ON WAVE PROPAGATION IN SCHWARZSCHILD SPACETIME

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18.12.2014

VII BLACK HOLES WORKSHOP,
AVEIRO

RESEARCH TRAINING GROUP
Models of Gravity

Universität Bremen*

*EXZELLENT.
Gewinnerin in der
Exzellenzinitiative

CENTER OF
APPLIED SPACE TECHNOLOGY
AND MICROGRAVITY

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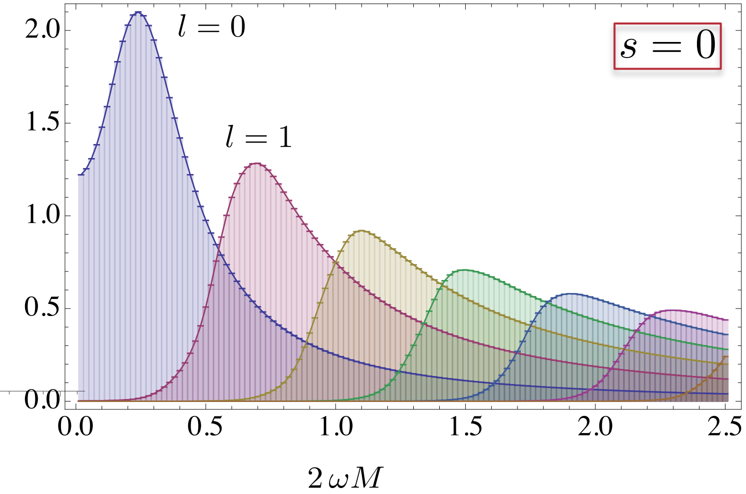
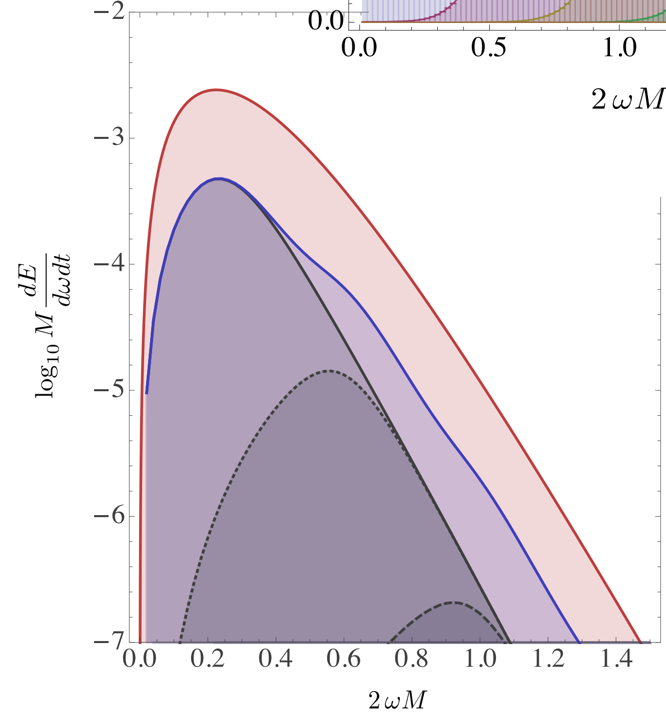
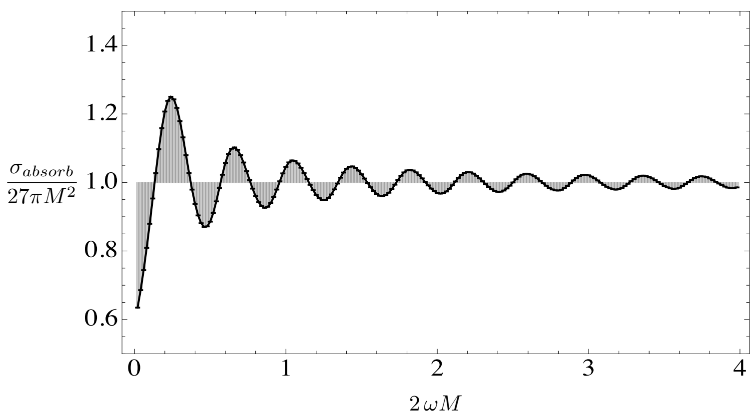
why wave propagation in black hole spacetimes?

more information than particle motion

- interference effects
- black hole scattering
- (quasi-)normal modes (QNM)
- Hawking radiation
- dark matter models
- black hole bombs
- …







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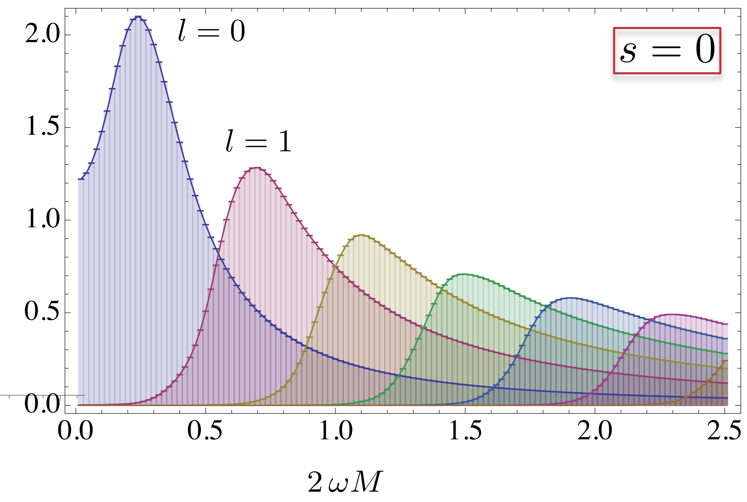

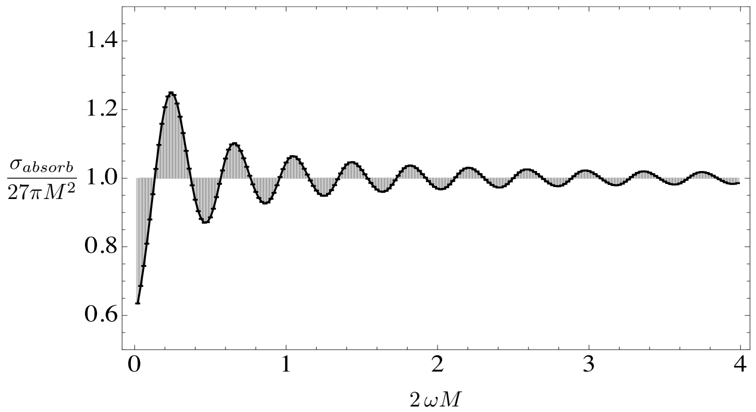

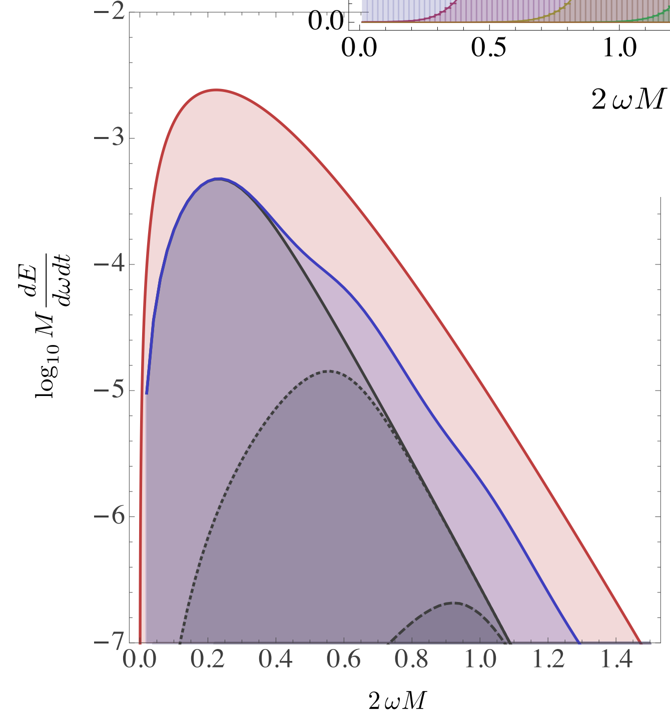

guidance

- choose a field (bosonic, fermionic) with spin s
- find the wave equation on fixed background spacetime:

spin	field	wave equation
0	scalar boson	Klein-Gordon
1	photon	Maxwell
2	graviton	linear Einstein eq.
1/2	fermion	Dirac

- choose a coordinate system, separate the angular and temporal parts
- solve the remaining radial equation with boundary conditions
  - scattering
  - Hawking radiation
  - QNM, ...

- create pretty pictures...

basics

- use Schwarzschild coordinates and units where $G = c = 1$

$$g = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \,, \quad f(r) = 1 - \frac{2M}{r}$$

- measure distances in units of the Schwarzschild radius $r_s = 2M$

$$\boxed{g = -\left(\frac{r-1}{r}\right) dt^2 + \left(\frac{r}{r-1}\right) dr^2 + r^2 d\Omega^2}$$

- tortoise coordinate $r_* = r + \log(r-1)$ shifts the horizon to $-\infty$

- (ingoing) Eddington-Finkelstein coordinates $v = t + r_* = t + r + \log(r-1)$

$$\boxed{g = -\left(\frac{r-1}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2}$$

- PG coordinates $v = t + 2\sqrt{r} + \log\left|\frac{\sqrt{r}-1}{\sqrt{r}+1}\right|$

$$\boxed{g = -dv^2 + \left(dr + \frac{1}{\sqrt{r}}dv\right)^2 + r^2 d\Omega^2}$$

wave equations

- spin s = 0: Klein-Gordon equation

$$\Box\Phi \equiv \nabla^\mu\nabla_\mu\Phi = \sqrt{-g}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}\; g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right]\Phi = 0$$

- spin s = 1: (source-free) Maxwell equations

$$\left.\begin{aligned}\nabla_\mu F^{\mu\nu} &= 0\\ \epsilon^{\mu\nu\rho\sigma}\nabla_\nu F_{\rho\sigma} &= 0\end{aligned}\right\} \quad \frac{r-1}{r^3}\overbrace{\left\{\cot\vartheta\frac{\partial\Pi}{\partial\vartheta}+\frac{\partial^2\Pi}{\partial\vartheta^2}+\frac{1}{\sin^2\vartheta}\frac{\partial^2\Pi}{\partial\varphi^2}\right\}}^{-\mathbf{L}^2\Pi}+\frac{\partial^2\Pi}{\partial r_*^2}-\frac{\partial^2\Pi}{\partial t^2}=0$$

transform into single scalar equation: Debye equation (Stephani, 1974)

- spin s = 2: linear perturbation theory, vacuum field equation (Regge + Wheeler, 1957)

$$g = \bar{g} + h, \quad \delta R_{\mu\nu} = 0$$

reduce all equations to the general time dependent Regge-Wheeler equation (RWE)

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Regge-Wheeler equation

- separate the angular part by expansion in spherical harmonics (e.g. for s=0)

$$\Phi = \Phi(t, r_*, \vartheta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\Phi_l(t, r_*)}{r} Y_{lm}(\vartheta, \varphi)$$

$$\boxed{\Rightarrow \frac{\partial^2 \Phi_{sl}(t, r_*)}{\partial r_*^2} - \frac{\partial^2 \Phi_{sl}(t, r_*)}{\partial t^2} - V_{sl}(r)\Phi_{sl}(t, r_*) = 0}$$

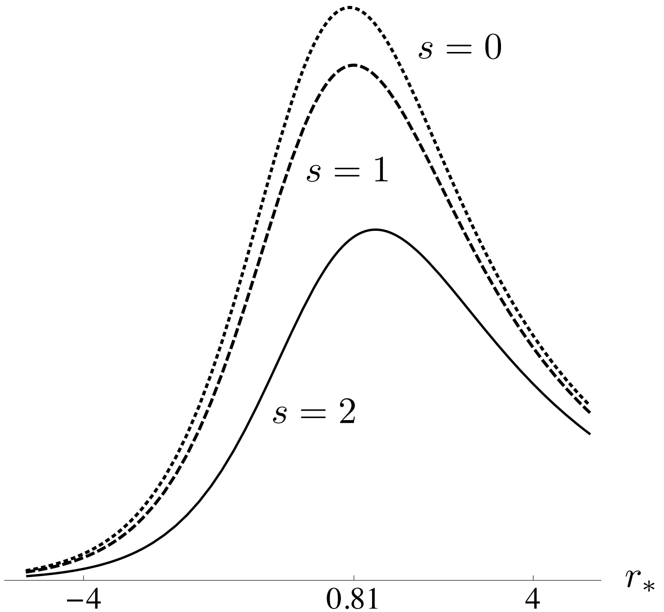
- the spin-dependent potential decays for $r_* \to \pm\infty$

$$V_{sl}(r) = \left(\frac{r-1}{r^3}\right)\left(l(l+1) + \frac{1-s^2}{r}\right)$$

- separate the time dependence

$$\Phi_{sl}(t, r_*) = \mathrm{e}^{-i\omega t} R_{\omega sl}(r_*)$$

$$\boxed{\Rightarrow \frac{\mathrm{d}^2 R_{\omega sl}(r_*)}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{sl}(r)\right] R_{\omega sl}(r_*) = 0}$$



Regge-Wheeler potential for $l = 2$

Regge-Wheeler equation II

$$\boxed{\frac{\mathrm{d}^2 R_{\omega sl}(r_*)}{\mathrm{d}r_*^2} + \left[\omega^2 - V_{sl}(r)\right] R_{\omega sl}(r_*) = 0}$$

- asymptotic solutions

$$R_{\omega sl}(r_*) = A(\omega,s,l)\,\mathrm{e}^{-i\omega r_*} + \cancel{B(\omega,s,l)}\,\mathrm{e}^{i\omega r_*},\ r_* \to -\infty$$
$$R_{\omega sl}(r_*) = C(\omega,s,l)\,\mathrm{e}^{-i\omega r_*} + D(\omega,s,l)\,\mathrm{e}^{i\omega r_*},\ r_* \to \infty$$

- define the IN-mode

$$R^{IN}_{\omega sl}(r_*) = \begin{cases} A^{trans}_{\omega l}\,\mathrm{e}^{-i\omega r_*}, & r_* \to -\infty \\ A^{in}_{\omega l}\,\mathrm{e}^{-i\omega r_*} + A^{out}_{\omega l}\,\mathrm{e}^{i\omega r_*}, & r_* \to +\infty \end{cases}$$

- scattering: ingoing, transmitted and reflected wave parts

$$A^{trans}_{\omega l} \equiv 1 \qquad \mathbb{T}_{\omega l} = |T|^2_{\omega l} = \left|\frac{1}{A^{in}_{\omega l}}\right|^2, \quad \mathbb{R}_{\omega l} = |R|^2_{\omega l} = \left|\frac{A^{out}_{\omega l}}{A^{in}_{\omega l}}\right|^2$$

- QNM: purely ingoing at the horizon and outgoing at spatial infinity

- problem: no analytic solution of the RWE due to the potential!
  - approximations: Born, WKB, Poeschl-Teller potential,...
  - numerical solutions

wave equation, again ...

$$\frac{\partial^2\Phi_{sl}(t,r_*)}{\partial r_*^2} - \frac{\partial^2\Phi_{sl}(t,r_*)}{\partial t^2} - V_{sl}(r)\Phi_{sl}(t,r_*) = 0$$

- step back: use **other coordinates** in the **wave equation**
- in usual Schwarzschild coordinates we get (separate time dependence as before)

$$\boxed{\left[r(r-1)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (r-1)\frac{\mathrm{d}}{\mathrm{d}r} + \left[r^3\omega^2 - (r-1)\left(l(l+1)+\frac{1-s^2}{r}\right)\right]\right]R_{\omega sl}(r)=0}$$

- introduce new coordinates $v = f(r,t)\,,\ u = g(r,t)$

$$\left[2\left(\frac{r-1}{r}\right)^2 f'g' - 2\dot f\dot g\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial u\partial v} + \left[\left(\frac{r-1}{r}\right)^2 f'^2 - \dot f^2\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial v^2} + \left[\left(\frac{r-1}{r}\right)^2 g'^2 - \dot g^2\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial u^2}$$
$$+\left[\left(\frac{r-1}{r}\right)^2 f'' + \left(\frac{r-1}{r^3}\right)f' - \ddot f\right]\frac{\partial\Phi_{sl}(u,v)}{\partial v} + \left[\left(\frac{r-1}{r}\right)^2 g'' + \left(\frac{r-1}{r^3}\right)g' - \ddot g\right]\frac{\partial\Phi_{sl}(u,v)}{\partial u} = V_{sl}(r)\Phi_{sl}$$

- EF coordinates

$$\boxed{\left[r(r-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (1-2i\omega r^2)\frac{\mathrm{d}}{\mathrm{d}r} - \left[l(l+1)+\frac{1-s^2}{r}\right]\right]R_{\omega sl}(r)=0}$$

- PG coordinates

$$\boxed{\left[r(r-1)\frac{\partial^2}{\partial r^2} + \left[1-2i\omega r^{3/2}\right]\frac{\partial}{\partial r} + \left[r^2\omega^2 + i\omega\frac{\sqrt{r}}{2} - l(l+1) - \frac{1-s^2}{r}\right]\right]R_{\omega sl}(r)=0}$$

$$\Phi_{sl}(v,r) = \mathrm{e}^{-i\omega v}R_{\omega sl}(r)$$

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wave equation, again ...

$$\frac{\partial^2 \Phi_{sl}(t,r_*)}{\partial r_*^2} - \frac{\partial^2 \Phi_{sl}(t,r_*)}{\partial t^2} - V_{sl}(r)\Phi_{sl}(t,r_*) = 0$$

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$$\left[2\left(\frac{r-1}{r}\right)^2 f'g' - 2\dot{f}\dot{g}\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial u\partial v} + \left[\left(\frac{r-1}{r}\right)^2 f'^2 - \dot{f}^2\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial v^2} + \left[\left(\frac{r-1}{r}\right)^2 g'^2 - \dot{g}^2\right]\frac{\partial^2\Phi_{sl}(u,v)}{\partial u^2}$$

$$+\left[\left(\frac{r-1}{r}\right)^2 f'' + \left(\frac{r-1}{r^3}\right) f' - \ddot{f}\right]\frac{\partial\Phi_{sl}(u,v)}{\partial v} + \left[\left(\frac{r-1}{r}\right)^2 g'' + \left(\frac{r-1}{r^3}\right) g' - \ddot{g}\right]\frac{\partial\Phi_{sl}(u,v)}{\partial u} = V_{sl}(r)\Phi_{sl}$$

- EF coordinates

$$\boxed{\left[r(r-1)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (1-2i\omega r^2)\frac{\mathrm{d}}{\mathrm{d}r} - \left[l(l+1) + \frac{1-s^2}{r}\right]\right] R_{\omega sl}(r) = 0}$$

- PG coordinates

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$$\Phi_{sl}(v,r) = \mathrm{e}^{-i\omega v} R_{\omega sl}(r)$$

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radial equation in EF coordinates

$$\frac{\mathrm{d}^2 R_{\omega sl}(r)}{\mathrm{d}r^2} + \left[\frac{1-2i\omega r^2}{r(r-1)}\right] \frac{\mathrm{d}R_{\omega sl}(r)}{\mathrm{d}r} - \left[\frac{l(l+1)+(1-s^2)/r}{r(r-1)}\right] R_{\omega sl}(r) = 0$$

- belongs to class of Heun equations: singly confluent case (CHE)
- two regular singularities at $r = 0, 1$ and one irregular singularity at $r = \infty$
- at singularities: two local solutions in terms of Frobenius solutions or asymptotic series

- consider the region $r \in [1, \infty]$, local solutions at the horizon are

$$R^{I}_{\omega sl}(r;1) = r^{s+1}\mathrm{HeunC}(2i\omega, -2i\omega, 2s, -2\omega^2, s^2 - l(l+1) + 2\omega^2, 1-r)$$
$$R^{II}_{\omega sl}(r;1) = r^{s+1}(r-1)^{2i\omega}\mathrm{HeunC}(2i\omega, 2i\omega, 2s, -2\omega^2, s^2 - l(l+1) + 2\omega^2, 1-r)$$

- the confluent Heun function can locally be defined as a regular Frobenius series

$$\mathrm{HeunC}(a,b,c,d,e,z) := \sum_{k=0}^{\infty} u_k(a,b,c,d,e)\, z^k$$

with the normalization $\mathrm{HeunC}(a,b,c,d,e,0) = 1$

$$\frac{d}{dz}\mathrm{HeunC}(a,b,c,d,e,z)\,|_{z=0} = \frac{(1+c-a)b+c-a+2e}{2(b+1)}$$

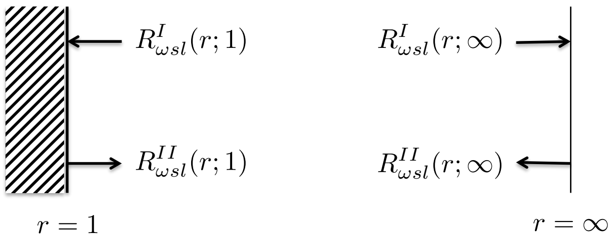
radial equation in EF coordinates II

- asymptotic solutions at spatial infinity

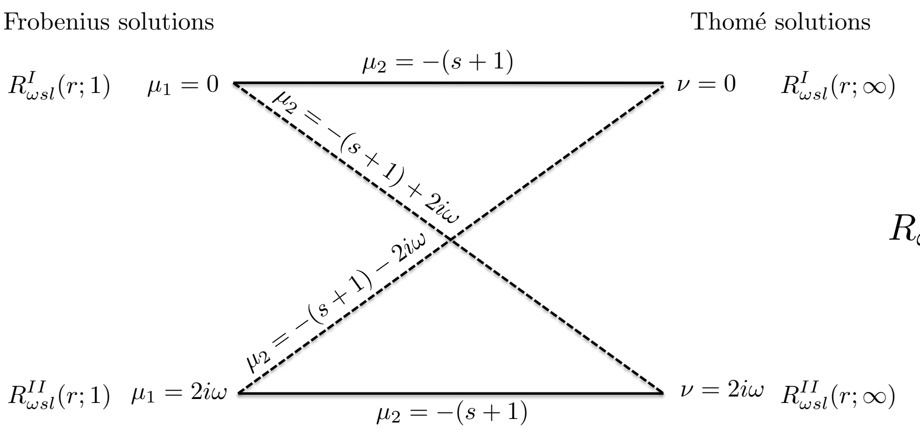
$$R^{I}_{\omega sl}(r;\infty) = \mathrm{e}^{2i\omega(r+\log(r))} \sum_{k=0}^{\infty} d_k(\omega,s,l)\; r^{-k}$$

$$R^{II}_{\omega sl}(r;\infty) = \sum_{k=0}^{\infty} c_k(\omega,s,l)\; r^{-k}$$

behavior of local solutions



- problem: representation of HeunC-function before only useful for $r \in]0,2[$
- scattering, QNM, Hawking radiation: boundary conditions also at spatial infinity!
- can "combine" local solutions from two singularities: CTCP (Slavyanov + Lay, 2002)
  - analytic continuation of Frobenius solutions and repr. of HeunC on larger domain
  - 4 possibilities of "connecting" the local solutions



$$R_{\omega sl}(r) = \mathrm{e}^{\nu(r-1)}(r-1)^{\mu_1} r^{\mu_2+s+1} \sum_{k=0}^{\infty} \xi_k(\omega,s,l)\, \frac{(r-1)^k}{r^k}$$

What are these solutions good for?

scattering

- define a plane wave as input and evaluate scattered wave part

$$\phi = \phi_{plane} + f(\vartheta)\frac{e^{i\omega r}}{r}$$

- transmission and reflection amplitudes

$$\mathbb{T}_{\omega l} = |T|^2_{\omega l} = \left|\frac{1}{A^{in}_{\omega l}}\right|^2, \quad \mathbb{R}_{\omega l} = |R|^2_{\omega l} = \left|\frac{A^{out}_{\omega l}}{A^{in}_{\omega l}}\right|^2$$

- absorption cross section

$$\sigma_{abs} = \frac{\pi}{\omega^2}\sum(2l+1)|T_{\omega l}|^2$$

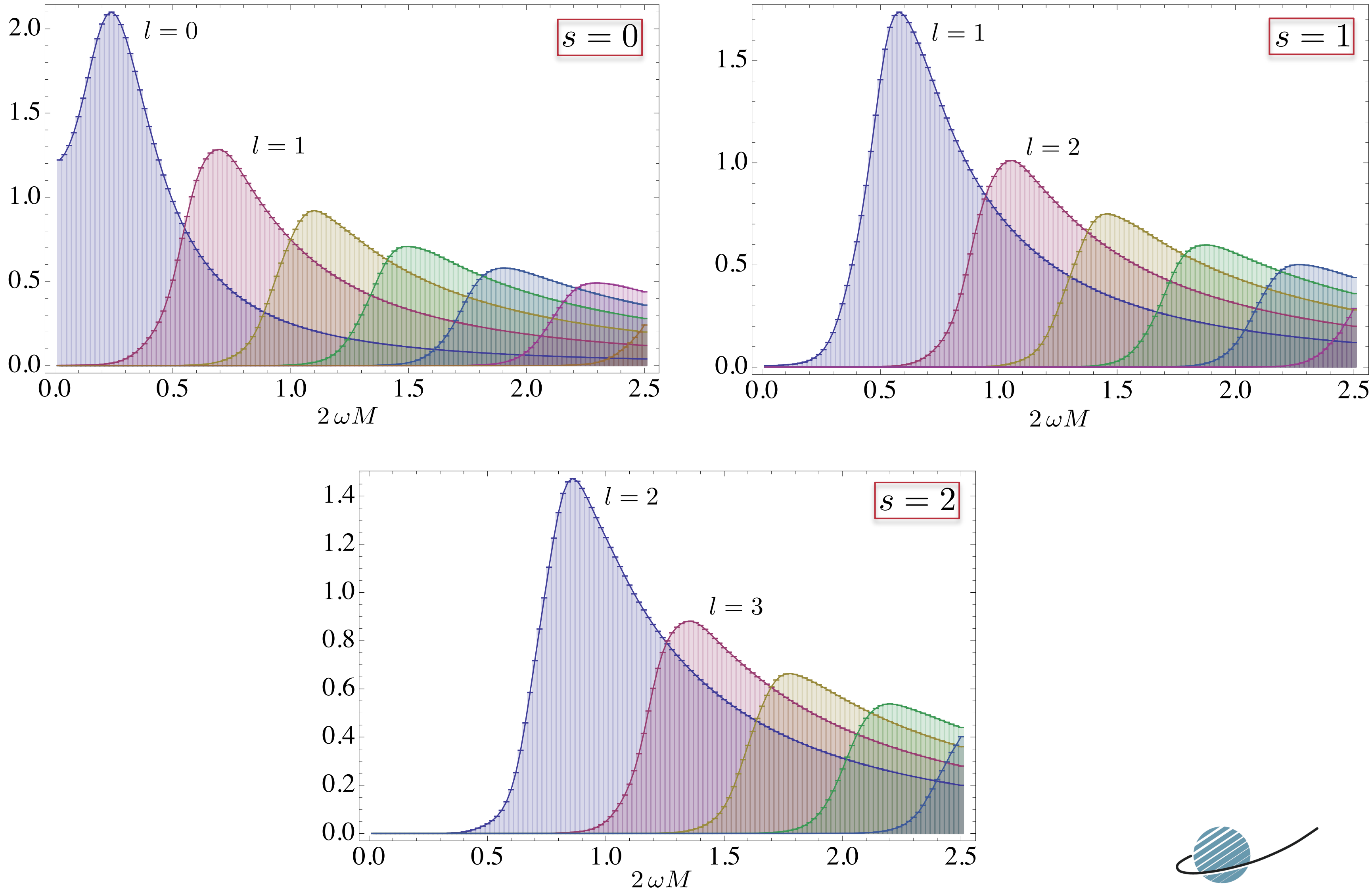
- scattering amplitude

$$f(\vartheta) = \frac{1}{2i\omega}\sum(2l+1)(e^{2i\delta_l}-1)P_l(\cos\vartheta), \quad e^{2i\delta_l} = (-1)^{l+1}\frac{A^{out}_{\omega l}}{A^{in}_{\omega l}}$$

- scattering cross section

$$\frac{d\sigma}{d\Omega} = |f(\vartheta)|^2$$

scattering: partial absorption cross sections $\frac{\sigma^l_{absorb}}{A_h}$



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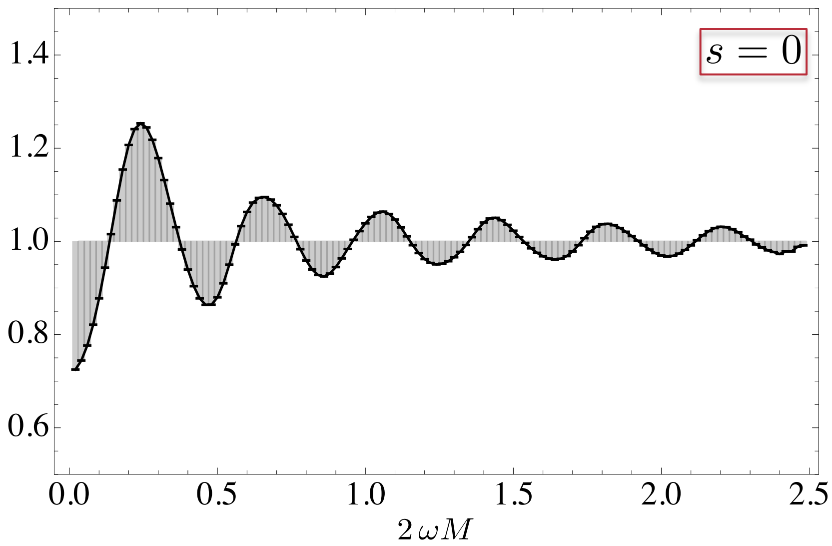
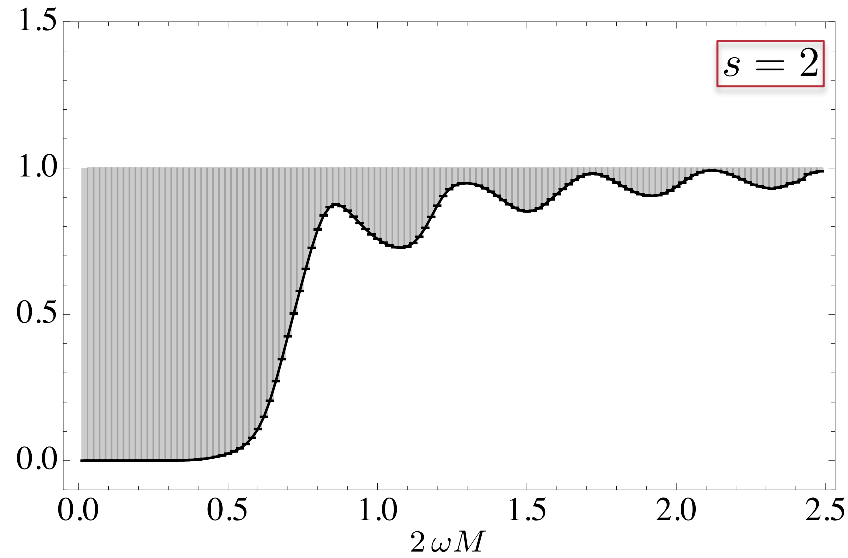
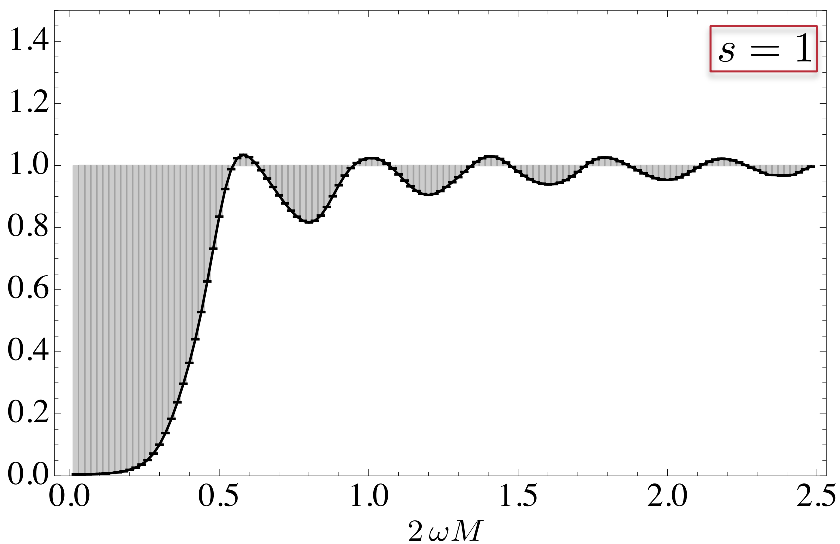
scattering: total absorption cross sections $\frac{\sigma_{absorb}}{27\pi M^2}$







- scalar cross section does not vanish for small frequencies
- for large frequencies the geometrical optics limit is valid
- the cross section gives the known geometrical value

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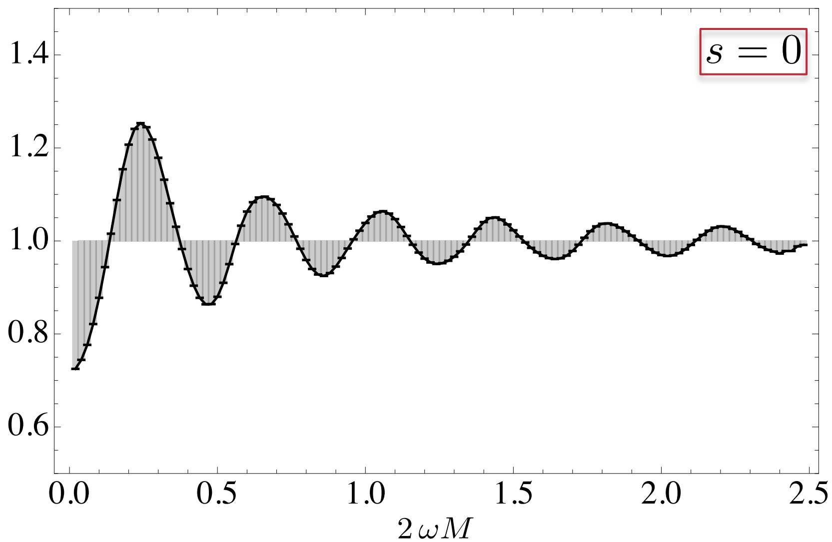
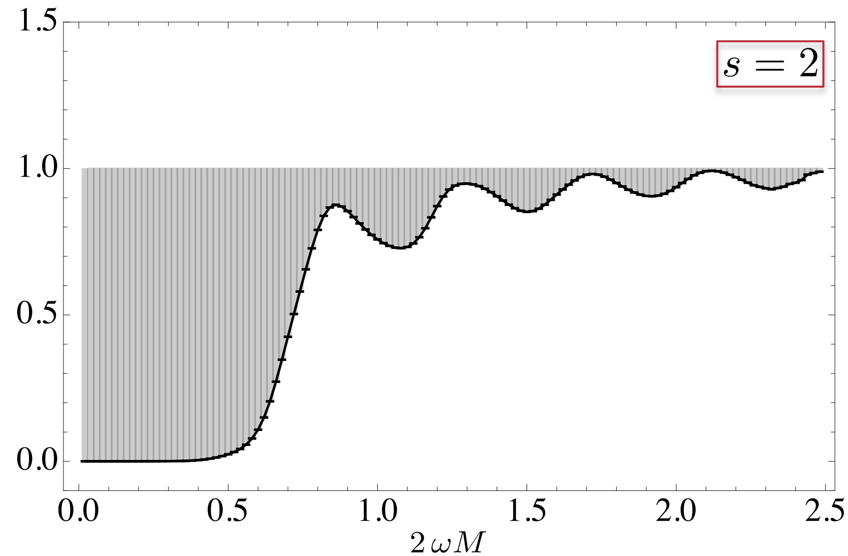
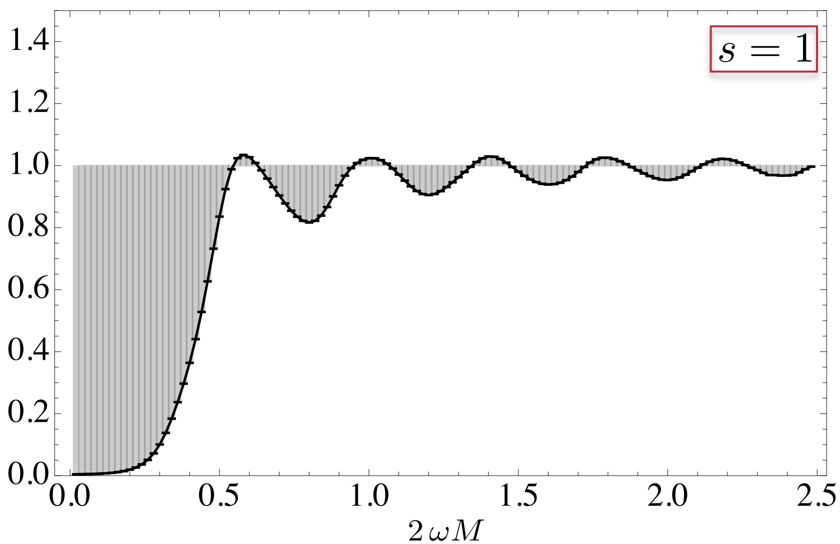
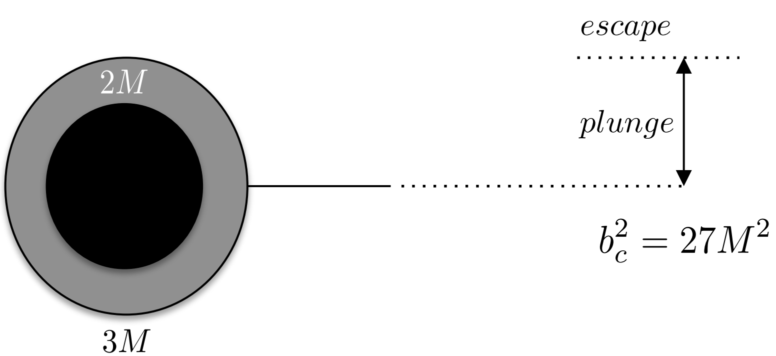
scattering: total absorption cross sections $\frac{\sigma_{absorb}}{27\pi M^2}$









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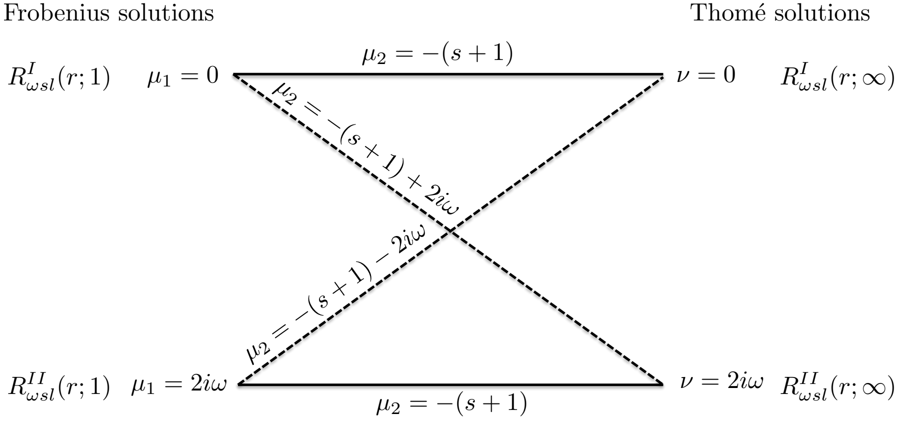
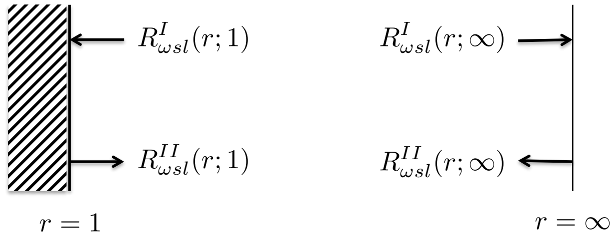
QNM

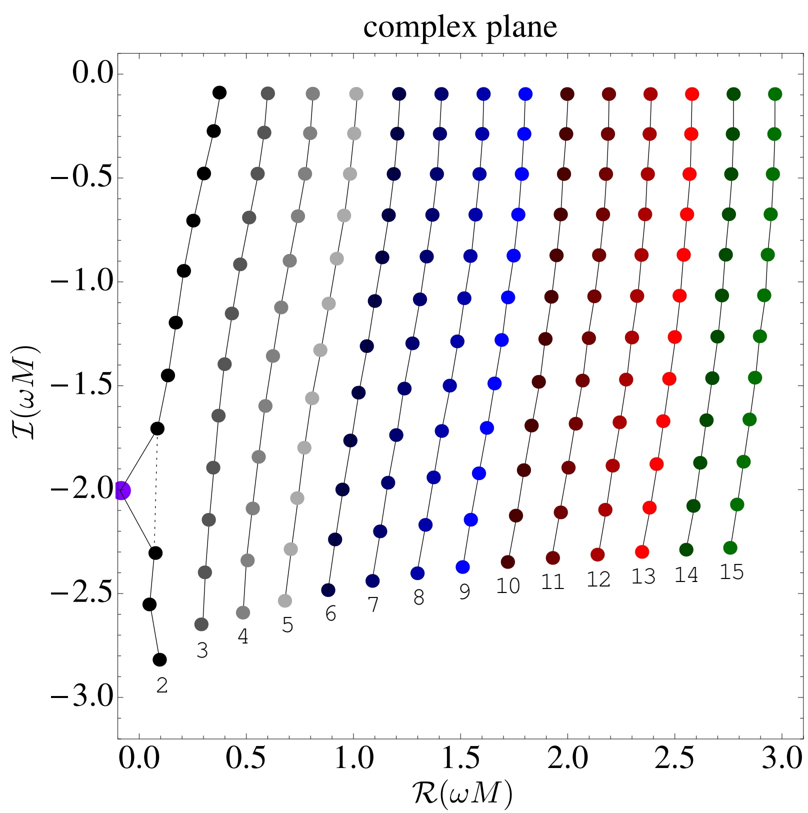




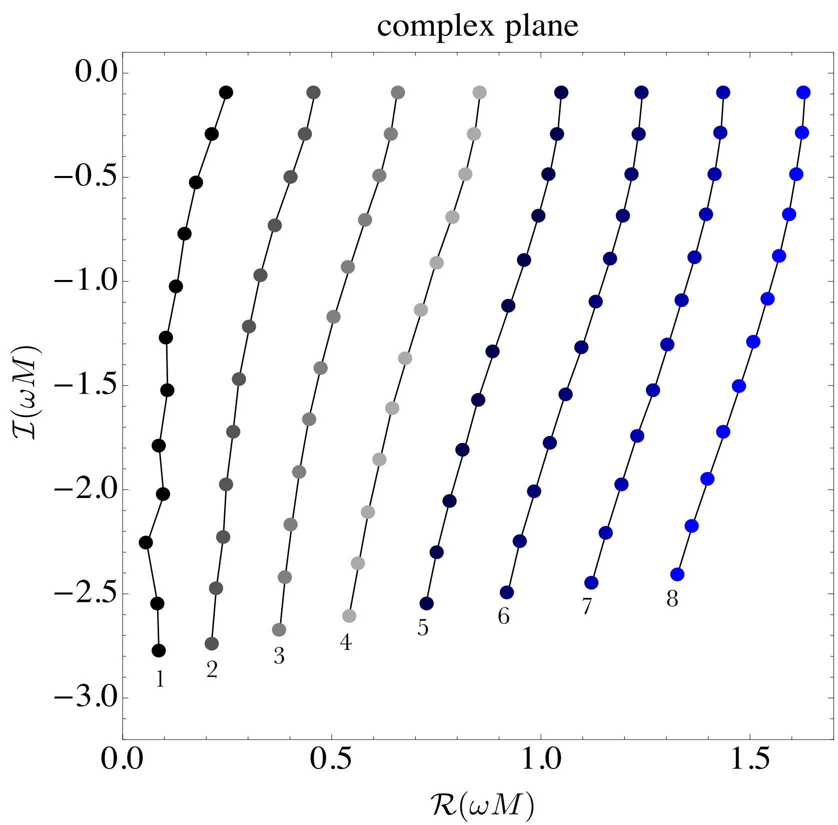
$$R_{\omega sl}(r) = \mathrm{e}^{\nu(r-1)}(r-1)^{\mu_1} r^{\mu_2+s+1} \sum_{k=0}^{\infty} \xi_k(\omega, s, l) \frac{(r-1)^k}{r^k}$$

- convergence at spatial infinity: Leaver's continued fraction equation (Leaver, 1985)
- the $\xi_k(\omega, s, l)$ have to be a minimal solution of the 3-term recurrence relation
- for each spin and wave mode we can find the QNM-frequencies
  - complex frequencies with the right sign of imaginary part
  - special QNM with vanishing real part (no oscillation)

QNM



$\boxed{s = 2}$



$\boxed{s = 1}$

Hawking radiation

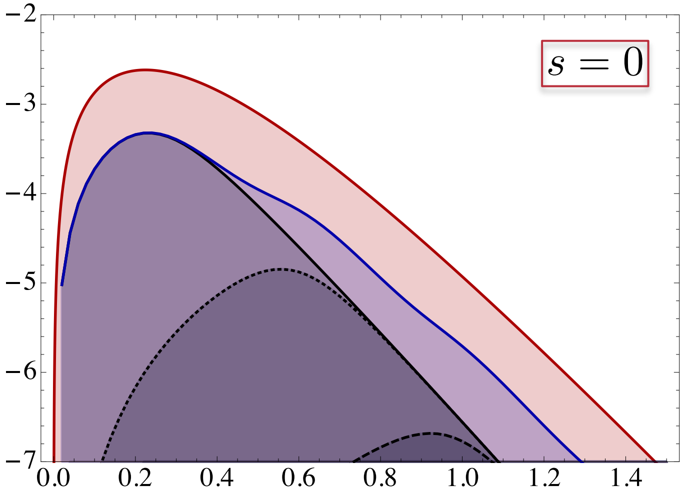
- method of Damour and Ruffini (1976)
- at the horizon: do not neglect outgoing solution
  - inside: antiparticle with negative energy
  - outside: particle escaping
- the solution is not regular at the horizon but admits a continuation
- relative amplitude gives mean particle number/probability flux carried away

$$N_{\omega slm} = \frac{\Gamma_{\omega slm}}{\mathrm{e}^{8\pi\omega M} \pm 1}$$

- derive a radiated energy per unit of time and frequency interval

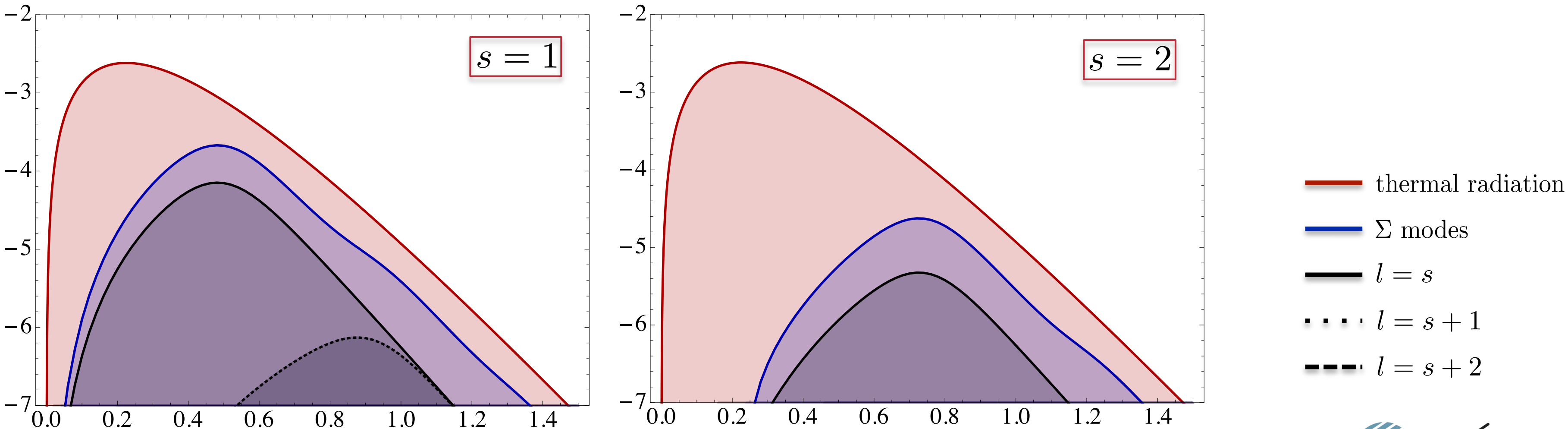
$$\frac{dE}{d\omega dt} = \frac{1}{2\pi}\omega \sum_{l,m} N_{\omega lm}$$

Hawking radiation $\log_{10} M \frac{dE}{d\omega dt}$

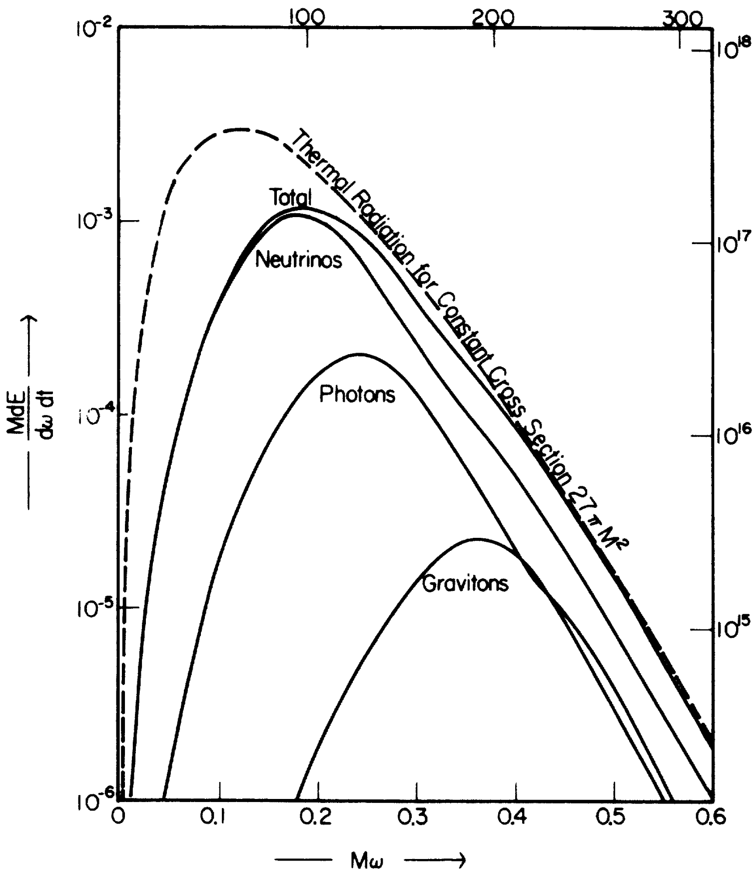


radiation spectra as received at spatial infinity

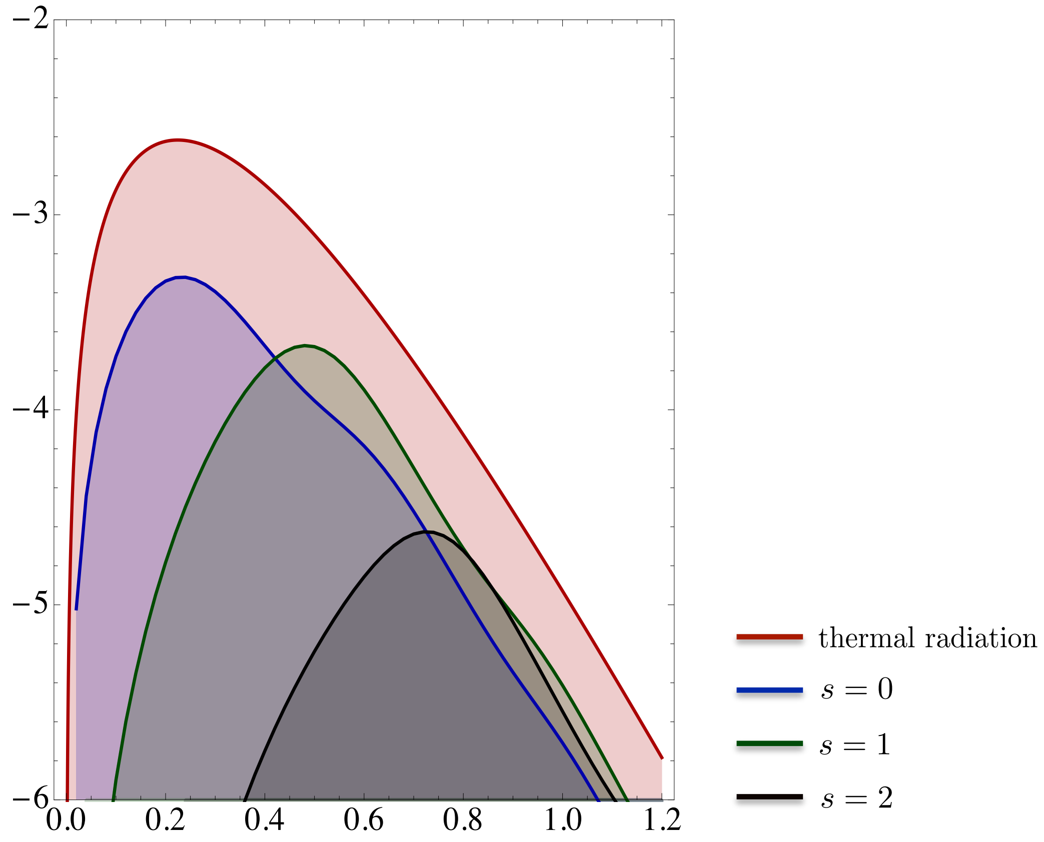
- first mode dominant
- small bumps from higher mode contributions
- blue: sum over all modes weighted with $2l + 1$
- clear deviation from thermal radiation



Hawking radiation II $\log_{10} M \frac{dE}{d\omega dt}$



Don S. Page, 1976



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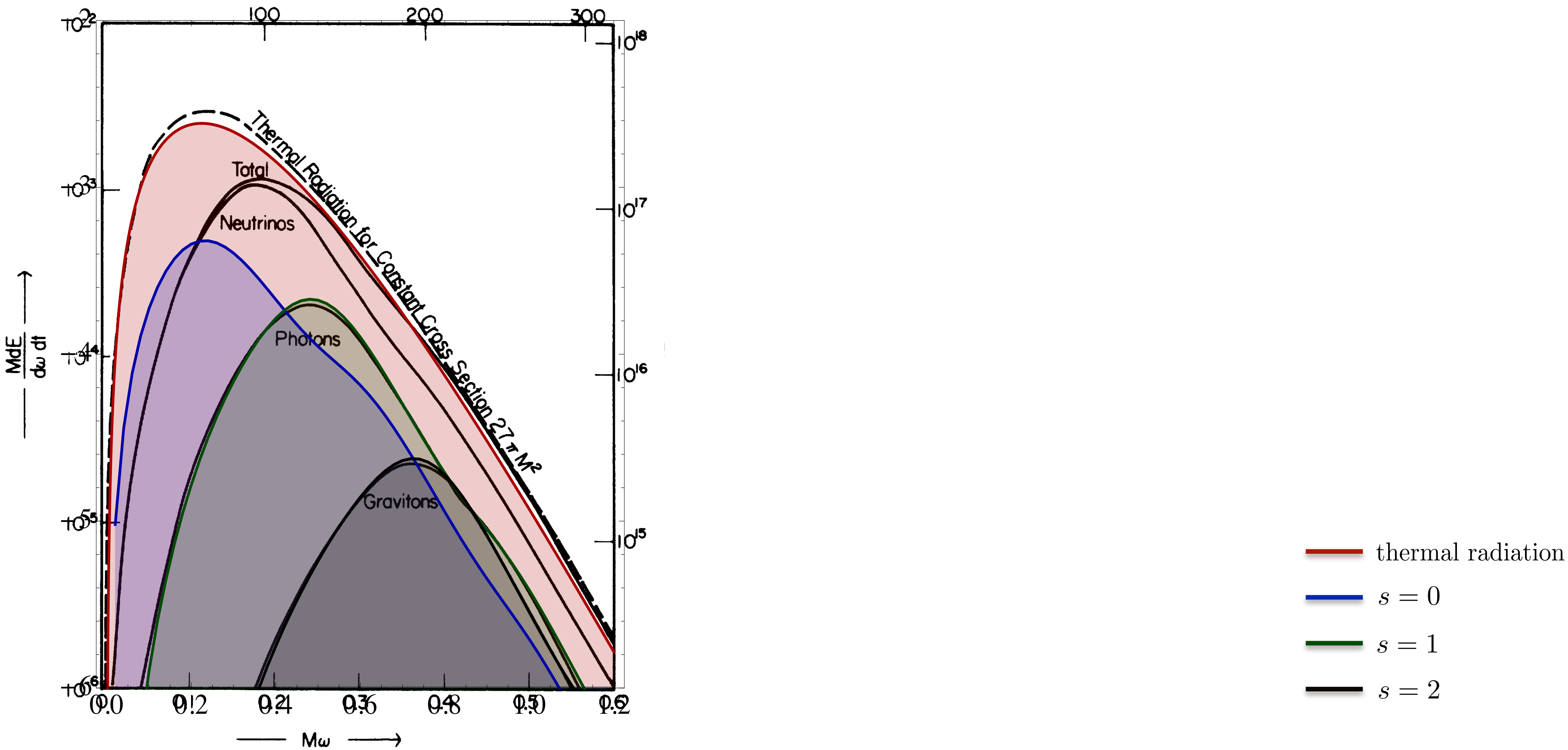

Don S. Page, 1976

some thoughts

- possible to obtain identities of HeunC for better representation?
  - faster convergence?
  - combine different representation (Frobenius series, hypergeom. series, Coulomb wave function series, ...)?
- analytic equations for graybody factors?
  - maybe as integrals from the above idea
  - express them as basis transitions coefficients (analog to QNM calc. of Fiziev)
- other black hole spacetimes
  - rotation, charge of BH no problem for solution scheme, compare to prev. numerical solutions of radial equations
  - regular black hole spacetimes?
- complete the Hawking radiation spectra for more species of particles
  - include massive fields
  - change of background metric due to emission
  - lifetime calculation

(some) references

Heun class of equations

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