Schematically $x(t,\alpha) = x(t,0) + \delta x(t)$

δ operator means variation by an arbitrary function!

Proof:

$$f(x(t,\alpha),\ \dot{x}(t,\alpha),\ t)$$

$$= f(x_0(t) + \delta x,\ \dot{x}_0(t) + (\delta \dot{x}),\ t)$$

Taylor expand:

$$\simeq f(x_0(t), \dot{x}_0(t), t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}(\delta \dot{x})$$

$$\Rightarrow \delta J = \int_{t_1}^{t_2} \delta\left(f(x,\dot{x},t)\right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}(\delta \dot{x})\right) dt$$

but $\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\delta x\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\delta x + \frac{\partial f}{\partial \dot{x}}(\delta \dot{x})$

$$\Rightarrow \delta J = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right) \delta x\, dt$$ using $\delta x(t_1) = \delta x(t_2) = 0$ ⚠

Exercise: Using that the distance along a curve $y(x)$ on a plane is given by

$$I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

write the Euler-Lagrange equations which ~~extremise~~ must be obeyed to minimize the distance between two points ~~x_1, x_2~~ $(x_1, y(x_1))$, $(x_2, y(x_2))$ and show that the solution is a straight line.

~~Hamiltonian formulation:~~

▷ This is all similar for many degrees of freedom:

$$q_a(t) \rightarrow q_a(t,0) + \delta q_a$$

$$I[q_a(t)] \equiv \int_{t_1}^{t_2} L(q_a, \dot{q}_a, t)\, dt$$

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \sum_a \underbrace{\left(\frac{\partial L}{\partial q_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a} \right) \right)}_{\text{each} = 0 \;\Rightarrow\; \text{Euler-Lagrange equations}} \delta q_a \, dt = 0$$

Note: If $\frac{\partial L}{\partial q_a} = 0 \Rightarrow p_a \equiv \frac{\partial L}{\partial \dot{q}_a}$ is conserved!

(usual momentum if $q_a \to \vec{r}_a$
angular " if $q_a \to \theta, \phi, \ldots$)

Advantages of Lagrangian formulation

- Generic coordinate systems (principle is independent of coordinates we use)

  $\Rightarrow$ symmetries are easier to deal with using appropriate coordinates (ex: angular momentum)

- In Field theory $\Leftrightarrow$ Many particle theory @ each space-time point

  $\Rightarrow \sum_a \rightarrow \int d^3x$ and

  we will get easily $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

  field

Hamiltonian formulation

General idea: $\vec{r}_a, \dot{\vec{r}}_a$ or $q_a, \dot{q}_a$ } 2nd order equations in n-variables

$\underbrace{q_a \; ; \; p_a}_{2n\text{-variables}} \equiv \frac{\partial L}{\partial \dot{q}_a}$ → conjugate momenta

2n-variables but first order equations

Achieved by a Legendre transformation:

$$H(q_a, p_a, t) \equiv \underbrace{\sum_a \dot{q}_a p_a}\; - L$$

→ Note: $\dot{q}_a$ needs to be eliminated for p_a

Then, the evolution equations are:

$$\Rightarrow \begin{cases} \dot{q}_a = \dfrac{\partial H}{\partial p_a} \\ -\dot{p}_a = \dfrac{\partial H}{\partial q_a} \end{cases}$$

Exercise: Noting that $L(q_a, \dot{q}_a, p_a, \dot{p}_a, t) = -H(q_a, p_a, t) + \sum_a \dot{q}_a p_a$ obtain the Hamiltonian equations above.

Note: If forces are conservative & $\vec{r}_i(q_1, q_2, \ldots q_n)$ → not t
& T is at most quadratic in $\dot{q}_a$

→ $\boxed{H = T + V}$ = total energy of the system when evaluated on the trajectory

Exercise: Solve the exercise for the shortest distance between two points on a plane, using the Hamiltonian formulation & contrast.

Poisson Bracket formulation

Consider generic functions $U(q_a, p_a, t)$ these will be observables
$V(q_a, p_a, t)$

define $\boxed{\{U, V\}_{q,p} \equiv \frac{\partial U}{\partial q_a}\frac{\partial V}{\partial p_a} - \frac{\partial U}{\partial p_a}\frac{\partial V}{\partial q_a}}$ sum over repeated indices

This implies that $\{q_a, p_a\} = \delta_{ab}$

(In QM this will become $[q_a, p_b] = i\hbar\,\delta_{ab}$ ⚠)

& $\Rightarrow$ $\boxed{\frac{dU}{dt} = \{U, H\} + \frac{\partial U}{\partial t}}$

particular cases, E.O.M. (obtained before)

$$\begin{cases} \dot{q}_a = \{q_a, H\} \\ \dot{p}_a = \{p_a, H\} \end{cases}$$

Ex: Check these give the Hamilton equations we had before

In QFT, quantisation of operators $\Rightarrow \{\ ,\ \} \to \frac{1}{i\hbar}[\ ,\ ]$

4 Revision of Quantum Mechanics & the harmonic oscillator:

4.1 The postulates of Quantum Mechanics:

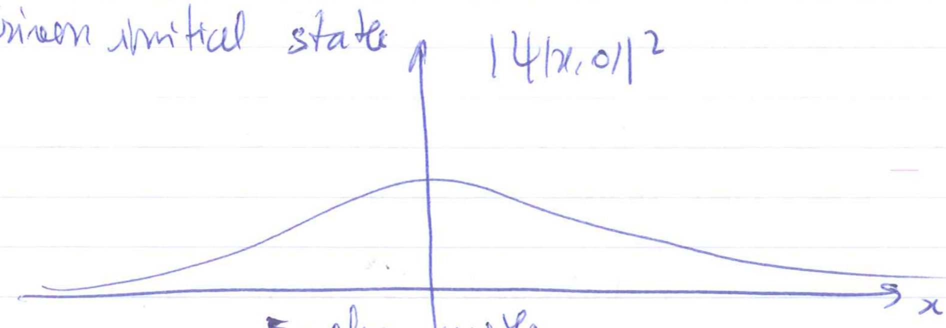
- Newtonian Physics:

Given:
- $\vec{F}(x)$ & Eqs of motion
- Initial positions (x_0) and velocities ($\dot{x}_0$)

$\Rightarrow$ We know $x(t)$ for all times or any other observable (Ex: $p(t) \equiv \dot{x}(t)$)

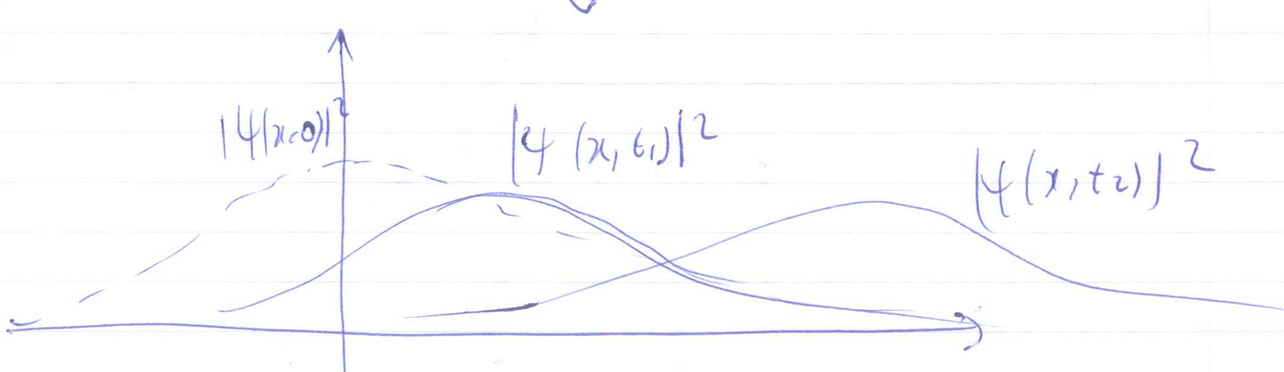
- In QM $\rightarrow$ particles do not follow localized trajectories
$\Delta x \, \Delta p \sim \hbar$

Given initial state



Evolve with Schrödinger equation

$\Downarrow$



$\psi(x,t) \rightarrow$ probabilistic nature

Postulates of Quantum Mechanics:

P1 The state of a quantum system (m particles) is specified by a complex function $\psi(\vec{r}_1, \ldots, \vec{r}_m, t)$, the wave function, which gives the probability

$$dP(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m, t) = \psi^* \psi \, d\vec{r}_1 \ldots d\vec{r}_m$$, of finding the particles in the corresponding positions.

(Corollary: $\int dP = 1 \Rightarrow \int \underbrace{\psi^* \psi}_{} \, d\vec{r}_1 \ldots d\vec{r}_m = 1$

actually a vector on a Hilbert space which is square integrable

P2: To every observable A in classical mechanics (H, $\vec{p}$, etc...) there exists a corresponding linear hermitian operator $\hat{A} = \hat{A}^+$

Note: An operator A acts on ψ to give another normalizable complex wave function. The definition of adjoint is such that

$$\int d\vec{r}_1 \ldots d\vec{r}_m \, \chi^* \hat{A} \psi = \int d\vec{r}_1 \ldots d\vec{r}_m \, (\psi^* A^+ \chi)^*$$

$\chi(\vec{r}, t)$
$\psi(\vec{r}, t)$

Example: $H = T + V \xrightarrow{Q.M.} \hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$$p \rightarrow \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$\Rightarrow \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

By definition $\hat{H} = \hat{H}^+ \Leftrightarrow \int \chi^* \hat{H} \psi = \int (\psi^* \hat{H}^+ \chi)^*$

$\hat{H} = \hat{H}^+$

Let us check if this is true...

$$\int \chi^* \hat{H} \psi$$

$$= \int \chi^* \left(-\frac{\hbar^2 \partial^2}{2m \partial x^2} + V(x) \right) \psi$$

Can we write this as $\int (\psi^* \hat{H}^+ \chi)^*$?

~~But~~ Let's integrate by parts:

$$\frac{\partial}{\partial x}\left(-\chi^* \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \right) = -\chi^* \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\psi - \frac{\partial}{\partial x}\chi^* \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \int dx\, \chi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi$$

$$= \int dx \frac{\partial}{\partial x}\left(-\chi^* \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \right) + \int dx \frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x}$$

$$= -\chi^* \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x}\Bigg|_{-\infty}^{+\infty} + \int dx \frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x}$$

zero for square integrable functions

Let's integrate by parts again:

$$\frac{\partial}{\partial x}\left(\frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \psi \right) = \frac{\partial^2 \chi^*}{\partial x^2} \frac{\hbar^2}{2m} \psi + \frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x}$$

$$\Rightarrow = \int dx \frac{\partial}{\partial x}\left(\frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \psi \right) - \int dx \frac{\partial^2 \chi^*}{\partial x^2} \frac{\hbar^2}{2m} \psi$$

$$= \frac{\partial \chi^*}{\partial x} \frac{\hbar^2}{2m} \psi \Bigg|_{-\infty}^{+\infty} + \int dx \left(\psi^* \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \chi \right)^*$$

zero again

collecting we get (assuming $V(x)$ is real)

$$\int dx \left(\psi^* \underbrace{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right)}_{\text{by definition } \hat{H}^\dagger \text{, but equal to } \hat{H}\,!} \chi \right)^*$$

$$\Rightarrow \quad \hat{H}^\dagger = \hat{H}$$

→ Alternatively, this could have been just checked as in the lectures.

$\boxed{P_3}$: In any measurement of an observable A, the only values that can be obtained are the eigenvalues $\{a\}$ of $\hat{A}$ satisfying

$$\hat{A}\,\psi_a = a\,\psi_a$$

ψ_a → eigenfunction

Example: Free particle $\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$$\hat{H}\psi = E\psi \iff -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi$$

Eigenfunction → $\psi = e^{ikx} \Rightarrow \boxed{\frac{k^2}{2m}\psi = E\psi}$

Continuum energy spectrum.

$\boxed{P_4}$: If the system (normalised) state is ψ and $\{a\}$ are the eigenvalues of $\hat{A}$, then the probability to observe a is

$$P(a) = \left| \int_{\text{all space}} \psi_a^* \psi \, d\vec{r} \right|^2$$

$\boxed{P_5}$ (Collapse of the wave function): As a result of a measurement of observable $\hat{A}$, in which the value $\underline{a}$ is obtained, the state of the system becomes ψ_a

$$\psi \to \psi_a$$

$\boxed{P_6}$ (Evolution): Between measurements, the wave function evolves according to the time dependent Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi,$$ $\hat{H}$ is the Hamiltonian of the system