5 Lagrangian Field theory

5.1 Lorentz transformations

1. A Lorentz transformation
$$x^\mu \to x'^\mu \Lambda^\mu_{\ \nu} x^\nu$$
preserves the standard Minkowski metric $\eta_{\mu\nu}$, i.e.
$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$$
   (a) Using this fact, show that an infinitesimal Lorentz transformation around the identity$\Lambda^\mu_{\ \nu} = \delta^\mu_\nu + \omega^\mu_{\ \nu}$ must satisfy
$$\omega_{\mu\nu} = -\omega_{\nu\mu.}$$
   (b) Write down the matrix transformation that corresponds to an infinitesimal rotation with angle θ around the 3rd spatial axis, identifying the corresponding $\omega_{\mu\nu}$. Do the same for a Lorentz boost along the 3rd axis with infinitesimal velocity v.
   (c) Exponentiate each of the infinitesimal transformations deduce above, using the basis of matricex introduced in the lectures and obtain the corresponding finite transformations. Compare with the standard form of rotations and boosts introduced in the begining of the course.

2. Verify that:
   (a) the basis of generators $\left(M^{\alpha\beta}\right)^\mu_{\ \nu}$ introduced in the lecture obey the Lorentz algebra commutation relations.
   (b) Show that
$$\left[M^{0i}, M^{jk}\right] = 0$$
   for any permutation of i, j, k all different.
   (c) Show that $J_i \equiv \frac{1}{2} i\epsilon_{ijk} M^{jk}$, where $\epsilon_{ijk}$is the usual totally anti-symmetric Levi-Civita tensor, obey the angular momentum commutation relations
$$[J_i, J_k] = i\epsilon_{ijk} J_k$$
   (d) Define also $K_i \equiv iM^{01}$ and show that the matrix argument of the exponential, when obtaining a finite Lorentz transformation is written as
$$\frac{1}{2}\omega_{\alpha\beta} M^{\alpha\beta} = -i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}$$
   (e) Show that the two combinations $J_i^\pm \equiv \frac{1}{2}\left(J_i \pm iK_i\right)$ are such that
$$\left[J_i^s, J_j^{s'}\right] = i\delta^{ss'}\epsilon_{ijk} J_k^s$$
   so J_i^+ commute with $J_k^-$and each set of three generators obey angular momentum commutation relations. Thus the Lorentz group finite-dimensional representations can be labelled by a pair of integers of half-integers $(j_{+,} j_{-})$.

5.2 Euler-Lagrange equations & Noether's theorem

1. Consider the non-relativistic example of the following lagrangian describing the vibration of a string with fixed ends in two possible directions y_1, y_2

$$L \equiv \int_0^a dx \sum_{i=1,2} \frac{1}{2} \left[\left(\frac{\partial y_i}{\partial t} \right)^2 - \left(\frac{\partial y_i}{\partial t} \right)^2 \right]$$

   (a) Derive the Euler-Lagrange equations and show that each displacement y_i obeys a wave equation with unit velocity.

   (b) Show that the Lagrangian is invariant under the following infinitesimal transformation and comment on what this infinitesimal transformation corresponds to:

$$y_1 \to y_1^{'} = y_1 - \theta y_2$$

$$y_2 \to y_2^{'} = y_2 + \theta y_1$$

      i. Derive the Noether current associated with the transformation above, and verify explicitely (using the equations of motion) that is is conserved. Use this result to show that the following quantity is time independent

$$h = \int_0^a dx \left(-\frac{\partial y_1}{\partial t} y_2 + \frac{\partial y_2}{\partial t} y_1 \right)$$

2. Consider a complex scalar field $\Phi(x^\mu)$ governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda \left(\Phi^* \Phi \right)^2$$

   (a) Write down the Euler Lagrange equations an write them as a wave equation with a source term.

   (b) Verify that the Lagrangian is invariant under the infinitesimal transformation below and determine the associated Noether current and conserved charge

$$\Phi \to \Phi' = (1 - i\alpha)\Phi$$

$$\Phi^* \to \Phi'^* = (1 + i\alpha)\Phi^*$$

3. Consider an isovector set of 3 real scalar fields ϕ_i, $i = 1, 2, 3$ with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i \phi_i$$

   (a) show that the Lagrangian is invariant under the inifinitesimal iso-rotation (n_i is a unit vector)

$$\phi_i \to \phi_i^{'} = \phi_i + \theta \epsilon_{ijk} n_j \phi_k$$

   (b) compute the corresponding Noether currents and check using the field equations that they are conserved:

$$j_i^\mu = \epsilon_{ijk} \phi_j \partial^\mu \phi_k$$

4. Consider a real scalar field $\phi(x^\mu)$ governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

(a) Write down the Euler Lagrange equations an write them as a wave equation with a source term.

(b) Write down the energy momentum tensor using the general formula derived in the lectures, and verify directly (using the equations of motion), the conservationn law.