4 Bra/Ket formalism & Quantum Harmonic Oscilator

1. Consider a complete orthornormal system $\{|a_n\rangle\}$ such that any ket is expanded as

$$|a\rangle = \sum_n \alpha_n |a_n\rangle$$

show that $\alpha_n = \langle a_n | a \rangle$

2. Consider a basis of physical quantum states $\{|a_n\rangle\}$which are eigenvectors of a given observable $\hat{A}$ with eigenvalues a_n. Using the postulate that the probability of the wave function ψ to be observed in a given state is $P(a_n) = |\langle \psi_n | \psi \rangle|^2$, show that:

   (a) $\sum P(a_n) = 1$

   (b) The expectation value defined from the rules of statistics as $\left\langle \hat{A} \right\rangle = \sum_n a_n P(a_n)$ is $= \langle \psi | \hat{A} | \psi \rangle$

3. Using the x-space representation, show that $[\hat{x}, \hat{p}] = i\hbar$

4. Consider the quantum harmonic oscilator in the Bra/Ket abstract notation. Show by recurrence that the states

$$|\psi_n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}} |0\rangle$$

are normalised and are eigenvectors of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ with eigenvalues $n \in \mathbb{N}_0$