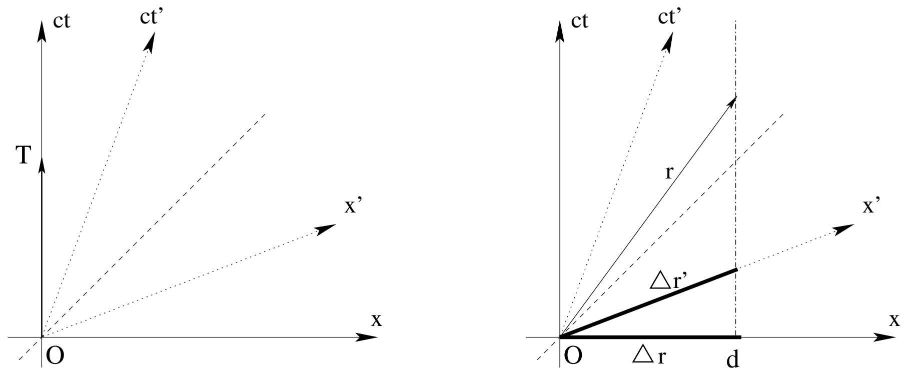

Figure 6: Space-time diagramatic construction to derive time dilation (left) and space contraction (right).

Time dilation In the Minkowski space-time formulation we have just developed, it becomes straightforward to identify the phenomenon of time dilation (Fig. 6, left). Consider two events in a reference frame, occuring at a fixed spatial coordinate but at different times. Onde can describe the separation between such two events by a space-time vector given by

$$\Delta r^{\mu} = (cT, 0, 0, 0) - (0, 0, 0, 0) = (cT, 0, 0, 0) \tag{41}$$

where we enphasize that this separation vector is given by the difference between the locations, in space-time, of the two events. One can see what a moving observer sees by applying a Lorentz boost with velocity v.

$$\Delta r^{\mu'} = \Lambda^{\mu'}_{\nu}\, \Delta r^{\nu} \rightarrow \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cT \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma cT \\ -\gamma\dfrac{v}{c}T \\ 0 \\ 0 \end{pmatrix} \tag{42}$$

One sees immediately that the new time separation between the two events is

$$T' = \gamma T = \frac{T}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{43}$$

thus, a moving observer will see the time between the two events slowing down, in its rest frame.

Space contraction In a very similar way, one can derive the phenomenon of space contraction. This consists of a shortening of lengths measured between two events which are spatially separated, when measured by a moving observer in its rest frame. (Fig. 6, right). Let us consider the following vector

$$\Delta r^{\mu} = (0, d, 0, 0) \Rightarrow \Delta r^{\mu} \eta_{\mu\nu}\, \Delta r^{\nu} = d^2 > 0 \tag{44}$$

that is, the distance between the two events in the frame where they are simultaneous is given by d. One can associate to these spatially separated events a rigid rod, with two timelike world line trajectories (vertical lines in the diagram). The world line of the end of the rod at the origin is represented by a vector $w^\mu$and the world line of the other end is represented by a vector r^μ. Thus, the separation vector between the two events is actually obtained from

$$\Delta r^\mu = r^\mu - w^\mu \to (t, d, 0, 0) - (t, 0, 0, 0) = (0, d, 0, 0) \tag{45}$$

To determine the distance between ends of the rod in motion, one needs to intersect the world lines of each end with a spatial axis of the moving frame so that we measured the distance when the two ends are simultaneous in the moving frame. First, we transform the vector associated with the trajectory of each end

$$r^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, r^\nu \to \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ d \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma t - \gamma\frac{v}{c} d \\ -\gamma\frac{v}{c} t + \gamma d \\ 0 \\ 0 \end{pmatrix} \tag{46}$$

$$w^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, w^\nu \to \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma t \\ -\gamma\frac{v}{c} t \\ 0 \\ 0 \end{pmatrix} \tag{47}$$

Now, the events will be simultaneous in motion, when $t' = 0$ for both. For the vector $r^{\mu'}$ one needs

$$t' = 0 \tag{48}$$

$$\Leftrightarrow \gamma t - \gamma \frac{v}{c} d = 0 \tag{49}$$

$$\Rightarrow t = \frac{v}{c} d \tag{50}$$

and for $w^{\mu'}$

$$\gamma t = 0 \Rightarrow t = 0 \tag{51}$$

The spacelike vector between the two events which are simultaneous in the moving frame is now

$$\Delta r^{\mu'} = r^{\mu'}(t = \frac{v}{c} d) - w^{\mu'}(t = 0) = (0, -\gamma \frac{v}{c}(\frac{v}{c} d) + \gamma d, 0, 0) - (0, 0, 0, 0) = (0, \sqrt{1 - \frac{v^2}{c^2}} d, 0, 0) \tag{52}$$

which means that the length of the rod as seen in the moving frame is shorter.

An important observation of such constructions in the diagrams of figure 6, is that events which are simultaneous in a frame are not necessarily simultaneous in a moving frame.

Lecture exercise

To obtain the 4-force in a moving frame, one needs to transform the Newtonian force that we have obtained in the lecture to a moving frame with 3-velocity $\vec{v}$ such that an observer in such frame sees the particle with such a velocity (note that in fact, that since we are observing the particle in motion, we must use the inverse transformation $\vec{v} \to -\vec{v}$, that is, the particle rest frame is moving with velocity $-\vec{v}$ with respect to our rest frame). Since the force is a 4-vector, it must transform according to the corresponding Lorentz transformation. Thus we conclude that in general

$$F^{\mu} \to (\gamma \vec{F}_N \cdot \frac{\vec{v}}{c}, \gamma \vec{F}_N) \tag{53}$$

The spatial part of the relativistic Newton equations is then

$$\begin{aligned}
m_0 \frac{d^2\vec{r}}{d\tau^2} &= \gamma \vec{F}_N && (54)\\
\Leftrightarrow m_0 \frac{dt}{d\tau}\frac{d^2\vec{r}}{dt d\tau} &= \gamma \vec{F}_N && (55)\\
\Leftrightarrow m_0 \gamma \frac{d}{dt}\left(\frac{dt}{d\tau}\frac{d\vec{r}}{dt}\right) &= \gamma \vec{F}_N && (56)\\
\Leftrightarrow \frac{d}{dt}\left(\gamma m_0 \vec{v}\right) &= \vec{F}_N && (57)\\
\Leftrightarrow \frac{d}{dt}\left(m(v)\vec{v}\right) &= \vec{F}_N && (58)
\end{aligned}$$

This equation is similar to the non-relativistic Newton equation except that the mass $m(v)$ depends on the velocity of the particle. As a consequence, the effective mass of the particle increases as we approach the speed of light since the particle acquires more and more kinetic energy(though never being able to reach exactly c which requires an infinite amount of energy)

Regarding the F^0 component one gets

$$\begin{aligned}
m_0 \frac{d^2x^0}{d\tau^2} &= \gamma \vec{F}_N \cdot \frac{\vec{v}}{c} && (59)\\
\Leftrightarrow m_0 \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{d(ct)}{d\tau}\right) &= \gamma \vec{F}_N \cdot \frac{\vec{v}}{c} && (60)\\
\Leftrightarrow m_0 c \gamma \frac{d}{dt}\left(\gamma\right) &= \gamma \vec{F}_N \cdot \frac{\vec{v}}{c} && (61)\\
\Leftrightarrow m_0 c \gamma \frac{d}{dt}\left(\gamma\right) &= \gamma \vec{F}_N \cdot \frac{\vec{v}}{c} && (62)\\
\Leftrightarrow \frac{d}{dt}\left(mc^2\right) &= \vec{F}_N \cdot \vec{v} && (63)
\end{aligned}$$

This equation is similar to the equation for the variation of the kinetic energy in non-relativistic Newtonian dynamics $E_{cinética} = mv^2/2$, but now it is replaced by mc^2. This quantity describes the variation of the total energy of the particle in special relativity which is interpreted as being equivalent to m (in natural units $c = 1$ it is even more manifest). One can verify this is the case by expanding in the small velocity limit

$$mc^2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} m_0 c^2 \simeq m_0 c^2 + \frac{1}{2} m_0 v^2 + \dots \tag{64}$$

So besides the kinetic energy we get the famous rest energy $E_0 = m_0 c^2$. The total kinetic energy is the difference between the total energy and the rest energy

$$E_k = mc^2 - m_0 c^2 \tag{65}$$

This equivalence between the rest mass of the particle and energy was one of the biggest surprises of special relativity.