ metric at } \Sigma$$

Parametric equations of the shell: $\begin{cases} r \equiv R(\tau) \\ t \equiv T(\tau) \end{cases}$

Perfect fluid

$$S^a{}_b = (\sigma + p) u^a u_b + p h^a{}_b$$

3.1 - The (d-1)-dimensional shell spacetime

Non-null components of $S^a{}_b$ in the static limit:

$$\sigma = \frac{(d-2)}{8\pi G_d} \frac{1-\sqrt{1-\frac{2m\mu}{R^{d-3}}}}{R}$$

$$p = \frac{(d-3)}{8\pi G_d} \frac{\sqrt{1-\frac{2m\mu}{R^{d-3}}}-1}{R} + \frac{(d-3)}{8\pi G_d} \frac{\frac{m\mu}{R^{d-2}}}{\sqrt{1-\frac{2m\mu}{R^{d-3}}}}$$

Shell's rest mass: $M = A\sigma$ where A is the area of the shell $A = \Omega_{d-2}R^{d-2}$

Mechanical equations of the shell:

$$M = \frac{R^{d-3}}{\mu}(1-k)$$

$$p = \frac{(d-3)(1-k)^2}{2(d-2)\Omega_{d-2}Rk\mu}$$

$$k = \left(1-\left(\frac{r_+}{R}\right)^{(d-3)}\right)^{1/2}$$ (redshift factor)

$$r_+ = (2\mu m)^{1/(d-3)}$$

3.2 - The entropy of a (d-1)-dimensional shell

First Law of Thermodynamics: $TdS = dM + pdA$

The integrability condition must be satisfied ($\beta \equiv 1/T$)

$$\left(\frac{\partial \beta}{\partial A}\right)_M = \left(\frac{\partial \beta p}{\partial M}\right)_A$$

From it we can obtain the differential for the inverse temperature

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+} = \frac{(d-3)(1-k^2)}{2k^2 R}\beta$$

which can be integrated to give the analytic solution

$$\beta(r_+, R) = b(r_+)k \, .$$

Inserting this and the mechanical equations of the shell in the first law, we obtain

$$dS = \frac{(d-3)}{2\mu} b(r_+) r_+^{d-4} dr_+ \, .$$

To further advance, one needs to specify the function $b(r_+)$.

3.3 - Intrinsic stability of a (d-1)-dimensional shell

Thermodynamic stability equations

$$\partial_M^2 S \leq 0$$

$$\partial_A^2 S \leq 0$$

$$\left(\partial_M^2 S\right)\left(\partial_A^2 S\right) - \left(\partial_M \partial_A S\right)^2 \geq 0$$

The most simple suggestion for $b(r_+)$ is a power-law equation of the form

$$b(r_+) = \frac{\eta_0}{\hbar} \frac{r_+^{a(d-2)+1}}{l_p^{a(d-2)}}$$

Inserting in the differential for the entropy gives the explicit expression

$$S(M,R) = \frac{\eta}{(a+1)(d-2)} \left(\frac{r_+}{l_p}\right)^{(a+1)(d-2)} .$$

All together, the stability conditions imply the restrictions

$$a \leq 2\frac{d-3}{d-2} \quad \Rightarrow \quad 0 \leq k \leq \sqrt{\frac{d-3}{(2a+1)d-(4a+1)}}$$

$$a \geq 2\frac{d-3}{d-2} \quad \Rightarrow \quad \frac{a - 2\frac{d-3}{d-2}}{a+2} \leq k \leq \sqrt{\frac{d-3}{(2a+1)d-(4a+1)}}$$

3.4 - Entropy of a (d-1)-dimensional shell in the BH limit

- Take the shell to its horizon radius $R \to r_+$, which doesn't affect the form of the entropy. Quantum fields must be present and their backreaction diverges unless we choose the inverse of the Hawking temperature for $b(r_+)$;
- Hence, fix $b(r_+) = \frac{1}{T_H}$ where $T_H = \frac{\hbar}{4\pi}\frac{(d-3)}{r_+}$.

Inserting this specific form for $b(r_+)$ in the differential of the entropy and integrating, leads to the entropy

$$S(M,R) = \frac{\Omega_{d-2}\, r_+^{d-2}}{4G_d\hbar} = \frac{A}{4l_p^{d-2}}$$

and so we obtain in this limit the Bekenstein-Hawking entropy of a d-dimensional Schwarzschild black hole. Note that this corresponds to the case $a = 0$ and $\eta = (d-2)\Omega_{d-2}/4$ in the previous ansatz.

4 - Conclusions

- The entropy differential for a thin shell in a d-dimensional Schwarzschild spacetime
$$dS = \frac{(d-3)}{2\mu} b(r_+) r_+^{d-4} dr_+$$
was obtained, where an arbitrary function $b(r_+)$ related to the temperature of the shell naturally appeared.
- By fixing the temperature with a phenomenological function, it was possible to obtain an exact expression for the entropy of the shell, which in turn led to an intrinsic stability analysis of its thermodynamics.
- When the shell was taken to the black hole limit, it returned the Bekenstein-Hawking entropy of the black hole in that corresponding spacetime. This leads to the suggestion that the degrees of freedom of a black hole are located at the horizon. Other spacetime choices can also be shown to give the Bekenstein-Hawking entropy for the black holes in the corresponding spacetimes, which supports the conclusion made here.