Note that
Note que

$$\omega = \frac{v'}{|\vec{r}_1'|} \Rightarrow \text{Força centrífuga} = \frac{v'^2}{|\vec{r}_1'|}\frac{\vec{r}_1'}{|\vec{r}_1'|}$$

Centrifugal Force

como esperado

As expected

For the Coriolis Force, one expects that
Para a força de coriolis, esperamos que

$$\vec{F}_{coriolis} = 2m\vec{\Omega}\times\vec{v}'$$

IGNORE THIS PART!!!!!!!

$$\vec{\Omega} = \omega\hat{e}_z \qquad \vec{\Omega}\times\vec{v}_1' = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 0 & 0 & \omega \\ v_x' & v_y' & 0 \end{vmatrix}$$

$$= \vec{e}_x(-\omega v_y') + \vec{e}_y\,\omega v_x$$

$$= \omega\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{v}_1'$$

2

Exercise 1

A primeira transformação leva-nos do referencial O para O'

The first transformation takes us from frame O to O'

$$\vec{r}' = \vec{r}_0 + \vec{v}_0 t + \vec{r}$$

The second one, from O' to O''

A segunda, leva-nos do referência O' para O''

$$\vec{r}'' = \vec{r}_1 + \vec{v}_1 t + \vec{r}'$$

The composed operation takes us from O to O'', and is obtained,...

A operação composta é aquela que nos leva do referêncial O para O'' e é obtida por substituição na equação anterior:

... by replacing in the first equation:

$$\vec{r}'' = \vec{r}_1 + \vec{v}_1 t + (\vec{r}_0 + \vec{v}_0 t + \vec{r}) = \underbrace{(\vec{r}_1 + \vec{r}_0)}_{\vec{r}\,\text{composto}} + \underbrace{(\vec{v}_1 + \vec{v}_0)}_{\vec{v}\,\text{composta}} t + \vec{r}$$

composed translation and velocity

d) $\alpha = \frac{\pi}{2}$

$$\Rightarrow \delta F = \frac{2L}{\lambda}\left(\frac{n_0}{c}\right)^2 \cos(2\theta)$$

IGNORE THIS!!!!!!!!!!

~~Por exemplo para $\theta = \frac{\pi}{4}$ o efeito cancela completamente por isso é necessário garantir que o dispositivo aponta em direções diferentes para que não estejamos numa orientação desfavorável.~~

Exercise $\boxed{3}$ A teoria da Relatividade restrita:

1. a)

$$R_1^T R_1 = \mathbb{1} \quad e \quad \det R_1 = 1$$

$$R_2^T R_2 = \mathbb{1} \quad e \quad \det R_2 = 1$$

$$R \equiv R_1 R_2$$

Let's compute

Vamos determinar $R^T R$

$$= (R_1 R_2)^T R_1 R_2$$

$$= R_2^T R_1^T R_1 R_2$$

$$= R_2^T (\mathbb{1}) R_2$$

$$= R_2^T R_2$$

So this condition is verified!

$= \mathbb{1}$ ou seja esta condição verifica-se ($R^T R = \mathbb{1}$)

For the other condition we compute the determinant

para a outra condição

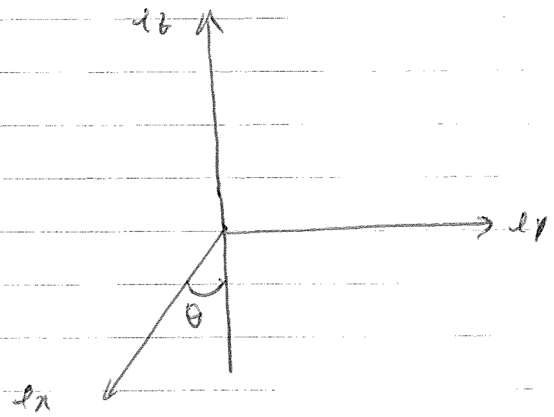
$$\det(R)$$
$$= \det(R_1 R_2)$$
$$= \det R_1 \det R_2$$
$$= 1 \times 1$$
$$= 1 \qquad \text{ou seja } \det R = 1 \checkmark$$

Thus R is also a rotation matrix

Logo $R$ também é uma matriz de rotação.

**d)**

(Note different ordering of the solutions in this exercise!!!!)




$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**e)** $R_1 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad R_2 = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\cancel{R_1 R_2 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha & \\ -2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha & \\ 0 & 0 & 1 \end{pmatrix}}$$

$$R_1R_2 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & \cos\alpha\sin\alpha + \sin\theta\cos\alpha & 0 \\ -\sin\theta\cos\alpha - \cos\theta\sin\alpha & -\sin\theta\sin\alpha + \cos\theta\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\alpha) & \sin(\theta+\alpha) & 0 \\ -\sin(\theta+\alpha) & \cos(\theta+\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As expected the angle of the composed transformation is sum

Logo, como esperado o ângulo da transformação composta é $\theta+\alpha$. of the two angles

Section 2
Exercise 1)

a) obeys

$L_1$ obedece $L_1^T \eta L_1 = \eta$ and e $\det L_1 = 1$

obeys

$L_2$ obedece $L_2^T \eta L_2 = \eta$ and e $\det L_2 = 1$

$$L \equiv L_1 L_2$$

$$L^T \eta L = (L_1 L_2)^T \eta (L_1 L_2)$$

$$= L_2^T L_1^T \eta L_1 L_2$$

$$= L_2^T ( \eta ) L_2$$

$$= \eta$$

$$\det(L_1 L_2) = \det L_1 \det L_2$$
$$= 1 \times 1$$
$$= 1$$

Thus L is also a Lorentz transformation

Logo L é também uma transformação de Lorentz

(index notation proof on the next page!)

**d)**

$$L = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 \\ -\sinh\phi & \cosh\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**e)**

1) $$L_1 = \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_1 L_2 = \begin{pmatrix} \cosh\phi_1 & -\sinh\phi_1 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & -\sinh\phi_2 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{= \begin{pmatrix} \cosh\phi_1 \cosh\phi_2 + \sinh\phi_1 \sinh\phi_2 & -2\cosh\phi_1 \sinh\phi_2 \\ -2\cosh\phi \end{pmatrix}}$$

$$\Leftrightarrow L_1 L_2 = \begin{pmatrix} \cosh\phi_1 \cosh\phi_2 + \sinh\phi_1 \sinh\phi_2 & -\cosh\phi_1 \sinh\phi_2 - \sinh\phi_1 \cosh\phi_2 & 0 \\ -\sinh\phi_1 \cosh\phi_2 - \cosh\phi_1 \sinh\phi_2 & \sinh\phi_1 \sinh\phi_2 + \cosh\phi_1 \cosh\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\phi_1+\phi_2) & -\sinh(\phi_1+\phi_2) & 0 \\ -\sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the pseudo-angle of the composed transformation

$\phi = \phi_1 + \phi_2$ é o ângulo da transformação composta.

ii) $v_1 = c \tanh\phi_1$

$v_2 = c \tanh\phi_2$

$$v = c\tanh\phi = c\tanh(\phi_1+\phi_2)$$

$$= c\,\frac{\tanh\phi_1 + \tanh\phi_2}{1+\tanh\phi_1 \tanh\phi_2}$$

$$= c\,\frac{\left(\frac{v_1}{c} + \frac{v_2}{c}\right)}{1+\frac{v_1 v_2}{c^2}} = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$

The composition of velocities is no longer a sum & we have that |v|<c always.

iii) A composição das velocidades não é mais uma soma, de modo a que $|v| < c$ sempre.

iv) $\frac{v_1}{c}, \frac{v_2}{c} \ll 1 \Rightarrow v = v_1 + v_2 + \dots$

Ou seja recuperamos a soma das velocidades

That is, we recover the Galilean sum of velocities at small speeds.

**f)** $L_{Rotação} =$ rotation

$$L_{Rotação} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_{boost} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

u)

$$L_1 = L_{rotação}\, L_{boost}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c}\cos\theta & \gamma\cos\theta & 0 & 0 \\ +\gamma\frac{v}{c}\sin\theta & -\gamma\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c}\cos\theta & \gamma\cos\theta & \sin\theta & 0 \\ \gamma\frac{v}{c}\sin\theta & -\gamma\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ii)

$L_2 = L_{boost} \times L_{rotação}$

$$= \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\frac{v}{c}\cos\theta & -\gamma\frac{v}{c}\sin\theta & 0 \\ -\gamma\frac{v}{c} & \gamma\cos\theta & \gamma\sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

iii) Fazendo primeiro o boost, o resultado final da ~~tra~~ transformação é apenas uma rotação dos eixos obtidos no novo referencial intermédio, enquanto que efetuando primeiro a rotação dos eixos, o boost é depois ao longo do eixo $x'$ novo. Deste modo as duas operações não comutam, ou seja, o resultado final depende da ordem.

If we first perform the boost, the final result is simply a rotation of the axes obtained in the intermediate reference frame. However, when one performs first the rotation of the axes, the boost is then along the new x' axis. Thus, the two operations do not commute, that is, the final result depends on the order of the operations.

**g)** i)

$$\begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Para inverter esta matriz:

$$\left(\begin{array}{cc|cc} \gamma & -\gamma\frac{v}{c} & 1 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{v}{c} & \frac{1}{\gamma} & 0 \\ -1 & \frac{c}{v} & 0 & \frac{c}{v}\frac{1}{\gamma} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{v}{c} & \frac{1}{\gamma} & 0 \\ 0 & -\frac{v}{c}+\frac{c}{v} & \frac{1}{\gamma} & \frac{c}{v}\frac{1}{\gamma} \end{array}\right)$$

$$-\frac{v}{c}+\frac{c}{v} = \frac{-v^2+c^2}{vc}$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{v}{c} & \frac{1}{\gamma} & 0 \\ 0 & 1 & \frac{vc}{c^2-v^2}\frac{1}{\gamma} & \frac{c}{v}\left(\frac{vc}{c^2-v^2}\right)\frac{1}{\gamma} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{v}{c} & \frac{1}{\gamma} & 0 \\ 0 & 1 & \frac{vc}{c^2-v^2}\sqrt{\frac{c^2-v^2}{c^2}} & \frac{c^2}{c^2-v^2}\sqrt{1-\frac{v^2}{c^2}} \end{array}\right)$$

$$\Leftrightarrow \left( \begin{array}{cc|cc} 1 & -\frac{v}{c} & \frac{1}{\gamma} & 0 \\ 0 & 1 & \frac{v}{c}\gamma & \gamma \end{array} \right)$$

$$\Leftrightarrow \left( \begin{array}{cc|cc} 1 & 0 & \frac{1}{\gamma} + \frac{v^2}{c^2}\gamma & \frac{v}{c}\gamma \\ 0 & 1 & \frac{v}{c}\gamma & \gamma \end{array} \right)$$

$$\frac{1}{\gamma} + \frac{v^2}{c^2}\gamma = \frac{1}{\gamma}\left(1 + \frac{v^2}{c^2}\gamma^2\right)$$

$$= \sqrt{1-\frac{v^2}{c^2}}\left(1 + \frac{v^2}{c^2}\frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$= \sqrt{1-\frac{v^2}{c^2}}\left(\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right)$$

$$= \gamma$$

$$\Leftrightarrow \left( \begin{array}{cc|cc} 1 & 0 & \gamma & \gamma\frac{v}{c} \\ 0 & 1 & \gamma\frac{v}{c} & \gamma \end{array} \right)$$

iii) A matriz inversa tem a mesma forma com

As expected, the inverse has the same form with v-> -v.

$v \to -v$ como seria de esperar visto que

This is because it corresponds to going to a frame, moving in the opposite direction

a inversa corresponde a ir para um referencial em movimento na direção oposta.

iv)

$L$ obeys
$L$ obedece $L^T \eta L = \eta$ and
e $\det L = 1$

The inverse matrix is defined such that
a matriz inversa é definida tal que

$$L^{-1} L = \mathbb{1}$$

Let's compute
Vamos determinar

$$(L^T \eta L)^{-1} = \eta^{-1}$$

but
Mas $\eta^{-1} = \eta$

$$\Leftrightarrow (L^T \eta L)^{-1} = \eta$$

$$\Leftrightarrow L^{-1} \eta^{-1} (L^T)^{-1} = \eta$$

$$\Leftrightarrow L^{-1} \eta (L^{-1})^T = \eta$$

~~Tomando a transposta desta equação~~

But noting that eta is such that
Mas notando que $\eta$ é tal que $A^T \eta A = A \eta A^T$

$$\Rightarrow (L^{-1})^T \eta L^{-1} = \eta$$

Thus L^-1 also obeys the first condition
logo $L^{-1}$ obedece a primeira condição também.

As for the determinant conditionm
Para a condição do determinante

$$\det(L^{-1} L) = \det \mathbb{1}$$

$$\Leftrightarrow \det(L^{-1}) \det L = 1$$

$$\Leftrightarrow \det(L^{-1}) \times 1 = 1$$

Thus the inverse of a Lorentz transformation is also a Lorentz transformation.

$\Rightarrow \det L^{-1} = 1$ logo $L^{-1}$ é também uma transformação de Lorentz

(NOTE: See also index notation solution on the next page!)

**2)** Consider two future directed timelike vectors defined:

Consideremos dois vectores do tipo tempo ~~espaço~~ dirigidos para o futuro

$$V_1^T \eta V_1 < 0 \quad e \quad V_2^T \eta V_2 < 0$$

$$\Leftrightarrow V_1^T \eta V_1 = -\delta_1^2 \quad e \quad V_2^T \eta V_2 = -\delta_2^2$$

$$\Leftrightarrow |V_1^0|^2 = |\vec{V_1}|^2 + \delta_1^2 \quad e \quad |V_2^0|^2 = |\vec{V_2}|^2 + \delta_2^2$$

$$\Leftrightarrow V_1^0 = \sqrt{|\vec{V_1}|^2 + \delta_1^2} \quad e \quad V_2^0 = \sqrt{|\vec{V_2}|^2 + \delta_2^2}$$

$$(V_1 + V_2)^T \eta (V_1 + V_2)$$

$$= V_1^T \eta V_1 + 2 V_1^T \eta V_2 + V_2^T \eta V_2$$

$$= -\delta_1^2 + 2\left(-\sqrt{|\vec{V_1}|^2+\delta_1^2}\sqrt{|\vec{V_2}|^2+\delta_2^2} + \vec{V_1}\cdot\vec{V_2}\right) - \delta_2^2$$

$$= -\delta_1^2 - \delta_2^2 - 2\left(\underbrace{\sqrt{|\vec{V_1}|^2+\delta_1^2}\sqrt{|\vec{V_2}|^2+\delta_2^2} - \vec{V_1}\cdot\vec{V_2}}_{>0}\right)$$

Thus, this shows that the sum is also future directed timelike.

Logo está provado que $V_1 + V_2$ é também do tipo tempo ~~[illegible]~~ e dirigido para o futuro.

Section 1 | Ex 2 → See pdf of lecture notes.

Ex: 3 | a), d) & e) see ~~[illegible]~~ some pages ahead!

b) $v^i w_i = v^1 w_1 + v^2 w_2 + v^3 w_3$

$= 1 \times 0 - 1 \times 0 + 0 \times (-5)$

$= 0 \Rightarrow v^i$ and $w^i$ are orthogonal

$v^i M_{ij} = M_{ji}^T v^i$

$$\rightarrow \begin{pmatrix} -2 & 0 & 4 \\ 7 & 0 & -5 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix}$$

$$\Rightarrow \boxed{w^i M_{ij} \rightarrow (-2, 7, -1)}$$

$$M_{ij} v^j \rightarrow \begin{pmatrix} -2 & 7 & 2 \\ 0 & 0 & 3 \\ 4 & -5 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{M_{ij} v^j \rightarrow (-5 \ 0 \ 1)}$$

c) $R^T_{1\, ik'} R_1{}^{k'}{}_j = \delta_{ij}$ defines rotation $R_1$

$R^T_{2\, ik'} R_2{}^{k'}{}_j = \delta_{ij}$ defines rotation $R_2$

$R^{i'}{}_j = R_1{}^{i'}{}_{k''} R_2{}^{k''}{}_j$ is the product

14.2

We need to compute $R^T R$, which in index notation reads:

$$R^T{}_{j i'} R^{i'}{}_l = R_{i' j} R^{i'}{}_l$$

$$= \left(R_{1\, i' k''} R_2{}^{k''}{}_j\right)\left(R_1{}^{i'}{}_{l''} R_2{}^{l''}{}_l\right)$$

$$= R_{1\, i' k''} R_1{}^{i'}{}_{l''} R_2{}^{k''}{}_j R_2{}^{l''}{}_l$$

because $R_1$ is a rotation

$$= \underbrace{R_1^T{}_{k'' i'} R_1{}^{i'}{}_{l''}} R_2{}^{k''}{}_j R_2{}^{l''}{}_l$$

$$= \delta_{k'' l''} R_2{}^{k''}{}_j R_2{}^{l''}{}_l$$

$$= R_{2\, l'' j} R_2{}^{l''}{}_l$$

$$= \underbrace{R_2^T{}_{j l''} R_2{}^{l''}{}_l}$$ because $R_2$ is a rotation

$$= \delta_{j l}$$

Thus $R^T R = \mathbb{1}$ as expected.

Note: The determinant condition is more elaborate in index notation so we do not show that here.

Section 2

Ex 1   a) index notation:

$$L_1{}^{\alpha'}{}_{\beta} \quad \text{obeys} \quad L_1^T{}_{\beta}{}^{\alpha'} \eta_{\alpha'\beta'} L_1{}^{\beta'}{}_{\gamma} = \delta_{\beta\gamma}$$

$$L_2{}^{\alpha'}{}_{\beta} \quad \text{obeys} \quad L_2^T{}_{\beta}{}^{\alpha'} \eta_{\alpha'\beta'} L_2{}^{\beta'}{}_{\gamma} = \delta_{\beta\gamma}$$

$L = L_1 L_2$ is defined $\quad L^{\alpha'}{}_{\beta} = L_1{}^{\alpha'}{}_{\beta''} L_2{}^{\beta''}{}_{\beta}$

we need to compute

$$L^T{}_{\beta}{}^{\alpha'} \eta_{\alpha'\beta'} L^{\beta'}{}_{\gamma}$$

$$= L^{\alpha'}{}_{\beta} \eta_{\alpha'\beta'} L^{\beta'}{}_{\gamma}$$

$$= L_1{}^{\alpha'}{}_{\beta''} L_2{}^{\beta''}{}_{\beta} \eta_{\alpha'\beta'} L_1{}^{\beta'}{}_{\gamma''} L_2{}^{\gamma''}{}_{\gamma}$$

$$= L_1{}^{\alpha'}{}_{\beta''} \eta_{\alpha'\beta'} L_1{}^{\beta'}{}_{\gamma''} L_2{}^{\beta''}{}_{\beta} L_2{}^{\gamma''}{}_{\gamma}$$

$$= \underbrace{L_1^T{}_{\beta''}{}^{\alpha'} \eta_{\alpha'\beta'} L_1{}^{\beta'}{}_{\gamma''}} L_2{}^{\beta''}{}_{\beta} L_2{}^{\gamma''}{}_{\gamma}$$

$$= \eta_{\beta''\gamma''} L_2{}^{\beta''}{}_{\beta} L_2{}^{\gamma''}{}_{\gamma}$$

$$= L_2{}^{\beta''}{}_{\beta} \eta_{\beta''\gamma''} L_2{}^{\gamma''}{}_{\gamma}$$

$$= \underbrace{L_2^T{}_{\beta}{}^{\beta''} \eta_{\beta''\gamma''} L_2{}^{\gamma''}{}_{\gamma}}$$

$$= \eta_{\beta\gamma}$$

so $L$ is also a Lorentz transformation

Section 2) Ex 2) index notation

$V_1^\mu \eta_{\mu\nu} V_1^\nu < 0$ defines $V_1$ as timelike

$V_2^\mu \eta_{\mu\nu} V_2^\nu < 0$ " $V_2$ " "

~~$\Leftrightarrow (V_1^0)^2 = |V_1|^2 + d_1^2$~~

$\Rightarrow V_1^\mu \eta_{\mu\nu} V_1^\nu = -d_1^2$ & $V_2^\mu \eta_{\mu\nu} V_2^\nu = -d_2^2$

$$\Leftrightarrow (V_1^0)^2 = \underbrace{|\vec{V}_1|^2}_{\equiv V_1^i V_1^i} + d_1^2 \qquad \& \qquad (V_2^0)^2 = |\vec{V}_2|^2 + d_2^2$$

Now the sum is $V^\mu = V_1^\mu + V_2^\mu$

$$\Rightarrow V^\mu \eta_{\mu\nu} V^\nu = (V_1^\mu + V_2^\mu)\,\eta_{\mu\nu}\,(V_1^\nu + V_2^\nu)$$

$$= \underline{V_1^\mu \eta_{\mu\nu} V_1^\nu} + V_1^\mu \eta_{\mu\nu} V_2^\nu + V_2^\mu \eta_{\mu\nu} V_1^\nu + \underline{V_2^\mu \eta_{\mu\nu} V_2^\nu}$$

$$= -d_1^2 + 2 V_1^\mu \eta_{\mu\nu} V_2^\nu - d_2^2$$

$$= -d_1^2 + d_2^2 + 2(-V_1^0 V_2^0 + \vec{V}_1 \cdot \vec{V}_2)$$

$$= -d_1^2 - d_2^2 - 2\Big(\underbrace{\sqrt{|\vec{V}_1|^2+d_1^2}\sqrt{|\vec{V}_2|^2+d_2^2} - \vec{V}_1 \cdot \vec{V}_2}_{>0}\Big)$$

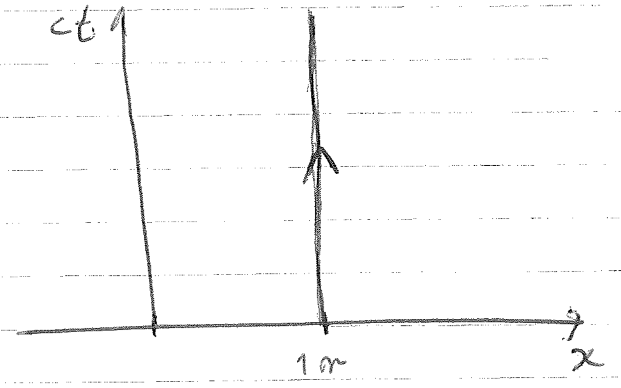
$$\Rightarrow V^\mu \eta_{\mu\nu} V^\nu < 0$$

~~Since $(V_1)^0 + (V_2)^0 > 0$ then $V$ is al~~

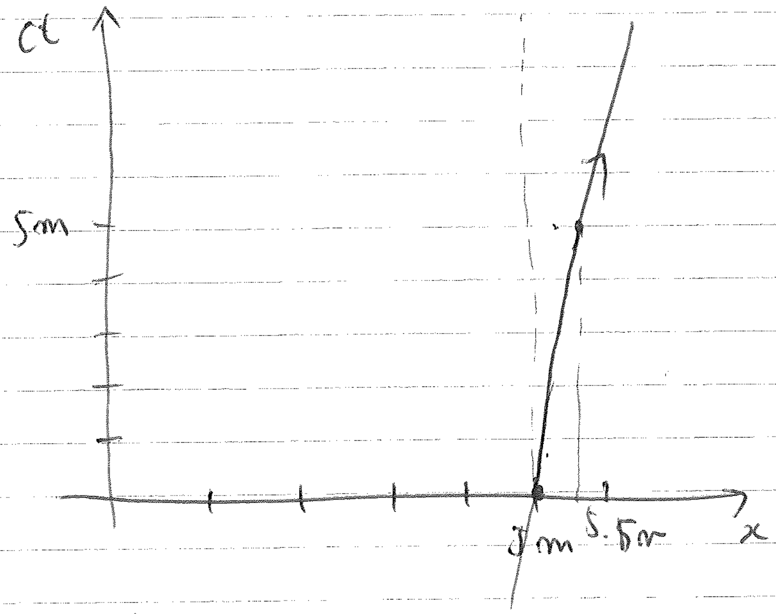
Since $(V_1)^0 + (V_2)^0 > 0$ then $V$ is also future directed.

**3)**

a)



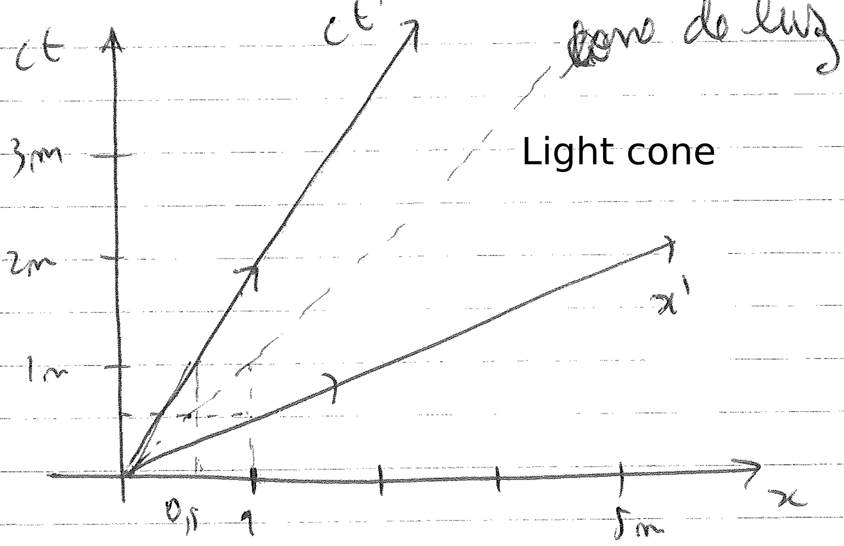

b)




$$x = x_0 + v t \Leftrightarrow x = x_0 + \frac{v}{c} ct$$

$$\Leftrightarrow x = x_0 + 0{,}1\, ct$$

c)

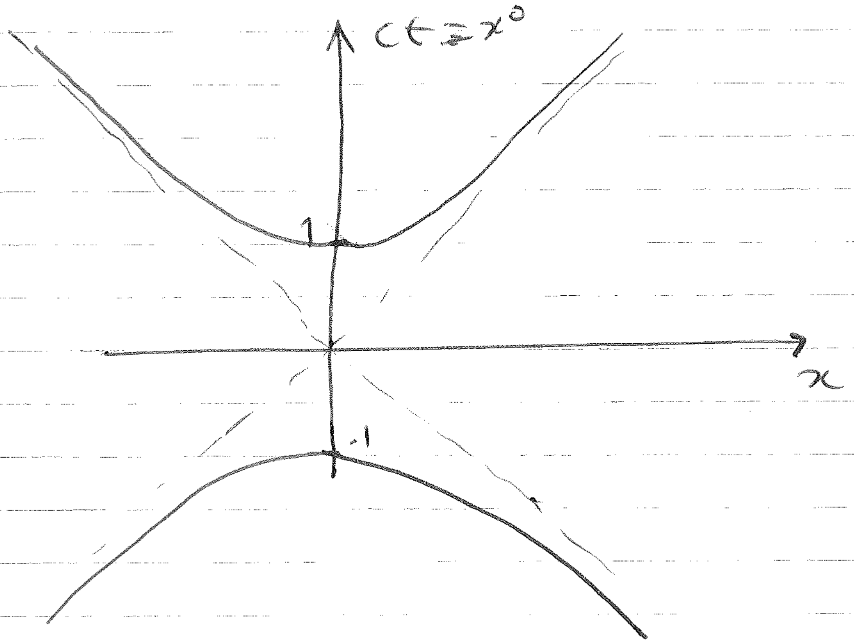


d) i) $\Delta r \equiv \cancel{\phantom{xx}}\ (x^0, x)$

$\Delta r^T \eta \Delta r = -1$ (index notation:

$\Leftrightarrow -(x^0)^2 + x^2 = -1$

$\Leftrightarrow x^0 = \pm\sqrt{1+x^2}$



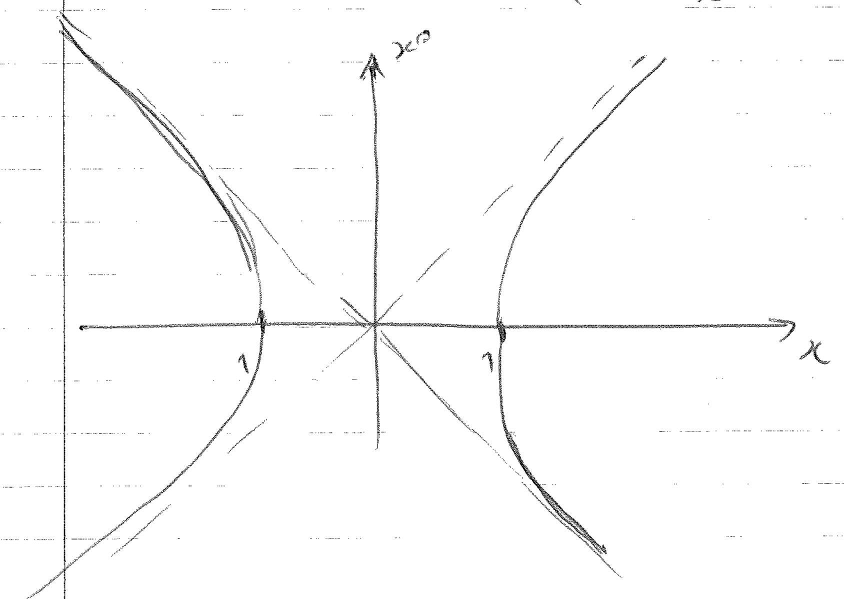

ii) $\Delta r \equiv (x^0, x)$

$\Delta r^T \eta \Delta r = 1 \Leftrightarrow -(x^0)^2 + x^2 = 1$

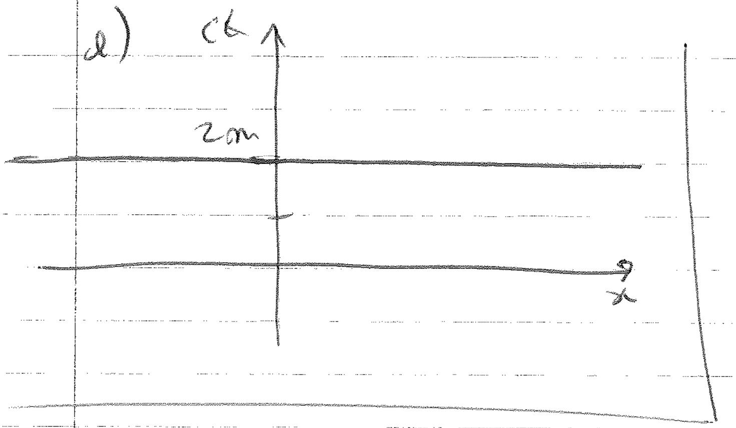
(index notation:

$\Leftrightarrow (x^0)^2 = x^2 - 1$

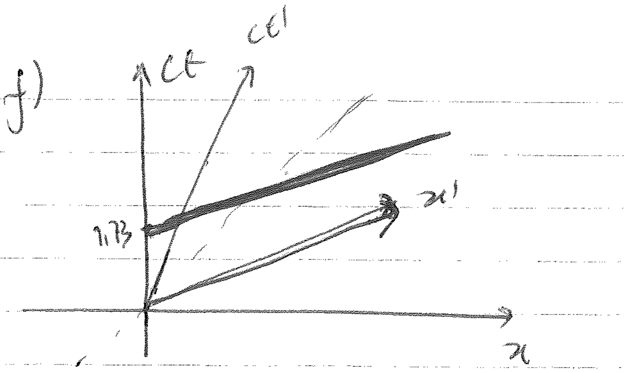
$\Leftrightarrow x^0 = \pm\sqrt{x^2 - 1}$



d)




f)




$\cancel{\Leftrightarrow \gamma ct - \gamma \frac{v}{c} x = 2}$

$\cancel{\Leftrightarrow \gamma x^0 = 0{,}5x \quad \sqrt{1-0{,}25}}$

Using the Lorentz transformations:

Usando as transformações de Lorentz

$$x^{0'} = \gamma x^0 - \gamma \frac{v}{c} x$$

$$\Leftrightarrow x^{0'} = 2m \Leftrightarrow \quad 2 = \gamma x^0 - \gamma \frac{v}{c} x$$

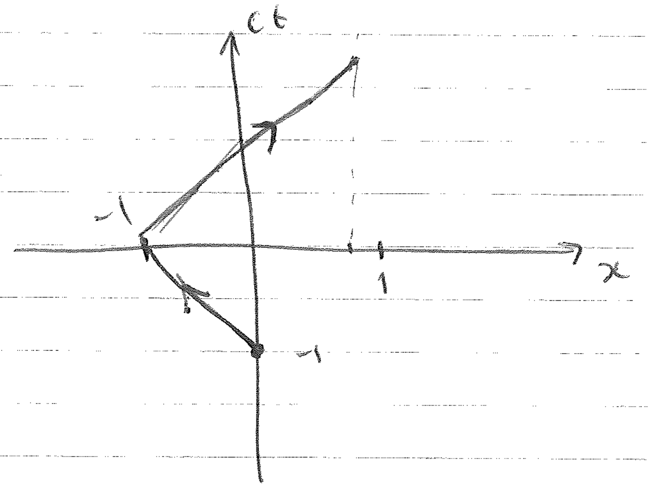
$$\Leftrightarrow x^0 = \frac{2}{\gamma} + \frac{v}{c} x$$

$$\Leftrightarrow x^0 = 2\sqrt{1-0{,}5^2} + 0{,}5x$$

$$\Leftrightarrow x^0 = \sqrt{3} + 0{,}5x$$

$$\Leftrightarrow x^0 = 1{,}73 + 0{,}5x$$

g)



## Solution of exercise 1-c) of section 2

Before addressing the exercise, let just note that if one considers time independent transformations, that is in the form

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix} \tag{22}$$

we recover the condition for the group of rotations for $\mathbf{R}$, if we use the Lorentz conditions . This means that the Lorentz group contains as a subgroup the group of rotations.

Let us now consider a general transformation involving $x$ and $t$, i.e $x^1 \to x^{1'} = Cx^0 + Dx^1$ e $x^0 \to x^{0'} = Ax^0 + Bx^1$, where the coefficients are for now, arbitrary. In matrix notation

$$\mathbf{\Lambda} = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

Inserting in the orthogonality condition with respect to $\eta_{\mu\nu}$we have

$$\begin{pmatrix} A & C & 0 & 0 \\ B & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

so we obtain 3 conditions

$$C^2 - A^2 = -1 \tag{25}$$
$$D^2 - B^2 = 1 \tag{26}$$
$$CD = AB \tag{27}$$

These can be solved with a single parameter, similarly to rotations in 2D

$$A = D = \cosh\phi \tag{28}$$
$$C = B = -\sinh\phi \tag{29}$$

which we call pseudo-rotations, due to the presence of hyperbolic, rather than trigonometric, functions. This result can be interpreted physically using the definition of the velocity of the frame $O'$ as seen in the frame $\mathcal{O}$. First we write again the transformations

$$x^{0'} = \cosh\phi\, x^0 - \sinh\phi\, x^1 \tag{30}$$
$$x^{'} = -\sinh\phi\, x^0 + \cosh\phi\, x^1 \tag{31}$$

Consider the position of the origin of the frame $\mathcal{O}'$ as seen in the frame $\mathcal{O}$ (Fig. 4). The equation for such point is

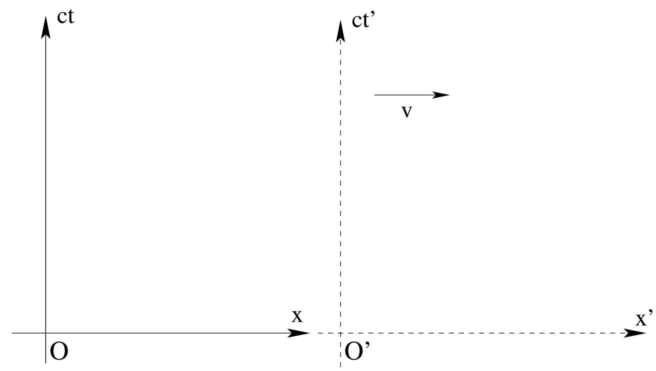



Figure 4: Frame $\mathcal{O}'$ in motion with respect to $\mathcal{O}$. Note that the position of the origin $\mathcal{O}'$ is by definition $x^{1'} = 0$.

$$\begin{aligned} x^{1'} &= 0 && (32)\\ \Leftrightarrow -\sinh\phi\, x^0 + \cosh\phi\, x^1 &= 0 && (33)\\ \Rightarrow x^1 &= \tanh\phi\, ct && (34) \end{aligned}$$

Thus $\mathcal{O}'$ moves with velocity $v = c\tanh\phi$ as seen in frame $\mathcal{O}$. Solving for $v$ we end up with a more conventional form of this Lorentz transformation

$$\begin{aligned} x^{0'} &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\, x^0 - \frac{v}{\sqrt{c^2-v^2}}\, x^1 && (35)\\ x^{1'} &= -\frac{v}{\sqrt{c^2-v^2}}\, x^0 + \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\, x^1 && (36) \end{aligned}$$

In the small speed limit, this reduces to a Galileo transformation. Note at this point once again, the similarity of such pseudo-rotations with standard rotations: The pseudo-rotation is done on the $xt$ plane of space-time (similarly a rotation is performed on an $xy$ plane as discussed in a previous exercise). In matrix form one writes

$$\mathbf{L} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (37)$$

where we define $\gamma = \sqrt{1-\frac{v^2}{c^2}}$. The left hand side representation is particularly useful to compose Lorentz transformations on the same plane, since similarly to rotations, we just need to add pseudo angles. Thus, consider two pseudo-rotations with pseudo-angles $\phi_1, \phi_2$ and associated velocities $v_1$ e $v_2$. That is, consider a frame $\mathcal{O}'$ in motion with respect to $\mathcal{O}$ with velocity

$$v_1 = c\tanh\phi_1 \qquad (38)$$

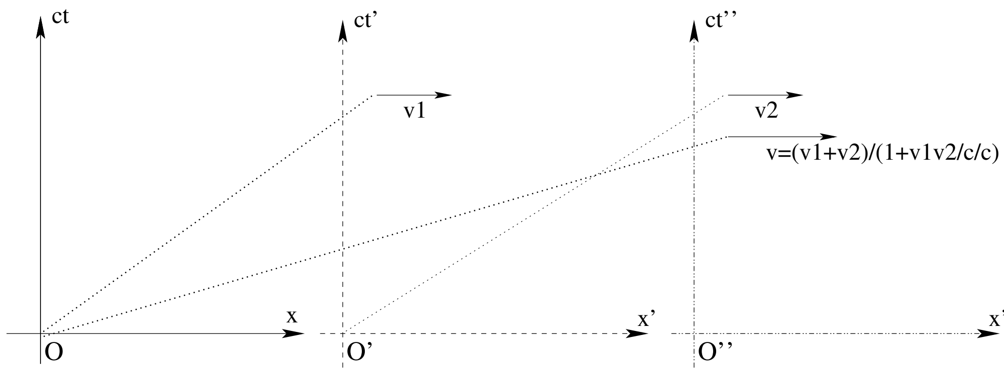



Figure 5: Composition of two Lorentz transformations. Note the dotted lines connecting the velocity vectors to the origin of the reference frame with respect to which they are measured. Thus $v_1$ is the velocity of $\mathcal{O}'$ with respect to $\mathcal{O}$, $v_2$ the velocity of $\mathcal{O}''$ with respect to $\mathcal{O}'$ and $v$ is the composed velocity of $\mathcal{O}''$ with respect to $\mathcal{O}$.

and a frame $\mathcal{O}''$ in motion with respect to $\mathcal{O}'$ with velocity

$$v_2 = c \tanh \phi_2 \tag{39}$$

(Fig. 5) thus the composed operation has a pseudo-angle $\phi = \phi_1 + \phi_2$ and velocity

$$v = c \tanh(\phi_1 + \phi_2) = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \, . \tag{40}$$

(verify this by making the matrix product). An immediate consequence is that the speed of light cannot be exceeded since $-1 < \tanh \phi < 1$. In particular, if the velocity of $\mathcal{O}$" with respect to $\mathcal{O}'$ is $v_2 = c$, then it is $v = c$ with respect to $\mathcal{O}$ (this again expressed the invariance of light rays propagating at the speed of light $c$ with respect to all frames). The speed of light is thus the limiting speed in the theory.