5.3 Quantized field theory

1. Consider a quantized vibrating string, whose Hamiltonian is expressed as a mode sum

$$H = \sum_n \frac{1}{2} p_n^2 + \frac{\omega_n^2}{2} q_n^2$$

where ω_n is the frequency of the nth mode and q_n, p_n are respectively the generalised coordinate and momentum in the classical theory of the corresponding mode. Assuming the standard poisson bracket of the classical theory is promoted to the following commutation relations of the corresponding operators

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \& \quad [q_n, p_m] = \delta_{nm}$$

show that:

(a) if you define the ladder operators

$$a_n = \frac{i}{\sqrt{1}} \left(\frac{1}{\sqrt{\omega_n}} p_n - i\sqrt{\omega_n} q_n \right)$$

$$a_n^\dagger = -\frac{i}{\sqrt{1}} \left(\frac{1}{\sqrt{\omega_n}} p_n + i\sqrt{\omega_n} q_n \right)$$

then they obey standard decoupled harmonic oscilator commutation relations. Obtain such relations.

(b) Show that the Hamiltonian can then be expressed in the form

$$H = \sum_n \frac{1}{2} \omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n \right)$$

2. Consider the free real Klein Gordon theory detailed in the lectures with canonical commutation relations for the field operator with its canonically conjugate momentum. Using the field expansions introduced in the lectures (in ladder operators) and imposing the commutation relations below, show that the canonical commutation relations are obeyed.

$$[a_{\vec{p}}, a_{\vec{p}'}] = \left[a_{\vec{p}}^\dagger, a_{\vec{p}'}^\dagger \right] = 0 \quad \& \quad \left[a_{\vec{p}}, a_{\vec{p}'}^\dagger \right] = (2\pi)^3 \delta^{(3)} \left(\vec{p} - \vec{p}' \right)$$

Contrast with the previous exercise and explain why in this case we have a continuum of operators labelled by momentum, whereas for the vibrating string we only have a countable tower of operators.

3. Verify that the energy momentum 4-vector operator for the scalar Klein-Gordon free theory can be put in the form

$$P^\mu = \int \frac{d^3\vec{p}}{(2\pi)^3} p^\mu a_{\vec{p}}^\dagger a_{\vec{p}}.$$

From this expression and the expression for the Heisenberg fiel operator, show that

$$[P^\mu, \phi_H(x^\alpha)] = -i\partial^\mu \phi_H(x^\alpha)$$

Verify that this is consistent with

$$\phi_H(x^\alpha) = e^{iP.x} \phi_H(0) e^{-iP.x}$$

4. The operator N is defined in the quantised Klein-Gordon free theory as

$$N = \int \frac{d^3\vec{p}}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}}$$

Calculate the commutators with the creation and anihilation operators and deduce N is the particle number operator.

5. Let $\phi_H(x^\alpha)$ be the Heisenberg field of the Klein-Gordon free theory, and consider the 1-particle states defined as

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}} a^\dagger_{\vec{p}} |0\rangle$$

Show that

$$\langle 0| \phi_H(x^\alpha) |\vec{p}\rangle = e^{ip.x}$$

and

$$\langle \vec{q}\, |\vec{p}\rangle = 2E_{\vec{p}}(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$$

and deduce that

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}\, |\vec{q}\rangle = |\vec{q}\rangle$$

Considering that we have chosen the normalisation of the states to be Lorentz invariant, how can you use this result to show that the following integration measure is also Lorentz invariant

$$\frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}$$