5.4 Interaction picture & Wick's theorem

1. Using the free scalar field expansion in the Heisenberg picture, and the commutation relations of the Hamiltonian operator with the creation and anihilation operators show that:

$$\dot{\phi}_H = [H, \phi_H] = \Pi_H(\vec{x}, t)$$

and that

$$\dot{\Pi}_H = (\nabla^2 - m^2)\phi_H$$

Use the previous relations to show that $\phi_H$obeys the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi_H = 0$$

2. Consider a generic quantum system where the Hamiltonian can be split as

$$H = H_0 + H_{int}$$

   (a) Define the interaction picture assuming an expansion around H_0.

   (b) Show that on general grounds (independently of the specific quantum degrees of freedom we work with) that the formal solution for the evolution operator in the interaction picture is

$$U(t_0, t_1) = \mathrm{T} \exp\left(\int_{t_0}^{t_1} dt\, H_I(t) \right)$$

   where you should define the interaction Hamiltonian $H_I$appropriately in the interaction picture.

   (c) verify directly by expanding the T-exponentials, to second order in the interaction, that

$$U(t_0, t_1) = U(t_0, \tau) U(\tau, t_1)$$

3. Show that the time ordered product $\mathrm{T}(\phi(x)\phi(y))$ and the normal ordered product $:\phi(x)\phi(y):$ are both symmetric under the interchange $x \leftrightarrow y$. Recalling the Wick theorem for this case, deduce that $D_F(x - y)$ has the same property.

4. Check Wick's theorem for 3-fields

$$\mathrm{T}(\phi(x_1)\phi(x_2)\phi(x_3)) =: \phi(x_1)\phi(x_2)\phi(x_3) : + \phi(x_1) D_F(x_2 - x_3) + \phi(x_2) D_F(x_3 - x_1) + \phi(x_3) D_F(x_1 - x_2)$$