Dynamics and thermodynamics of a rotating thin shell in a (2+1)-dimensional asymptotically AdS spacetime

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Preface

- Work done in colaboration with
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- Research done in group GRIT of CENTRA (Centro Multidisciplinar de Astrofísica) at the Physics Department of IST.
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Overview

1. Introduction

- Martinez (1996) presented the thermodynamics of a thin matter shell in (3+1)-dimensions.
- Lemos and Quinta (2014) obtained the thermodynamics of a thin matter shell in a (2+1)-dimensional asymptotically AdS spacetime.
- Both obtained the Bekenstein-Hawking entropy of a black hole when the shell is taken to its own gravitational radius.
- The interest in (2+1)-dimensional spacetimes suffered an increment after the discovery of a black hole solution in spacetimes asymptotically AdS, the Bañados-Teitelbom-Zanelli (BTZ) black hole.

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- In 2+1 dimensions, Einstein's equation with cosmological constant is

$$G_{\alpha\beta} = 8\pi G_3 T_{\alpha\beta} + \Lambda g_{\alpha\beta}$$

- The exterior metric is given by the BTZ line element written in coordinates (t_o, r, ϕ)

$$ds_o^2 = -\left(\frac{r^2}{l^2} - 8G_3 m\right) dt_o^2 + \frac{dr^2}{\left(\frac{r^2}{l^2} - 8G_3 m + \frac{16J^2 G_3^2}{r^2}\right)} - 8G_3 J dt_o d\phi + r^2 d\phi^2, \quad r \geq R$$

where $l^2 = -1/\Lambda$.

- We introduce the horizon radii $r_\pm$

$$r_\pm = 2l\sqrt{G_3 m \pm \sqrt{G_3^2 m^2 - \frac{J^2 G_3^2}{l^2}}}$$

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- The interior metric is $m = 0$ BTZ spacetime written in coordinates (t_i, ρ, ψ)
$$ds_i^2 = g^-_{\alpha\beta} dx^\alpha dx^\beta = -\frac{\rho^2}{l^2} dt_i^2 + \frac{l^2}{\rho^2} d\rho^2 + \rho^2 d\psi^2, \quad \rho \leq R$$
- The induced metric, as viewed from the exterior region, is
$$ds_\Sigma^2 = -\left(\frac{R^2}{l^2} - 8G_3 m\right) dt_o^2 - 8G_3 J dt_o d\phi + R^2 d\phi^2$$
- We define the new polar coordinate ψ by
$$\psi = \phi - \Omega t_o$$
- The line element is diagonal if
$$\Omega = \frac{4G_3 J}{R^2}$$
- Therefore, in coordinates $y^a = (t, \psi)$, where $t \equiv t_o$, the induced metric is
$$ds_\Sigma^2 = -\left(\frac{R^2}{l^2} - 8G_3 m + \frac{16J^2 G_3^2}{R^2}\right) dt^2 + R^2 d\psi^2$$

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- The induced metric, as viewed from the interior region, is

$$ds_\Sigma^2 = -\frac{R^2}{l^2}\, dt_i^2 + R^2 d\psi^2$$

- Applying the first junction condition, which states that the induce metric must be the same on both sides of the shell, yields

$$\left(\frac{R^2}{l^2} - 8G_3 m + \frac{16J^2G_3^2}{R^2}\right) dt_o^2 = \frac{R^2}{l^2}\, dt_i^2$$

- On the other hand, the second junction condition gives the components of the stress-energy tensor

$$S^a_b = -\frac{1}{8\pi G_d}\left([K^a_b] + [K]h^a_b\right)$$

where $K_{ab} = n_{\alpha;\beta} e^\alpha_a e^\beta_b$, with greek indices running from 0 to 2 and latin indices from 0 to 1, n_α is the normal vector to the shell and e^α_a are the two tangent vectors.

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- We want to put these components in a perfect fluid form

$$S^{ab} = \lambda u^a u^b + p\left(h^{ab} + u^a u^b\right)$$

- The shell must move rigidly in the ψ direction with an uniform angular velocity ω implying that the velocity vector is

$$u^a = \gamma(t^a + \omega\psi^a)$$

where $t^a = \frac{\partial y^a}{\partial t}$ and $\psi^a = \frac{\partial y^a}{\partial \psi}$.

- It is useful to define the redshift $k(R, r_+, r_-)$ as

$$k = \frac{R}{l}\sqrt{\left(1 - \frac{r_+^2}{R^2}\right)\left(1 - \frac{r_-^2}{R^2}\right)}$$

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

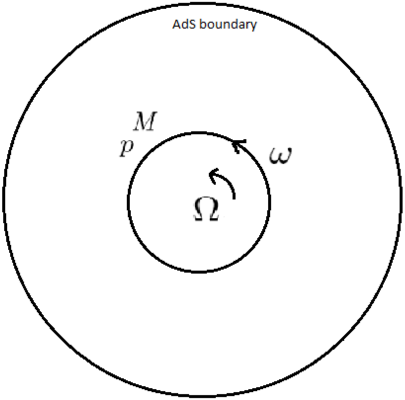
- Therefore

$$\lambda = \frac{1}{8\pi G_3 l}\left(1 - \frac{l}{R}k\right) + \frac{r_+^2 r_-^2}{R^4}\frac{\left(1 - R^2/r_+^2\right)}{8\pi G_3 l^2 k/R}$$

$$\begin{aligned} p =& \frac{1}{8\pi G_3 l}\left[\frac{R}{lk}\left(1 - \frac{r_+^2 r_-^2}{R^4}\right) - 1\right] \\ &+ \frac{r_-^2}{R^2}\frac{\left(R^2 - r_+^2\right)}{8\pi G_3 l^2 k/R\left(r_+^2 - r_-^2\right)}\left(-\frac{2r_-^2 r_+^2}{R^4} + \frac{r_+^2 + r_-^2}{R^2}\right) \end{aligned}$$

$$\omega = \frac{r_-}{r_+ l} - \frac{r_- r_+}{lR^2}$$

2. Dynamics of rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime



3. Thermodynamics of slowly rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- The thermodynamic variables are (M, A, J).
- $M \equiv 2\pi\lambda R$, $A \equiv 2\pi R$ and J is the angular momentum.
- The first law of thermodynamics is written as

  $$TdS = dM + p\,dA - \omega dJ$$

  where T is the temperature as measured locally.

- In order to dS be an exact differential $(\beta \equiv 1/T)$

$$\left(\frac{\partial\beta}{\partial A}\right)_{M,J} = \left(\frac{\partial\beta p}{\partial M}\right)_{A,J}$$

$$\left(\frac{\partial\beta}{\partial J}\right)_{M,A} = -\left(\frac{\partial\beta\omega}{\partial M}\right)_{A,J}$$

$$\left(\frac{\partial\beta p}{\partial J}\right)_{M,A} = -\left(\frac{\partial\beta\omega}{\partial A}\right)_{M,J}$$

3. Thermodynamics of slowly rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- The relevant equations in the slowly rotating limit, $J << ml$, are

$$M = 2\pi R\lambda = \frac{R}{4G_3 l}\left(1 - \frac{l}{R}k\right)$$

$$p = \frac{1}{8\pi G_3 l}\left(\frac{R}{lk} - 1\right)$$

$$\omega = \frac{r_-}{r_+ l} - \frac{r_- r_+}{lR^2}$$

- The first integrability condition is equivalent to

$$\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \frac{R}{l^2k^2}\beta \implies \beta(R, r_+) = k(R, r_+)\, b(r_+)$$

3. Thermodynamics of slowly rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- From the integrabiliy conditions we also get

$$\omega(R, r_+, r_-) = \frac{\omega_0(r_+, r_-)}{k} - \frac{r_+ r_-}{lR^2 k}$$

- From the dynamics ω_0 is fixed to be

$$\omega_0(r_+, r_-) = \frac{r_-}{lr_+}$$

- Combining the last results gives

$$dS = \frac{b(r_+) r_+}{4Gl^2} dr_+$$

- Taking the black hole limit fixes

$$b(r_+) = \frac{1}{T_{\mathrm{H}}} = \frac{2\pi l^2}{\hbar} \frac{1}{r_+} \implies S(r_+) = \frac{\pi r_+}{2l_{\mathrm{p}}^2} = \frac{A_+}{4l_{\mathrm{p}}^2}$$

with $l_{\mathrm{p}} = \sqrt{G\hbar}$ and $A_+ = 2\pi r_+$.

3. Thermodynamics of slowly rotating thin matter shells in a (2+1)-dimensional asymptotically AdS spacetime

- The local intrinsic thermodynamical stability of the shell is guaranteed as long as the entropy of the system stays in a maximum
- We prescribe the function $b(r_+)$ as

$$b(r_+) = 4G\alpha l^2 \frac{r_+^a}{l_{\mathrm{p}}^{a+2}}$$

- This gives

$$S = \frac{\alpha}{a+2}\left(\frac{r_+}{l_{\mathrm{p}}^{a+2}}\right)^{a+2}$$

- Therefore, this thermodynamic system is stable if obeys

$$a = -1 \quad \text{and} \quad \frac{R}{r_+} \to \infty$$

4. Conclusions

- By using the junction conditions we have obtained the rest mass, pressure and angular velocity (measured at infinity) of a rotating thin matter shell in a $(2+1)$-dimensional asymptotically AdS spacetime.
- Futhermore, we have studied the thermodynamics of that shell in the slowly rotating limit.
- Inserting those quantities in the first law of thermodynamics led us to the entropy for the thin matter shell up to an arbitrary function of the shell's gravitational radius.
- The matter contained in the shell specifies this function.

4. Conclusions

- We also take the shell to its gravitational radius which fixes the temperature to be the Hawking temperature and we recover the Bekenstein-Hawking entropy of a BTZ black hole

$$S_{\mathrm{BH}} = \frac{\pi r_+}{2 l_{\mathrm{p}}^2} = \frac{A_+}{4 l_p^2} \, .$$

- This seems to show some evidence that the degrees of freedom of a black hole are situated at its event horizon.
- The general case is an open problem to solve.
- The difficulty relies on interpretating the pressure terms that appear in the first law of thermodynamics.