Last Lecture:

- Motivation of looking @ multi-particle relativistic field theories.
- General E-L equation → trajectories in classical theory
- Proof of Noether's theorem.

Today: • Example 2 → $T^{\mu}{}_{\nu}$ & Hamiltonian

• Klein-Gordon field (Why?) & its free classical solution.

• Quantisation of the free K-G theory

Example 2: Infinitesimal translations in space-time (valid for any field)

Consider set of infinitesimal space-time translations:

$$x^{\mu} \to x'^{\mu} = x^{\mu} - a^{\mu}$$

For a generic field:

inverse transf.

$$\bullet\ \phi(x^{\alpha}) \to \phi'(x^{\alpha}) = \phi(x^{\alpha} + a^{\alpha})$$

$$\simeq \phi(x^{\alpha}) + a^{\nu} \underbrace{\partial_{\nu}\phi}_{(\Delta\phi)_{\nu}} + O(a^2)$$

$$\bullet\ \mathcal{L}(x) \to \mathcal{L}(x+a) \simeq \mathcal{L}(x) + a^{\nu}\partial_{\nu}\mathcal{L}(x) = \mathcal{L}(x) + a^{\nu}\partial_{\mu}\underbrace{(\delta^{\mu}_{\nu}\mathcal{L})}_{(X^{\mu})_{\nu}}$$

Note: a^{ν} is a vector ⇒ vector of conserved currents ⟺ tensor with 2 indices.

Using Noether current def.

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\Delta\phi)_\nu - (X^\mu)_\nu$$

label for each a^ν

$$\Leftrightarrow \boxed{T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}}$$

Energy momentum tensor

$$\Downarrow$$

$$\partial_\mu T^{\mu}{}_{\nu} = 0$$

$$\Downarrow$$

4-momentum $P^\mu \equiv \int d^3x \, T^{0\mu}$ is conserved

- Space translations symmetry $\Leftrightarrow$ 3-momentum conserved

$$P^i \equiv \int T^{0i} \, d^3x$$

- Time translations symmetry $\Leftrightarrow$ Energy conservation!

$$P^0 \equiv E = \int d^3x \, T^{00} = \int d^3x \, \mathcal{H}$$

$\mathcal{H} \rightarrow$ Hamiltonian density

⚠ Important when quantizing the theory

To see T^{00} is actually ~~Lagrangian~~ $\mathcal{H}$ consider Klein Gordon theory example:

$$L = \int d^3x \underbrace{\left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right)}_{\mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = \frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi \right) = \partial_\mu \phi$$

$$T^{00} = (\partial_t\phi)^2 - \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}m^2\phi^2\right)$$

$$= \underbrace{\frac{1}{2}\partial_t\phi}_{\text{energy cost of time evolution}} + \underbrace{\frac{1}{2}|\nabla\phi|^2}_{\text{energy cost of space deformation}} + \underbrace{\frac{1}{2}m^2\phi^2}_{\text{energy cost of having the field present at all}}$$

(mass term, we will see later)

Also recall: Lagrange/Hamiltonian Mechanics evolution.

→ Euler-Lagrange eqs $\Rightarrow$ "trajectories" of $\phi(x^\mu)$
$\mathcal{L}(\phi)$ given $\phi(0, x^i)$ $\dot\phi(0, x^i)$

→ Hamilton eqs $\Rightarrow$ trajectories of ϕ, π given
$H(\phi, \pi_\phi)$ $\phi(0, x^i), \pi(0, x^i)$
(π_ϕ: momentum)

$$\dot\phi = \{\phi, \mathcal{H}\}$$

$$\dot\pi = \{\pi, \mathcal{H}\}$$

In general $U \to \dfrac{dU}{dt} = \{U, H\} + \dfrac{\partial U}{\partial t}$

$\mathcal{H}$ generates time evolution/translations, so no wonder it is conserved if problem invariant under time translations.

To check $T^{00} \equiv \mathcal{H}$, let's define π:

$$L(t) = \int d^3x\, \mathcal{L}(x,t) \qquad \left(\sum_x \mathcal{L}(x,t)\right)$$
$$\mathcal{L}(\phi, \dot\phi, \nabla\phi)$$

Dynamical variable is now ϕ (whereas C. Mech. r & $p = \frac{\partial L}{\partial \dot r}$)

So naturally here: $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$

$$\Rightarrow H = \int d^3x \, \{ \Pi \dot{\phi} \quad - L$$
$$\Big(\sum_x \Pi_x \dot{\phi}_x \Big)$$

$$= \int d^3x \, \big(\underbrace{\Pi \dot{\phi} - \mathcal{L}} \big)$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \mathcal{L}$$
$$= \Pi \dot{\phi} - \mathcal{L} \quad ⚠$$

In summary, time translation symmetry $\Rightarrow$ conserved charge is energy

& $H = T^{00}$ generates time evolution.

$\boxed{\S 3}$ Classical solutions of the (free) Klein-Gordon theory

$$L = \int d^3x \Big(\underbrace{\tfrac{1}{2} \partial_\mu \phi \, \partial^\mu \phi}_{T} - \underbrace{\tfrac{1}{2} m^2 \phi^2}_{V(\phi)} \Big)$$

$L = T - V$

Why such a simple case $(V(\phi))$?

- Physically describes free (non-interacting) field theory ($\Leftrightarrow$ free multi-particle th. in QFT)

- Simple & exact solutions ⚠ which represents 1 single degree of freedom (1 type of particles)
  $\rightarrow$ Later can introduce other field (vector, spinor etc.) but basic structure is ~~s~~ similar.

- When introducing later interactions ($V(\phi_1, \phi_2, \ldots)$) can expand ~~exact~~ ~~around~~ perturbatively around free field solutions!
- Why classical first? Analogous to classical trajectory of particles ⇒, (think of it as classical trajectory of multi-particle th.)

Classical solution ("trajectory")

$$E\text{-}L \Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Leftrightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

Clearly $\phi = e^{-ip_\mu x^\mu}$ is a solution

$$\Leftrightarrow \quad p_\mu p^\mu + m^2 = 0 \quad \Leftrightarrow \quad \boxed{E^2 = |\vec{p}|^2 + m^2}$$

→ Energy momentum relation for particle with mass m.
(but so far classical wave)

Or, similarly, we can just Fourier transform space

$$\phi = a(t)\, e^{i\vec{p}\cdot\vec{x}}$$

$$\Rightarrow \boxed{\ddot{a} + (p^2 + m^2)\, a(t) = 0}$$

→ Harmonic oscillator! Of course $a(t) = e^{\pm i E_{\vec{p}} t}$
$E_{\vec{p}} = \sqrt{m^2 + p^2}$

⇒ Free K.G. field = ~~set~~ infinite set of decoupled harmonic oscillators labelled by $\vec{p}$ (or $\vec{k}$) with frequency $E_{\vec{k}}$

General solution:

$$\phi(x^\alpha) = \int d^3p \left(f(p)\, e^{-ip_\mu x^\mu} + g(p)\, e^{ip_\mu x^\mu} \right) \Big|_{p^0 \equiv E_p = \sqrt{p^2+m^2}}$$

Thus general solution, given

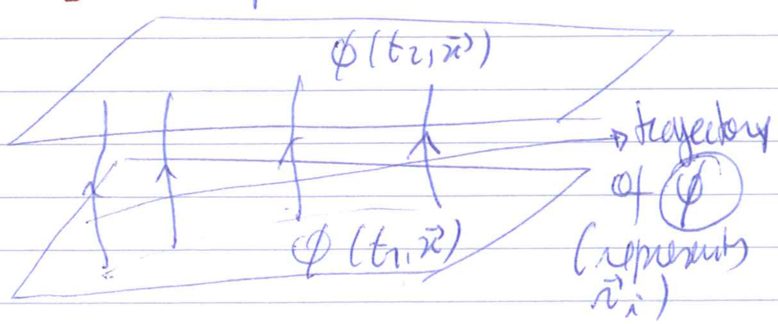
$\phi(0, x^i)$, $\dot\phi(0, x^i) \Rightarrow f, g$ defined

- Can be localized (physical) unlike plane waves
- if ϕ real $\Rightarrow g = f^*$

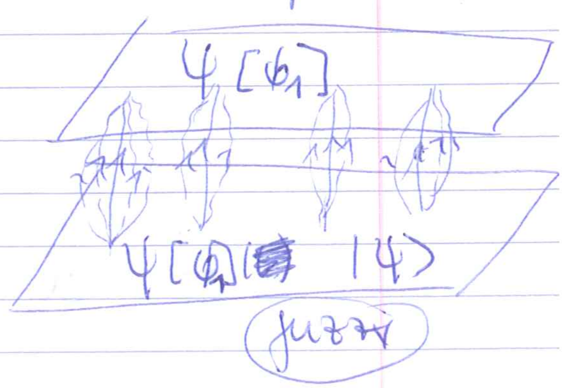
6 Quantization of the (free) K-G field in the Schrödinger picture

Recall $\quad H = \int d^3x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \right]$

Classical picture



Quantum picture



$\Psi[\phi] \rightarrow$ complicated functional, for each config.

$\phi_i \rightarrow P_i = |\psi_i|^2$

$\rightarrow$ Better work with abstract bra/ket to avoid explicitly find $\Psi[\phi]$

Fundamental postulate of QM applied:

$$i \frac{d}{dt} |\psi\rangle = \hat{H}(\hat{\phi}, \hat{\pi}) |\psi\rangle$$

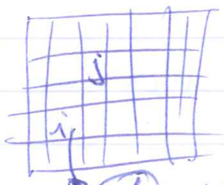
ϕ, π, H become operators acting on Hilbert space of states (Drop hat (^) for simplicity)

In QM, $[q_i, p_j] = i\delta_{ij}$ all others vanish

In QFT → continuum limit

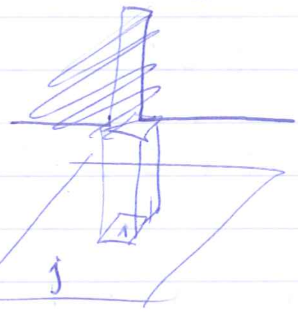
$$[\phi(\vec{x}), \pi(\vec{x}')] = i\, \delta^{(3)}(\vec{x} - \vec{x}')$$

↑ Dirac delta function



V volume

$$\delta_{ij} \to \frac{\delta_{ij}}{V}$$

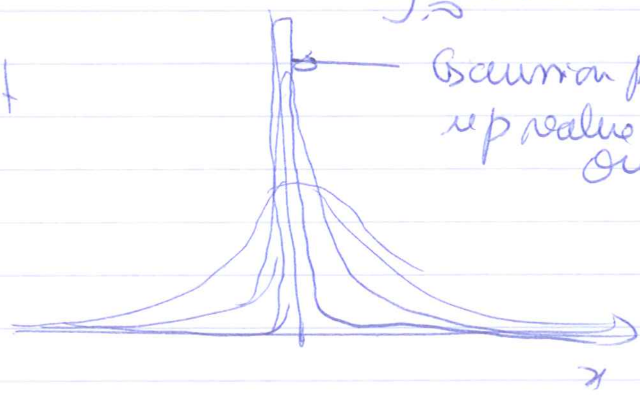


Dirac delta distribution:

$$\int_{-\infty}^{+\infty} dx \; f(x)\, \delta(x - x_0) = f(x_0)$$, in particular

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1 \text{ (normalised)}$$

Can be seen as limit



Gaussian picking up value at origin.

n- dimensional generalisation:

$$\int d^n x \; f(\vec{x})\, \delta^{(n)}(\vec{x}-\vec{x}_0) = f(\vec{x}_0)$$

Note $\delta^{(n)}(\vec{x}-\vec{x}_0) = \delta(x^1-x^1_0) \ldots \delta(x^n-x^n_0)$ in flat space

In summary, continuum limit

$[\phi(\underline{x}), \Pi(\underline{x}')] = i\,\delta^{(3)}(\underline{x}-\underline{x}')$, others vanish
just like in QM.

Just as for Harmonic oscillator, look for states
$|\psi\rangle = |\psi_E\rangle e^{-iEt}$ (find spectrum), so we need
to solve time independent Schrödinger Eq.:

$$H(\phi, \Pi)\, |\psi_E\rangle = E\, |\psi_E\rangle$$

But! We know classical sol of Klein-Gordon
= set of decoupled harmonic oscillators

So, recall Simple H.O.

$$x = \frac{1}{\sqrt{2\omega}}(a+a^+), \quad p = i\sqrt{\frac{\omega}{2}}(a-a^+)$$

$$\Rightarrow [a, a^+] = 1$$

Can we do something similar?

Yes! Expand

real Hermitian

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right]$$

$$\Pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} \left[a_p e^{ip\cdot x} - a_p^\dagger e^{-ip\cdot x} \right]$$

$\Downarrow$

Exercise: If we impose $[a_p, a_{p'}] = [a_p^\dagger, a_{p'}^\dagger] = 0$

$$[a_p, a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(p-p')$$

Then Canon. C. relations for ϕ & Π are obeyed

$\Rightarrow$ Free Quantum field ϕ = infinite set of Quantum Harmonic oscillators

Next lecture $\Rightarrow$ get spectrum etc...